The momentum equation.

Using z (height) as vertical coordinate, the horizontal momentum equation (in vector notation) is,

$$D_t\vec{v} = -f\vec{k}\times\vec{v} - \rho^{-1}\nabla_z p + \vec{F}$$

where D_t is the material derivative $D_t = \partial_t + \vec{v}\cdot\nabla_z + w\partial_z$,
$\vec{F}$ denotes friction forces and subscript z means derivation along a constant z-surface. The vertical component of the momentum equation is assumed simplified to the hydrostatic equation $\partial_z p = -g\rho$. Recall that in a general vertical coordinate system (s) the material derivative is

$$D_t = \partial_t + \vec{v}\cdot\nabla_s + (D_t s)\partial_s = \partial_t + \vec{v}\cdot\nabla_s + \dot{s}\,\partial_s \qquad (1)$$

and that the gradient transforms to,

$$\nabla_z() = \nabla_s() - \partial_s()\partial_z s\nabla_s z = \nabla_s() - \partial_z()\nabla_s z$$

The horizontal component of the momentum equation then transforms to:
$D_t\vec{v} = -f\vec{k}\times\vec{v} - \rho^{-1}(\nabla_s p - \partial_z p\nabla_s z) + \vec{F}$ and, by using the hydrostatic equation we obtain,

$$D_t\vec{v} = -f\vec{k}\times\vec{v} - \rho^{-1}(\nabla_s p + g\rho\nabla_s z) + \vec{F} = -f\vec{k}\times\vec{v} - \rho^{-1}(\nabla_s p + \rho\nabla_s\varphi) + \vec{F}$$

$$D_t\vec{v} = -f\vec{k}\times\vec{v} - \rho^{-1}\nabla_s p - \nabla_s\varphi + \vec{F} \qquad (2)$$

where $\varphi = gz$ is the geopotential.

We recognize that the horizontal pressure force is replaced by two terms. This comes from the fact that the s-surfaces may slope in relation to z. Taking derivatives along this sloping surface instead of the horizontal z-surface introduces an error which is compensated by the second term.

The first law of thermodynamics

The first law of thermodynamics in z-coordinates is,

$$D_t T = -\frac{p}{c_v}D_t\rho^{-1} + \frac{Q}{c_v} \qquad (3)$$

which becomes unaltered in general coordinates except that the we must remember to use the definition of the material derivative in s-coordinates (1). Here Q is diabatic heating.

The equation of state

The equation of state (for the atmosphere) is

$$p = \rho RT \qquad (4)$$

Since this equation contains no derivation is (of course) unaltered.

Transformation to specific coordinate systems

a) pressure coordinates

Using pressure as vertical coordinate is common in meteorology and oceanography because it simplifies the equations. We use s=p. The momentum equation then transforms into,

$$D_t\vec{v} = -f\vec{k}\times\vec{v} - \rho^{-1}\nabla_p p \nabla_p \varphi + \vec{F}$$

Since $\nabla_p p = 0$ (pressure does not change along a constant pressure surface!), we obtain The momentum equation in pressure coordinates

$$D_t\vec{v} = -f\vec{k}\times\vec{v} - \nabla_p\varphi + \vec{F} \tag{5}$$

The hydrostatic equation becomes,

$$\partial_p\varphi = -\rho^{-1} \tag{6}$$

The continuity equation in general coordinates is,

$$D_t(\partial_s p) = -\partial_s p(\nabla_s\cdot\vec{v} + \partial_s\dot{s})$$

introducing s=p and noticing that $\partial_p p = 1$ give,

$$\nabla\cdot\vec{v} + \partial_p\omega = 0 \tag{7}$$

The first law of thermodynamics and the equation of state are unchanged. Equations 5, 6 and 7 together with 3 and 4 are the pressure coordinate equations. Note that the material derivative is

$$D_t = \partial_t + \vec{v}\cdot\nabla + \omega\partial_p, \tag{8}$$

and that the vertical velocity in pressure coordinates ω is defined as

$$\omega = D_t p. \tag{9}$$

Note that the equations appear simplified; the continuity contains no explicit time derivative and density is removed from the pressure force term in the momentum equation. This set of equations is widely used in meteorological theory and was used in the first generation of numerical models. They have however some drawbacks: First of all, a constant pressure surface in general never coincides with the earth's surface so there exists no simple lower boundary condition. Secondly, high mountains will cut through the surfaces creating areas where the dependent variables are undefined (see figure below).

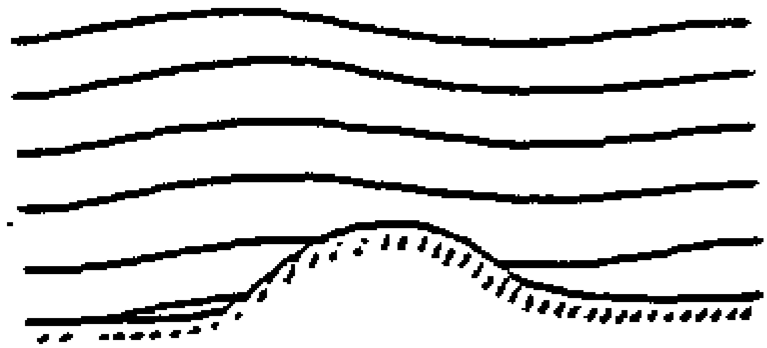


Figure 1. Sketch of surfaces of constant pressure and the underlying terrain.

Pressure varies between zero at the top of the atmosphere and p_0 (pressure at ground). The upper boundary (p=0) is therefore a nice property of the pressure surface coordinate system while for instance in z-coordinates one will have to put the top of the system at some height H (where the pressure is different from zero) or at infinity.

Today, other alternatives are used; some even combine the nice property of the pressure coordinates at the top of the atmosphere while having a terrain following system close to ground, see below.

The Sigma coordinate system
The Sigma coordinate system (or variants) of it is based on normalized pressure.
σ is defined as

$$\sigma = p / p_0 \tag{10}$$

where p_0 is pressure at the earth's surface $p_0 = p_0(x, y, t)$
This coordinate system has the property that σ varies between zero at the top of the atmosphere and 1 at ground. Since these coordinate surfaces also are material surfaces (air does not move through them) we also has the nice property that $\dot{\sigma} = 0$ at the upper and lower boundaries.

Using $s = \sigma$ in <u>the momentum equation,</u> (2) give

$$D_t\vec{v} = -f\vec{k} \times \vec{v} - \rho^{-1}\nabla p - \nabla\varphi + \vec{F} \tag{11}$$

<u>The hydrostatic equation</u> becomes,

$$g\partial_\sigma z = -\rho^{-1}\partial_\sigma p$$

which, by using the (10) is simplified to,

$$\partial_\sigma \varphi = -\rho^{-1} p_0 \tag{12}$$

<u>The continuity equation</u> becomes,

$$D_t(\partial_\sigma p) = -\partial_\sigma p(\nabla \cdot \vec{v} + \partial_\sigma \dot{s})$$

which, by use of the definition (10) changes to,

$$D_t p_0 = -p_0(\nabla \cdot \vec{v} + \partial_\sigma \dot{\sigma}) \tag{13}$$

As for the pressure coordinate system, <u>the first law of thermodynamics</u> and <u>the equation of state</u> are not changing, (except of course that the material derivative for the sigma coordinate system has to be used), i.e.

$$D_t = \partial_t + \vec{v} \cdot \nabla \ \ + \dot{\sigma}\partial_\sigma \tag{14}$$

and that we must remember to calculate horizontal derivatives along a constant σ surface. To solve these equations (11, 12, 13, 3, 4) we need to calculate the geopotential as well as the vertical velocity $\dot{\sigma}$ at the σ surfaces. The geopotential is computed from (12) by integrating upwards from $\sigma = 1$, i.e. the earth's surface where $\varphi = gh_0$, h_0 is the height of the terrain (above sea level). We may write (12) as,

$$\partial_\sigma \varphi = -RT / \sigma$$

so that

$$\varphi(\sigma) = \varphi(\sigma = 1) - \int_1^\sigma \frac{RT}{\sigma} d\sigma \tag{15}$$

or numerically,

$$\varphi(\sigma) = \varphi(1) - \sum_1^\sigma \frac{RT}{\sigma} \Delta\sigma \tag{16}$$

To calculate $\dot{\sigma}$ we manipulate the continuity equation. First it is integrated from $\sigma = 0$ to $\sigma = 1$, i.e.

$$\int_0^1 D_t(\partial_\sigma p) d\sigma = -\int_0^1 \partial_\sigma p(\nabla \cdot \vec{v} + \partial_\sigma \dot{s}) d\sigma$$

From the definition of σ, we have $p = \sigma p_0$ so that $\partial_\sigma p = p_0(x, y, t)$

The integral on the left hand side becomes simply $D_t p_0$ and the right hand side transforms into $-p_0 \int_0^1 (\nabla \cdot \vec{v} + \partial_\sigma \dot{\sigma}) d\sigma$

Thus, we obtain an equation for the change of surface pressure;

$$D_t p_0 = -p_0 \int_0^1 (\nabla \cdot \vec{v}) d\sigma \tag{17}$$

where we have used the property that $\dot{\sigma} = 0$ at $\sigma = 0$ and $\sigma = 1$.

By writing $D_t p_0 = \partial_t p_0 + v \cdot \nabla p_0$ and using the boundary conditions, (17) simplifies further,

$$\partial_t p_0 = -\int_0^1 (\nabla \cdot p_0 \vec{v} d\sigma \tag{18}$$

Since $p_0 \vec{v}$ is the flux of air into the grid column, this equation states the obvious fact that surface pressure increases (decreases) if more (less) air moves into the column above than out. In (18) we have derived an equation for the change of surface pressure with time.

To find the vertical velocity $\dot{\sigma}$ we apply the continuity equation once more, but now we integrate from the surface to a σ level, i.e.

$$\int_1^\sigma D_t p_0 d\sigma = -\int_1^\sigma p_0 (\nabla \cdot \vec{v} + \partial_\sigma \dot{\sigma}) d\sigma$$

which, since p_0 is not a function of σ, transforms into,

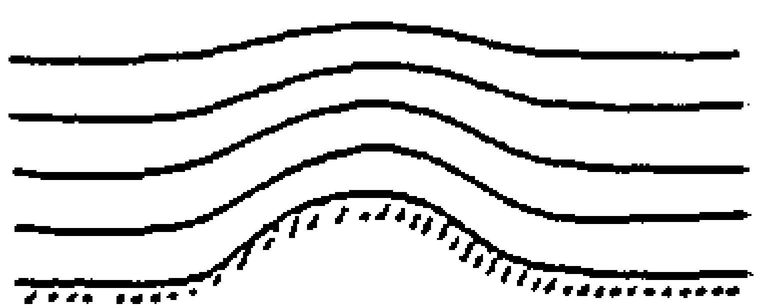
$$\sigma D_t p_0 = -\int_1^\sigma p_0 \nabla \cdot \vec{v} d\sigma - \int_1^\sigma p_0 d\dot{\sigma} = -\int_1^\sigma p_0 \nabla \cdot \vec{v} d\sigma - \dot{\sigma}(\sigma)$$

writing out the material derivative at the left hand side, gives finally,

$$\dot{\sigma}(\sigma) = \frac{1}{p_0}(\sigma \partial_t p_0 + \int_0^\sigma \nabla \cdot p_0 \vec{v} d\sigma) \tag{19}$$

The equations are solved by first integrating the hydrostatic equation (15) upwards from the surface to obtain the geopotential at all σ levels. Then the surface pressure tendency is computed from (18) and vertical velocity $\dot{\sigma}$ from (19). We have then sufficient information to calculate the material derivative and the rest of the terms of the horizontal momentum equation to obtain values for $\partial_t u$ and $\partial_t v$. From the first law of thermodynamics we compute temperature tendencies $\partial_t T$ and we have enough information to progress forward in time by for instance the Leapfrog scheme.

The σ system has the nice property of following the ground surface as the lower boundary by definition is at the ground. The surfaces undulate up and down according to the terrain



below.

Figure 2. Sketch of sigma coordinate surfaces and the underlying terrain.

There exists several variants of the σ system. At the Norwegian Meteorological Institute they used to have a model where σ was defined as

$$\sigma = (p - p_T)/(p_0 - p_T)$$

where p_T is the top of the model, different from zero.

The European Centre for Medium Range Weather Forecasts (ECMWF) uses a coordinate system which follows ground at the lower levels, but turns smoothly into a pressure coordinate system at upper levels. This is done in order to make use of the benefits of both the σ and the pressure systems. A problem with the σ system is that the coordinate surfaces bulb up and down all the way to the model top. Since the pressure force is computed as a difference between two (large) terms, this may introduce errors.
In their coordinate system, η, pressure in coordinate surfaces is defined as,

$$p(\eta) = a(\eta)p_0 + b(\eta) \tag{20}$$

Close to ground $b(\eta) = 0$ and the system have the same properties as the σ system. (Remember that in the σ system $p(\sigma) = \sigma p_0$). At high levels $a(\eta) = 0$ and $b(\eta) = p$, i.e. a pressure coordinate system. The same approach is used in the operational weather prediction model HIRLAM which at present is used at the Norwegian Meteorological Institute. Other alternatives employs using normalized height, i.e. variants of $\eta = \frac{z - h_0}{h_0}$.
Where h_0 is terrain height. The operational weather prediction model (and the climate model) at the UK Meteorological Office (Hadley centre) have chosen this approach.