Additions to chapter 5.2 (non-linear instability)

The effect of non-linear instability and aliasing is illustrated by the following simple example, We study the non-linear advection equation

$$\partial_t u = -u \partial_x u \tag{1}$$

Terms like this describe advection in the full non-simplified governing equation.
We now introduce one Fourier-component $u = A \sin \alpha x$ and obtain

$$\partial_t u = -A\alpha \sin \alpha x \cdot \cos \alpha x = -\frac{1}{2} A\alpha \sin 2\alpha x \tag{2}$$

The non-linear advection term has created a non-zero time derivative for a wave component with twice the initial wave-number, i.e. half wave length. In this way smaller and smaller waves are created (and obtain energy) so that there is an energy cascade from the larger to the smaller scales. This happens in non-linear models as well as in nature. *But*, in a model we are not able to describe waves which are shorter than a certain limit depending on the grid length. We can surely not resolve waves that are smaller than $L = 2\Delta x$.

So what happens if we say, started with a wave length $L = 3\Delta x$? After integration we would have created a wave with $L = \frac{3}{2}\Delta x$. This wave can not be represented in our grid but is instead folded into a wave on the "right side" of the lower limit $L = 2\Delta x$.

Since

$$\sin \alpha x_j = \sin \alpha\, j\Delta x = \sin(\alpha\, j\Delta x + 2\pi n j) =$$
$$\sin((\alpha + \frac{2\pi n j}{j\Delta x})\, j\Delta x) \equiv \sin(\alpha + 2\alpha_{max} n)x_j \tag{3}$$
$$(\alpha_{max} = \pi\big/\Delta x)$$

it is not possible to distinguish between wave-numbers α and $(\alpha + 2\alpha_{max} n)$. ($n$ is an integer.)

To see which wavelength wave number $(\alpha + 2\alpha_{max} n)$ represents we introduce wave lengths,
i.e., $\alpha = \dfrac{2\pi}{L}$
Then,

$$\alpha + 2\alpha_{max} n = \frac{2\pi}{L} + \frac{2\pi}{\Delta x} n \equiv \frac{2\pi}{L^*}$$

where the "false" or folded wavelength L^* is defined by

$$\frac{1}{L^*} = \frac{1}{L} + \frac{n}{\Delta x}$$

$$L^* = \frac{L}{1 + \dfrac{L}{\Delta x} n}$$

As an example we chose $L = 4\Delta x$. Then for $n = 1$, we obtain $L^* = \dfrac{4\Delta x}{1 + \dfrac{4\Delta x}{\Delta x} 1} = \dfrac{4}{5}\Delta x$

It is apparently not possible to distinguish between the two waves $L = \dfrac{4}{5}\Delta x$ and $L = 4\Delta x$. Figure 1 below shows this behaviour. The two waves pass through their common grid points.
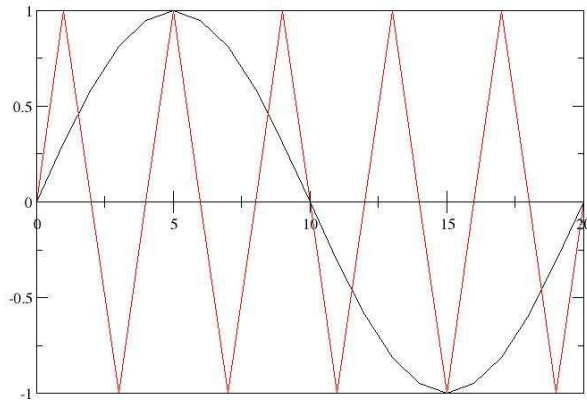

Figure 1. Two sin-waves with wavelengths $L = 4/5\Delta x$ and $L = 4\Delta x$ respectively. The grid length is plotted as ticks on the x-axis (5 corresponds to $1\Delta x$, 20 corresponds to $4\Delta x$ etc.). The longer wave therefore passes the x-axis at $x = 2\Delta x$ and $x = 4\Delta x$. The grid point values of the shorter wave exactly match the grid point values of the longer wave at these grid points. It is therefore impossible to distinguish the shorter wave from the longer wave when only the grid-point values are available.