A Brief Introduction to Atmospheric Dynamics

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1 General dynamics

The motion in the atmosphere is governed by a set of equations, known as the *Navier-Stokes* equations. These equations, solved numerically by computers, are used to produce our weather forecasts. While there are details about these equations which are uncertain (for example, how we parameterize processes smaller than the grid size of the models), the equations for the most part are accepted as fact. In the following, we will consider how these equations come about.

1.1 Derivatives

Weather forecasting specifies how fields (temperature, wind, atmospheric moisture) change in time and space. Thus we must first specify how to take derivatives.

Consider a scalar, ψ, which varies in both time and space. By the chain rule, the total change in the ψ is:

$$d\psi = \frac{\partial}{\partial t}\psi \, dt + \frac{\partial}{\partial x}\psi \, dx + \frac{\partial}{\partial y}\psi \, dy + \frac{\partial}{\partial z}\psi \, dz \tag{1}$$

so

$$\frac{d\psi}{dt} = \frac{\partial}{\partial t}\psi + u \, \frac{\partial}{\partial x}\psi + v \, \frac{\partial}{\partial y}\psi + w \, \frac{\partial}{\partial z}\psi = \frac{\partial}{\partial t}\psi + \vec{u} \cdot \nabla\psi \tag{2}$$

We refer to the left side as the *Lagrangian* derivative. The time derivative on the RHS is the *local* derivative, and the RHS is called the *Eulerian* formulation. The Lagrangian formulation applies to moving measurements, like balloons, while Eulerian applies to fixed measurements, like weather stations.

1.2 Continuity equation

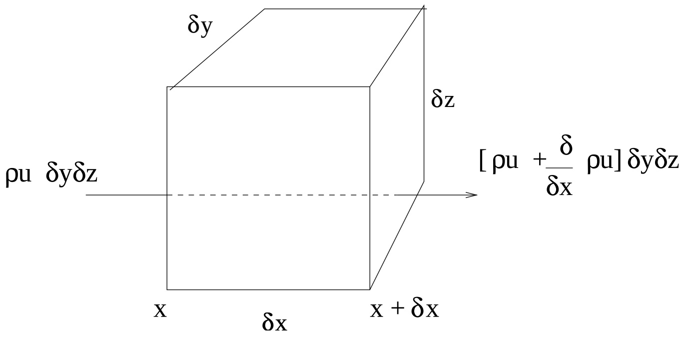

Figure 1: A infinitesimal element of air, with volume δV.

Consider a box fixed in space, with air flowing through it. The flux of density through the left side is:

$$(\rho u)\,\delta y\,\delta z \tag{3}$$

Through the right side the flux is:

$$[\rho u + \frac{\partial}{\partial x}(\rho u)\delta x]\,\delta y\,\delta z \tag{4}$$

The net rate of change in mass is:

$$\begin{aligned}\frac{\partial}{\partial t}M = \frac{\partial}{\partial t}(\rho\,\delta x\,\delta y\,\delta z) &= (\rho u)\,\delta y\,\delta z - [\rho u + \frac{\partial}{\partial x}(\rho u)\delta x]\,\delta y\,\delta z \\ &= -\frac{\partial}{\partial x}(\rho u)\delta x\,\delta y\,\delta z \end{aligned} \tag{5}$$

The volume of the box is constant, so:

$$\frac{\partial}{\partial t}\rho = -\frac{\partial}{\partial x}(\rho u) \tag{6}$$

Taking into account all the other sides of the box we have:

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \vec{u}) \tag{7}$$

We can rewrite the RHS and put it on the LHS:

$$\nabla \cdot (\rho \vec{u}) = \rho \nabla \cdot \vec{u} + \vec{u} \cdot \nabla \rho \tag{8}$$

Rearranging:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = \frac{\partial \rho}{\partial t} + \vec{u} \cdot \nabla \rho + \rho(\nabla \cdot \vec{u}) = \frac{d\rho}{dt} + \rho(\nabla \cdot \vec{u}) = 0 \tag{9}$$

We can also obtain the same result using a Lagrangian (moving) box. The box contains a fixed amount of air, so it conserves it mass. We can write:

$$\frac{1}{M}\frac{d}{dt}M = \frac{1}{\rho \delta V}\frac{d}{dt}(\rho \delta V) = \frac{1}{\rho}\frac{d\rho}{dt} + \frac{1}{\delta V}\frac{d\delta V}{dt} = 0 \tag{10}$$

Expanding the volume term:

$$\frac{1}{\delta V}\frac{d\delta V}{dt} = \frac{1}{\delta x}\frac{\partial \delta x}{dt} + \frac{1}{\delta y}\frac{\partial \delta y}{dt} + \frac{1}{\delta z}\frac{\partial \delta z}{dt} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \tag{11}$$

So:

$$\frac{1}{\rho}\frac{d\rho}{dt} + \nabla \cdot \vec{u} = 0 \tag{12}$$

which is the same as (9). Thus the change in the relative density is proportional to the velocity *divergence*. If the box expands, the density decreases, to preserve the box's mass.

1.3 Equations of motion

The continuity equation pertains to mass. Now we consider wind velocities. We can derive expressions for these from Newton's second law:

$$\vec{a} = \vec{F}/m \tag{13}$$

The forces acting on an air parcel (a vanishingly small box) are:

- pressure gradients
- gravity
- friction

For a parcel with density ρ, we can write:

$$\frac{d}{dt}\vec{u} = -\frac{1}{\rho}\nabla p + \vec{g} + \vec{F} \tag{14}$$

This is the *momentum equation*. We use the Lagrangian derivative because we are following the air parcel.

We have additional acceleration terms because the earth is rotating. Consider first a stationary object on a rotating sphere. Even though the object is not moving on the sphere, it appears to move when viewed from space (a fixed frame) because of the rotation. Consider the vector, $\vec{A}$, in Fig. (2). In time, δt, the vector rotates through an angle:

$$\delta\Theta = \Omega\delta t \tag{15}$$

We will assume the rotation rate is constant ($\Omega = const.$), which is reasonable for the earth on weather time scales. The change in A is equal to δA, the arc-length:

$$\delta\vec{A} = |\vec{A}|sin(\gamma)\delta\Theta = \Omega|\vec{A}|sin(\gamma)\delta t = (\vec{\Omega} \times \vec{A})\,\delta t \tag{16}$$

So:

$$\frac{d\vec{A}}{dt} = \vec{\Omega} \times \vec{A} \tag{17}$$

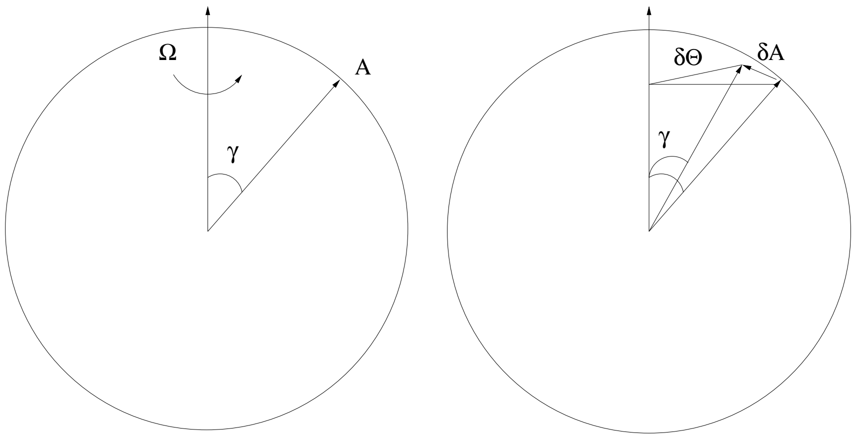

Figure 2: The effect of rotation on a vector, A, which is otherwise stationary. The vector rotates through an angle, $\delta\Theta$, in a time δt.

If the vector is not stationary but moving in the rotating frame, one can show that:

$$(\frac{d\vec{A}}{dt})_F = (\frac{d\vec{A}}{dt})_R + \vec{\Omega} \times \vec{A} \tag{18}$$

The F here refers to the fixed frame and R to the rotating one. If $\vec{A} = \vec{r}$, the position vector, then:

$$(\frac{d\vec{r}}{dt})_F \equiv \vec{u}_F = \vec{u}_R + \vec{\Omega} \times \vec{r} \tag{19}$$

So the acceleration is:

$$(\frac{d\vec{u}_F}{dt})_F = (\frac{d\vec{u}_F}{dt})_R + \vec{\Omega} \times \vec{u}_F = [\frac{d}{dt}(u_F + \vec{\Omega} \times \vec{r})]_R + \vec{\Omega} \times \vec{u}_F =$$

$$(\frac{d\vec{u}_R}{dt})_R + 2\vec{\Omega} \times \vec{u}_R + \vec{\Omega} \times \vec{\Omega} \times \vec{r} \tag{20}$$

We have two additional terms: the *Coriolis* and *centrifugal* accelerations. Plugging these into the momentum equation, we obtain:

$$(\frac{d\vec{u}_F}{dt})_F = (\frac{d\vec{u}_R}{dt})_R + 2\vec{\Omega} \times \vec{u}_R + \vec{\Omega} \times \vec{\Omega} \times \vec{r} = -\frac{1}{\rho}\nabla p + \vec{g} + \vec{F} \quad (21)$$

or:

$$(\frac{d\vec{u}_R}{dt})_R + 2\vec{\Omega} \times \vec{u}_R = -\frac{1}{\rho}\nabla p + \vec{g}' + \vec{F} \quad (22)$$

where:

$$g' = g - \vec{\Omega} \times \vec{\Omega} \times \vec{r} \quad (23)$$

We have absorbed the centrifugal force into the gravity term because both are constant in time and both act in the radial direction. We can understand this as follows. The spinning earth yields a centrifugal (negative centripetal) force outwards. If the spinning were too fast, we would all fly off into space. But because we don't fly off, gravity is strong enough to hold us down. We define the net downward force (gravity plus centrifugal) as g'.

There are three spatial directions and thus three momentum equations, in the corresponding directions. In what follows, we assume that we are in localized region of the atmosphere, centered at a latitude, θ. Then we can define local coordinates (x, y, z) such that:

$$\delta x = acos(\theta)\delta\phi, \; \delta y = a\delta\theta, \; \delta z = \delta R$$

where ϕ is the longitude, a is the earth's radius and R is the radius. We define:

$$u \equiv \frac{dx}{dt},\ v \equiv \frac{dy}{dt},\ w \equiv \frac{dz}{dt}$$

We will write expressions for du/dt, etc.

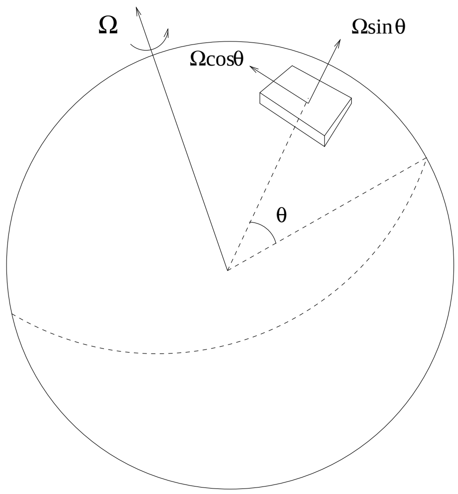

Figure 3: A region of the atmosphere at latitude θ. The earth's rotation vector projects onto the local latitudinal and radial coordinates.

Note first that the Coriolis term projects onto the y and z directions:

$$2\vec{\Omega} \times \vec{u} = 2\Omega(w\,cos\theta - v\,sin\theta,\ u\,sin\theta, -u\,cos\theta) \tag{24}$$

In addition, the use of a spherical coordinate system introduces some additional "curvature terms" (see Batchelor, *Fluid Dynamics*). These involve products of the velocities and can be derived using conservation of momentum arguments (see e.g. Holton, *An Introduction to Dynamical Meteorology*). Adding these, and the Coriolis terms, yields:

$$\frac{du}{dt} - \frac{uvtan\theta}{a} + \frac{uw}{a} + 2\Omega w\,cos\theta - 2\Omega v\,sin\theta = -\frac{1}{\rho}\frac{\partial}{\partial x}p + F_x \tag{25}$$

$$\frac{dv}{dt} + \frac{u^2tan\theta}{a} + \frac{vw}{a} + 2\Omega u\,sin\theta = -\frac{1}{\rho}\frac{\partial}{\partial y}p + F_y \tag{26}$$

$$\frac{dw}{dt} - \frac{u^2 + v^2}{a} - 2\Omega u \, cos\theta = -\frac{1}{\rho}\frac{\partial}{\partial z}p - g + F_z \qquad (27)$$

where F_i is the frictional force acting in the i direction.

1.4 Scaling the horizontal acceleration

Not all the terms in the momentum equations are important. Take the x-momentum equation:

$$\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u + w\frac{\partial}{\partial z}u - \frac{uvtan\theta}{a} + \frac{uw}{a} + 2\Omega w \, cos\theta - 2\Omega v \, sin\theta = -\frac{1}{\rho}\frac{\partial}{\partial x}p$$

$$\frac{U}{T} \quad \frac{U^2}{L} \quad \frac{U^2}{L} \quad \frac{UW}{D} \quad \frac{U^2}{a} \quad \frac{UW}{a} \quad 2\Omega W \quad 2\Omega U \quad \frac{\triangle p}{\rho L}$$

$$\frac{1}{2\Omega T} \quad \frac{U}{2\Omega L} \quad \frac{U}{2\Omega L} \quad \frac{W}{2\Omega D} \quad \frac{U}{2\Omega a} \quad \frac{W}{2\Omega a} \quad \frac{W}{U} \quad 1 \quad \frac{\triangle p}{2\Omega \rho U L}$$

For the moment, we neglect will friction (which is generally unimportant outside the planetary boundary layer). In the second line we have *scaled* the equation by assuming typical values for the variables. In the third line, we have divided through by the scaling for the scaling of the second Coriolis acceleration, $2\Omega U$.

Why have we done this? Note that the terms on the third line are all *dimensionless* parameters, i.e. they have no units. What we will do is to evaluate each and see how it compares to one (the size of the second Coriolis term). We have made an (educated) guess that this is one of the largest terms. If any of the preceding terms is much less than one, we can neglect it. If a term is much greater than one, than our assumption that the Coriolis term was the largest was wrong, and we have to divide again, using another term.

To evaluate the terms, we use scales which are typical of weather disturbances:

$$U \approx 10m/sec, \quad 2\Omega = \frac{4\pi}{86400\, sec} = 1.45 \times 10^{-4} sec^{-1},$$

$$L \approx 10^6 m, \quad D \approx 10^4 m, \quad T = L/U \approx 10^5 sec \quad a \approx 6400km$$

$$\triangle_H P/\rho \approx 10^3 m^2/sec^2, \quad W \approx 1cm/sec, \tag{28}$$

The horizontal scale, 1000 km, is known as the *synoptic scale* in the atmosphere. The time scale, proportional to the length scale divided by the velocity scale, is the *advective* time scale. This is what you'd expect, for example, if a front were advected by the winds past an observer. With an advective time scale, we have:

$$\frac{1}{2\Omega T} = \frac{U}{2\Omega L}$$

So the first term is the same size as the second and third terms. This ratio is the *Rossby number*, a fundamental parameter in dynamical meteorology. It has a value at synoptic scales of

$$\frac{U}{2\Omega L} = 0.067$$

Thus the first three terms are smaller than the second Coriolis term.

However, the other terms are even smaller. The fourth term:

$$\frac{W}{2\Omega D} = 0.0067$$

is about 10 times smaller than the Rossby number. The fifth and sixth terms:

$$\frac{U}{2\Omega a} = 0.01, \quad \frac{W}{2\Omega a} = 10^{-5}$$

are also smaller than the Rossby number. And the seventh term,

$$\frac{W}{U} = .001$$

is also small.

The pressure gradient term scales as:

$$\frac{\triangle p}{2\Omega \rho U L} = 0.70$$

Thus this term is comparable in size to the second Coriolis term.

Saving only the terms which are the size of the Rossby number or larger, the x-momentum equation is approximately:

$$\frac{du}{dt} - fv = -\frac{1}{\rho}\frac{\partial}{\partial x}p \tag{29}$$

The same reasoning yields:

$$\frac{dv}{dt} + fu = -\frac{1}{\rho}\frac{\partial}{\partial y}p \tag{30}$$

where the Lagrangian derivative:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$$

now does not include the vertical advection term. Thus the advection in these approximate equations is *quasi-horizontal* (two-dimensional). These equations are fairly accurate at synoptic scales, and we will focus on them from now on.

In addition, we have defined:

$$f \equiv 2\Omega sin\theta$$

This is the vertical component of the Coriolis parameter, the only one which is important at these scales.

1.5 Geostrophic balance

The scaling suggest that the first order balance in the momentum equations is between the Coriolis and pressure gradient forces:

$$-fv = -\frac{1}{\rho}\frac{\partial}{\partial x}p \tag{31}$$

$$+fu = -\frac{1}{\rho}\frac{\partial}{\partial y}p \tag{32}$$

This is called the *geostrophic* balance. It is one of two fundamental balances at synoptic scales. The balance implies that if we know the pressure, we can deduce the velocities. So the winds can be determined from maps of the pressure.

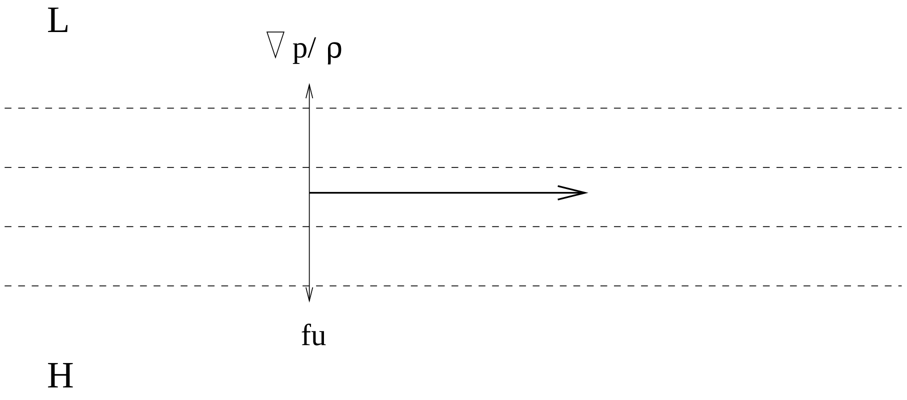

Figure 4: The geostrophic balance.

Consider the flow in Fig. (4). The pressure is high to the south and low to the north. Left alone, this would force the air to move north. Because $\frac{\partial}{\partial y}p < 0$, we have that $u > 0$ (eastward), from (32). The Coriolis force is acting to the right of the motion, exactly balancing the pressure gradient force. And because the two forces balance, the motion is constant in time.

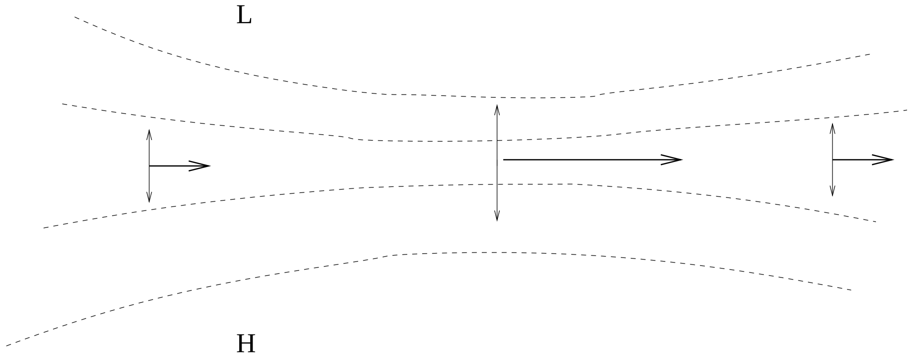

Figure 5: Geostrophic flow with non-constant pressure gradients.

If the pressure gradient changes in space, so will the geostrophic velocity. In Fig. (5), the flow accelerates into a region with more closely-packed pressure contours, then deaccelerates exiting the region.

Since $f = 2\Omega sin\theta$, it is *negative* in the southern hemisphere. So the flow in Fig. (4) would be westward, with the Coriolis force acting to the left. In addition, the Coriolis force is identically *zero* at the equator. So the geostrophic balance cannot hold there.

1.6 Other momentum balances

The geostrophic balance occurs at synoptic scales, but other balances are possible at smaller scales. To see this, consider a perfectly circular flow (Fig. 6). The momentum equation in cylindrical coordinates (e.g. Batchelor, *Fluid Mechanics*) for the velocity in the radial direction is given by:

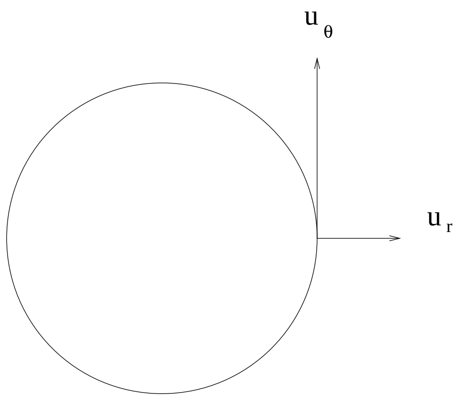

Figure 6: Circular flow.

$$\frac{\partial}{\partial t}u_r + (u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta})\,u_r - \frac{u_\theta^2}{r} - fu_\theta = \frac{d}{dt}u_r - \frac{u_\theta^2}{r} - fu_\theta = -\frac{1}{\rho}\frac{\partial}{\partial r}p \quad (33)$$

The term u_θ^2/r is called the *cyclostrophic* term and is a curvature term like those found with spherical coordinates. This is specifically related to the centripetal acceleration. If the flow is *steady* (not changing in time), then we have:

$$\frac{u_\theta^2}{r} + fu_\theta = \frac{1}{\rho}\frac{\partial}{\partial r}p \quad (34)$$

$$\frac{U^2}{R} \qquad 2\Omega U \qquad \frac{\triangle p}{\rho R}$$

$$\frac{U}{2\Omega R} \qquad 1 \qquad \frac{\triangle p}{2\rho\Omega U R}$$

We have scaled the equation as before. Note that the scale of the cyclostrophic term is determined by the Rossby number. Let's define that as ϵ. If $\epsilon \ll 1$, the cyclostrophic term is much smaller than the Coriolis term. Then, we must have:

$$\frac{\triangle p}{2\rho\Omega UR} \approx 1$$

and we have the geostrophic balance again:

$$fu_\theta = \frac{1}{\rho}\frac{\partial}{\partial r}p \tag{35}$$

Note that if the term on the RHS wasn't order one, the pressure gradient wouldn't be large enough to balance the Coriolis force and there would be no velocity.

Now consider if $\epsilon \gg 1$. For example, a tornado at mid-latitudes has:

$$U \approx 30m/s, \quad f = 10^{-4}sec^{-1}, \quad R \approx 300m,$$

So $\epsilon = 1000$. Then the cyclostrophic term dominates over the Coriolis term. As we noted earlier, that means that we shouldn't have divided the scaling parameters by $2\Omega U$, but rather U^2/R. So we would have:

$$1 \qquad \frac{2\Omega R}{U} \qquad \frac{\triangle p}{\rho U^2}$$

Now the second term, which is just one over the Rossby number, is very small (0.001 for the tornado) and we require:

$$\frac{\triangle p}{\rho U^2} \approx 1$$

In this case, we have the *cyclostrophic wind balance*:

$$\frac{u_\theta^2}{r} = \frac{1}{\rho}\frac{\partial}{\partial r}p \tag{36}$$

Notice that this is a *non-rotating* balance (because Ω doesn't enter). The pressure gradient is balanced by the centrifugal acceleration.

There is a third possibility, that there is no radial pressure gradient at all. This is called *inertial flow*. Then:

$$\frac{u_\theta^2}{r} + f u_\theta = 0 \quad \rightarrow \quad u_\theta = -fr \tag{37}$$

The velocity is negative, implying the rotation is clockwise (or "anticyclonic") in the Northern Hemisphere. The time for a parcel to complete a full circle is:

$$\frac{2\pi r}{u_\theta} = \frac{2\pi}{f} = \frac{0.5\ day}{|sin\theta|}\ , \tag{38}$$

which is known as the "inertial period". "Inertial oscillations" are frequently seen at the ocean surface, but are much rarer in the atmosphere.

The last possibility is that $\epsilon = 1$, in which case all three terms in (34) are important. This is the *gradient wind balance*. We can then solve for u_θ using the quadratic formula:

$$u_\theta = -\frac{1}{2} fr \pm \frac{1}{2}(f^2 r^2 + \frac{4r}{\rho}\frac{\partial}{\partial r}p)^{1/2} = -\frac{1}{2} fr \pm \frac{1}{2}(f^2 r^2 + 4\, r\, f\, u_g)^{1/2}\ , \tag{39}$$

after substituting in the definition of the geostrophic velocity. Note that if the pressure gradient vanishes, we recover the inertial velocity. The gradient wind estimate clearly differs from the geostrophic estimate.

This difference is typically about 10-20 % at mid-latitudes. To see this, we rewrite (34) thus:

$$\frac{u_\theta^2}{r} + f u_\theta = \frac{1}{\rho}\frac{\partial}{\partial r}p = f u_g \tag{40}$$

Then:

$$\frac{u_g}{u_\theta} = 1 + \frac{u_\theta}{fr} \tag{41}$$

The last term scales as the Rossby number. So if $\epsilon = 0.1$, the gradient wind estimate differs from the geostrophic value by 10 %. At low latitudes, where ϵ can be 1-10, the gradient wind estimate is more accurate.

An interesting point is that geostrophy flow is *symmetric to sign changes*. If we change the sign of the pressure gradient, we change the sign, but not the speed, of the wind. But this is not true with the gradient wind. If $u_g < 0$, which corresponds to anticyclonic (clockwise) rotation, we require:

$$|u_g| < \frac{fr}{4} \tag{42}$$

so that the root in (39) is positive. But there is no such limit with cyclonic (counter-clockwise) rotation. So cyclones can be much stronger under the gradient wind approximation.

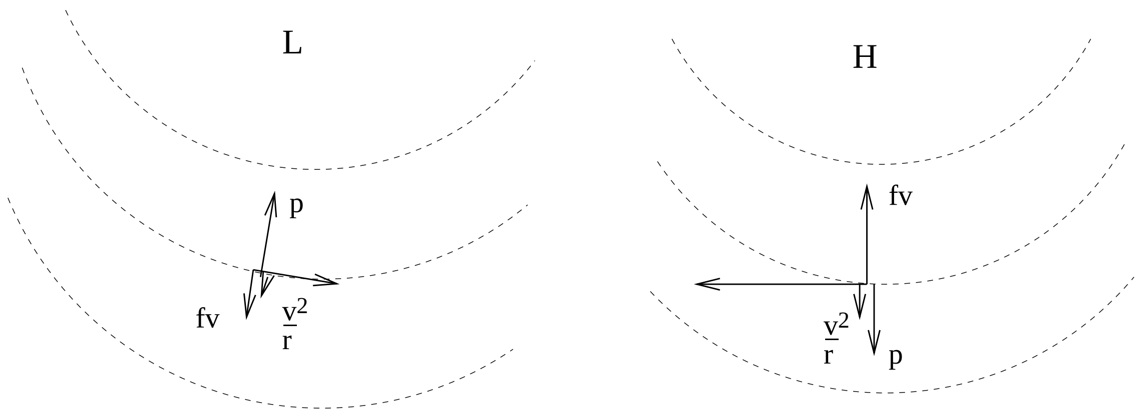

Figure 7: High and low pressure center flows under the gradient wind approximation.

The gradient wind balance alters the strength of the winds around high and low pressure systems. Consider the examples in Fig. (7). For a low pressure system (at left), the wind is somewhat weaker, because the cyclostrophic term acts in the same direction as the Coriolis term, to balance the pressure gradient force. With a high pressure system though, the Coriolis and cyclostrophic forces are opposed. So the Coriolis term must balance the other two and the wind is stronger.

If the cyclostrophic term is large enough, gradient wind vortices can actually have the pressure gradient and Coriolis forces in the *same* direction (Fig. 8). This can only occur for cyclones. Such "anomalous lows" (low pressure with clockwise flow) are usually found only near the equator.

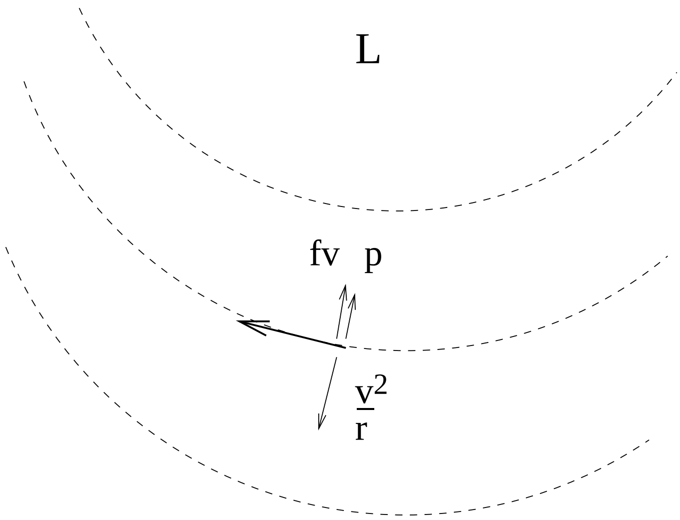

Figure 8: An anomalous low pressure system.

1.7 Hydrostatic balance

Now we will scale the vertical momentum equation. For this, we need an estimate of the vertical variation in pressure:

$$\triangle_V P/\rho \approx 10^5 m^2/sec^2$$

Neglecting the friction term, F_z, we have:

$$\frac{\partial}{\partial t}w+u\frac{\partial}{\partial x}w+v\frac{\partial}{\partial y}w+w\frac{\partial}{\partial z}w-\frac{u^2+v^2}{a}-2\Omega u cos\theta = -\frac{1}{\rho}\frac{\partial}{\partial z}p-g \quad (43)$$

$$\frac{WU}{L} \qquad \frac{UW}{L} \qquad \frac{UW}{L} \qquad \frac{W^2}{D} \qquad \frac{U^2}{a} \qquad 2\Omega U \qquad \frac{\triangle_V P}{\rho D} \qquad g$$

$$\frac{UW}{gL} \qquad \frac{UW}{gL} \qquad \frac{UW}{gL} \qquad \frac{W^2}{gD} \qquad \frac{U^2}{ga} \qquad \frac{2\Omega U}{g} \qquad \frac{\triangle_V P}{g\rho D} \qquad 1$$

$$10^{-8} \qquad 10^{-8} \qquad 10^{-8} \qquad 10^{-11} \qquad 2\times 10^{-6} \qquad 10^{-4} \qquad 1 \qquad 1$$

Notice that we used the advective time scale, L/U, for the time scale T and we have divided through by g, which we assume will be large. The vertical pressure gradient and gravity terms are much larger than any of the others. However this is somewhat misleading because we obtain the same balance if there is *no motion at all*! In particular, if $u = v = w = 0$, the vertical momentum equation is:

$$\frac{\partial}{\partial z}p = -\rho g \quad (44)$$

This is called the “hydrostatic balance” or literally the “non-moving fluid balance”. We aren’t particularly interested in this component of the flow, since weather comes from the dynamic (moving) component. So we separate the pressure and density into static and dynamic components:

$$p(x, y, z, t) = p_0(z) + p'(x, y, z, t)$$
$$\rho(x, y, z, t) = \rho_0(z) + \rho'(x, y, z, t) \quad (45)$$

Generally the dynamic components are much smaller than the static components, so that:

$$|p'| \ll |p_0| \tag{46}$$

Then we can write:

$$-\frac{1}{\rho}\frac{\partial}{\partial z}p - g = -\frac{1}{\rho_0 + \rho'}\frac{\partial}{\partial z}(p_0 + p') - g \approx -\frac{1}{\rho_0}\,(1 - \frac{\rho'}{\rho_0})\,\frac{\partial}{\partial z}(p_0 + p') - g$$

$$\approx -\frac{1}{\rho_0}\frac{\partial}{\partial z}p' + (\frac{\rho'}{\rho_0})\frac{\partial}{\partial z}p_0 = -\frac{1}{\rho_0}\frac{\partial}{\partial z}p' - \frac{\rho'}{\rho_0}g \tag{47}$$

Note we neglect terms proportional to the product of the dynamical variables, like $p'\rho'$.

Now the question is: how do we scale the dynamical pressure terms? Measurements suggest the vertical variation of p' is comparable to the horizontal variation, so:

$$\frac{1}{\rho_0}\frac{\partial}{\partial z}p' \propto \frac{\triangle_H P}{\rho_0 D} \approx 10^{-1} m/sec^2 \ .$$

The perturbation density, ρ', is roughly $1/100$ as large as the static density, so:

$$\frac{\rho'}{\rho_0}g \approx 10^{-1} m/sec^2 \ .$$

To scale these, we again divide by g. So both terms are of order 10^{-2}. So while they are smaller than the static terms, they are still *two orders of magnitude larger* than the next largest term in (43). Thus the approximate vertical momentum equation is still the hydrostatic balance, but for the perturbation pressure and density:

$$\frac{\partial}{\partial z}p' = -\rho' g \tag{48}$$

A weather model which uses this equation instead of the full vertical momentum equation is called a "hydrostatic model"; a model which uses the full vertical momentum is a "nonhydrostatic model". Notice that in the hydrostatic model, we have no information about $\frac{\partial}{\partial t}w$ and so have lost the ability to predict changes in the vertical velocity. Instead, w must be estimated from the other variables. We discuss this more later on.

1.8 Pressure coordinates

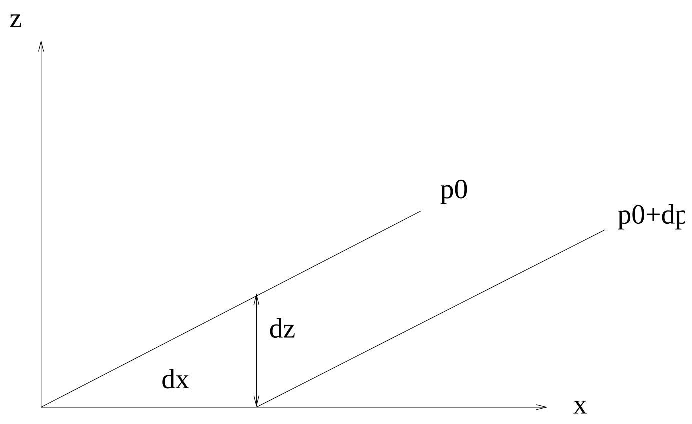

Figure 9: Surfaces of constant pressure. Note that the pressure increases here with increasing x, but decreases with increasing z.

The hydrostatic balance can be exploited to simplify the momentum and continuity equations. For this, we shift the vertical coordinate to one defined by pressure rather than height. Consider a 2-D pressure surface. By definition, the pressure doesn't change along the surface, so that:

$$\triangle p = \frac{\partial}{\partial x}p \triangle x + \frac{\partial}{\partial z}p \triangle z = 0 \tag{49}$$

Substituting the hydrostatic relation, we get:

$$\frac{\partial}{\partial x}p \triangle x - \rho g \triangle z = 0 \tag{50}$$

so that:

$$\frac{\partial p}{\partial x}|_z = -\frac{\partial p}{\partial z}\frac{\triangle z}{\triangle x}|_p = \rho g\frac{\triangle z}{\triangle x}|_p \equiv \rho\frac{\partial \Phi}{\partial x}|_p \tag{51}$$

where the subscripts indicate z and p coordinates, and where Φ is the *geo-potential*:

$$\Phi \equiv \int_0^z g\, dz \ . \tag{52}$$

Note that the geopotential is essentially a height, multiplied by g.

A big advantage of this coordinate change is that it removes the density from momentum equation, because:

$$-\frac{1}{\rho}\nabla p = -\nabla\Phi$$

So the horizontal momentum equations (29-30) can be written:

$$\frac{du}{dt} - fv = -\frac{\partial}{\partial x}\Phi \tag{53}$$

and

$$\frac{dv}{dt} + fu = -\frac{\partial}{\partial y}\Phi \tag{54}$$

and the geostrophic balance is simply:

$$fv = \frac{\partial}{\partial x}\Phi, \qquad fu = -\frac{\partial}{\partial y}\Phi \tag{55}$$

So if we know the geopotential on a given pressure surface, we can diagnose the velocities—without knowing the density.

In addition, the coordinate change simplifies the continuity equation. The easiest way to see this is to go back to our Lagrangian box. This has a volume:

$$\delta V = \delta x\, \delta y\, \delta z = -\delta x\, \delta y\, \frac{\delta p}{\rho g} \tag{56}$$

after substituting from the hydrostatic balance. The mass of the box is:

$$\rho\, \delta V = \rho\, \delta x\, \delta y\, \delta z = -\frac{1}{g} \delta x\, \delta y\, \delta p$$

again using the hydrostatic balance. Note that δp is negative here, so that the mass is positive. Conservation of the box's mass implies:

$$\frac{1}{\delta M} \frac{d}{dt} \delta M = \frac{g}{\delta x \delta y \delta p} \frac{d}{dt} (\frac{\delta x \delta y \delta p}{g}) = 0 \tag{57}$$

Rearranging:

$$\frac{1}{\delta x} \delta(\frac{dx}{dt}) + \frac{1}{\delta y} \delta(\frac{dy}{dt}) + \frac{1}{\delta p} \delta(\frac{dp}{dt}) = 0 \tag{58}$$

If we let $\delta \to 0$, we get:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0 \tag{59}$$

So the flow is *incompressible* in pressure coordinates. This equation is much easier to use than the conventional continuity equation because it doesn't involve the density.

In pressure coordinates, the velocity normal to pressure surfaces is referred to as ω. So for example, the three-dimensional Lagrangian derivative changes to:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{dx}{dt} \frac{\partial}{\partial x} + \frac{dy}{dt} \frac{\partial}{\partial y} + \frac{dp}{dt} \frac{\partial}{\partial p}$$

$$= \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + \omega \frac{\partial}{\partial p} \tag{60}$$

1.9 Thermal wind

Geostrophy tells us what the velocities are if we know the geopotential on a pressure surface. But say we want to estimate the velocities on a different surface. Then we need to know the velocity *shear*. We can obtain this using the "thermal wind" relation.

Consider that we have two pressure surfaces, with pressures p_1 and p_0. Using the geostrophic relations (55), we can write:

$$v_g(p_1) - v_g(p_0) = \frac{1}{f}\frac{\partial}{\partial x}(\Phi_1 - \Phi_0) \equiv \frac{g}{f}\frac{\partial}{\partial x}Z_{10}$$

$$u_g(p_1) - u_g(p_0) = -\frac{1}{f}\frac{\partial}{\partial y}(\Phi_1 - \Phi_0) \equiv -\frac{g}{f}\frac{\partial}{\partial y}Z_{10} \qquad (61)$$

where:

$$Z_{10} = \frac{1}{g}(\Phi_1 - \Phi_0) \qquad (62)$$

is the *thickness* of the layer between p_0 and p_1. Thus the velocity shear is proportional to gradients in layer thickness, and the thermal wind (the vertical shear vector) is aligned with surfaces of constant thickness.

There is another version of the thermal wind relation. From the definition of the geopotential and the hydrostatic relation, we have:

$$d\Phi = g\,dz = -\frac{dp}{\rho} \qquad (63)$$

Neglecting atmospheric moisture, we can relate the density and temperature using the *ideal gas law*, which reads:

$$p = \rho R T \qquad (64)$$

Substituting this into the expression for $d\Phi$ and rearranging, we infer:

$$\frac{\partial \Phi}{\partial p} = -\frac{1}{\rho} = -\frac{RT}{p} \tag{65}$$

We can use this, and the geostrophic relations, to determine how the geostrophic velocities vary with pressure:

$$\frac{\partial v_g}{\partial p} = \frac{1}{f}\frac{\partial}{\partial p}\frac{\partial \Phi}{\partial x}|_p = -\frac{R}{pf}\frac{\partial T}{\partial x}|_p \tag{66}$$

or:

$$p\frac{\partial v_g}{\partial p} = \frac{\partial v_g}{\partial \, ln(p)} = -\frac{R}{f}\frac{\partial T}{\partial x}|_p \tag{67}$$

Similarly, we obtain:

$$\frac{\partial u_g}{\partial \, ln(p)} = \frac{R}{f}\frac{\partial T}{\partial y}|_p \tag{68}$$

Thus the vertical shear is proportional to the gradients of temperature on a pressure surface. Integrating the v relation between two pressure levels, we find:

$$v_g(p_1) - v_g(p_0) = -\frac{R}{f}\int_{p_0}^{p_1} \frac{\partial T}{\partial x}|_p \, d\, ln(p) \tag{69}$$

and

$$u_g(p_1) - u_g(p_0) = \frac{R}{f}\int_{p_0}^{p_1} \frac{\partial T}{\partial y}|_p \, d\, ln(p) \tag{70}$$

So if we know the geostrophic velocities at p_0, can calculate them at p_1.

Think of p_0 and p_1 bounding a layer. We can define a *mean temperature* in the layer:

$$\overline{T} \equiv \int_{p_0}^{p_1} T\, d(lnp) / \int_{p_0}^{p_1} \, d(lnp) = \int_{p_0}^{p_1} T\, d(lnp)/ln(\frac{p_1}{p_0}) \tag{71}$$

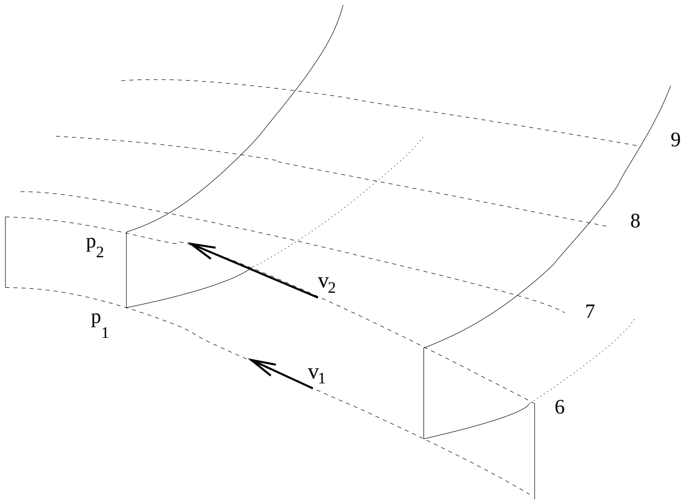

Figure 10:

.

Then:

$$v_g(p_1) - v_g(p_0) = \frac{R}{f}\,ln(\frac{p_0}{p_1})\,(\frac{\partial \overline{T}}{\partial x})_p \tag{72}$$

$$u_g(p_1) - u_g(p_0) = -\frac{R}{f}\,ln(\frac{p_0}{p_1})\,(\frac{\partial \overline{T}}{\partial y})_p \tag{73}$$

If we equate this estimate with the earlier one, we get:

$$Z_{10} = \frac{R}{g}\overline{T}\,ln(\frac{p_0}{p_1}) \tag{74}$$

So the layer thickness is *proportional to its mean temperature*.

Consider a idealized case in which the magnitude of the velocity changes with height, but not the direction (Fig. 10). This is known as an *equivalent barotropic* flow.[1] In the layer, the shear vector (which in this case is parallel to the velocities themselves) is parallel to the temperature isolines, which in turn are parallel to the thickness lines.

[1]In a purely barotropic flow, the wind speed is also constant with height.

Using thermal wind, we can estimate the strength of the jet stream. The zonally-averaged temperature decreases with latitude on the earth (the poles are colder than the equator). This means that $\partial T/\partial y < 0$, so that u should increase with height (Fig. 11).

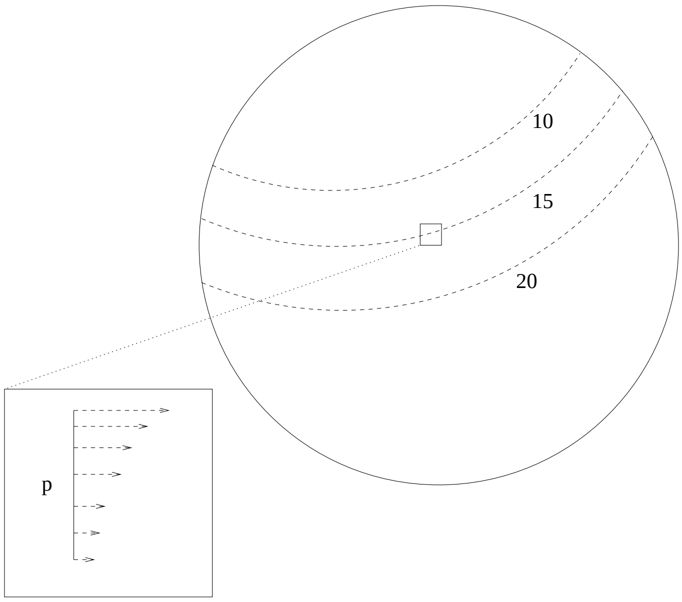

Figure 11: The jet stream on a zonally-average earth.

At 30N, the zonally-averaged temperature gradient is roughly $0.75\ Kdeg^{-1}$. Assuming the average wind is zero at the earth's surface, we can estimate the mean zonal wind at the level of the jet stream ($250\ hPa$), from (70):

$$u_g(p_1) - u_g(p_0) = u_g(p_1) = \frac{R}{f}\, ln(\frac{p_1}{p_0})\, \frac{\partial \overline{T}}{\partial y} \qquad (75)$$

or:

$$u_g(250) = \frac{287}{2\Omega sin(30)}\, ln(\frac{250}{1000})\,(-\frac{0.75}{1.11 \times 10^5\, m}) = 36.8\ m/sec \qquad (76)$$

This is roughly the speed of the jet at this height.

The equivalent barotropic assumption is often exploited in simplified models of the atmosphere. However, the actual atmosphere is usually more complicated, with both the wind speed and direction changing with height (the atmosphere is *baroclinic*). This implies that the temperature and geopotential isolines are not generally parallel, and therefore that the geostrophic wind can advect temperature (we speak of warm or cold winds).

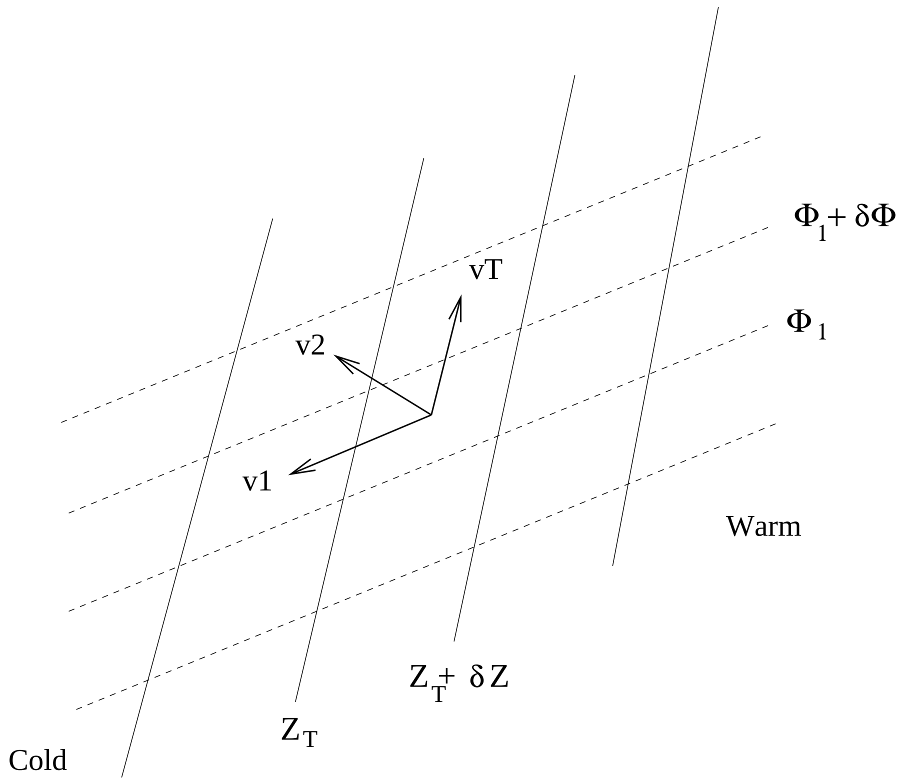

Figure 12: Thermal wind between two layers (1 and 2). The geopotential height contours for the lower layer, Φ_1, are the dashed lines and the thickness (temperature) contours are the solid lines.

Consider the case shown in Fig. (12). The geopotential lines for the lower surface of a layer are indicated by dashed lines. The wind at this level is parallel to these lines, with the larger values of Φ_1 to the right. The thickness (temperature) contours are the solid lines, with the temperature increasing to the right. The thermal wind vector is parallel to these contours, with larger temperatures on the right. We add the vectors v_1 and v_T to obtain the vector v_2, which is the wind at the upper surface. This is to the northwest, carrying the warm air towards the colder.

Notice then that the wind vector turns clockwise with height. This is called *veering* and is typical of warm advection. Cold advection produces counter-clockwise turning, called *backing*.

Note too the similarity between the geostrophic and thermal winds. The geostrophic wind is parallel to the geopotential contours with larger values to the right of the wind. The thermal wind is parallel to the thickness (mean temperature) contours, with larger values to the right. But recall that the thermal wind is not an actual wind, but a *difference* in wind vectors between the lower and upper levels.

1.10 Thermodynamics

The primary forcing agent for the atmosphere is the sun, which heats the earth's surface, changing the density of the overlying air. Thus thermodynamics, which describes how gases respond to heat, plays a central role in the dynamics. We require an additional equation, which links the temperature, density and pressure and describes their dependence on heating.

Consider a volume of air, as shown in Fig. (13). The First law of thermodynamics states that the change in internal energy of the volume equals the heat added minus the work done:

$$dq - dw = de \tag{77}$$

Work is done by expanding the volume against external forces (which press on the piston).

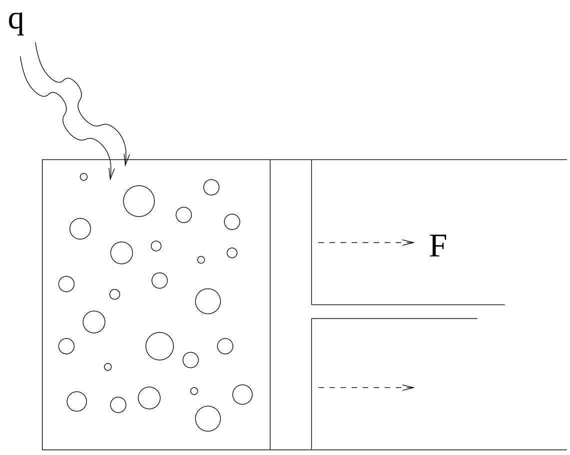

Figure 13: A volume being heated and doing work by forcing a piston.

$$dw = F dx = pA dx = p dV \tag{78}$$

If $dV > 0$, the volume is doing the work (otherwise the surroundings are compressing the volume and doing work on it). We can assume the volume has a unit mass, so that:

$$\rho V = 1 \tag{79}$$

so that

$$dV = d(\frac{1}{\rho}) \equiv d\alpha \tag{80}$$

where α is the *specific volume*. So:

$$dq = p\, d\alpha + de \tag{81}$$

If we add heat to the volume, its temperature rises. The extent of the rise is determined by the *specific heat*. If the volume is held constant, then:

$$c_v \equiv \frac{dq}{dT}|_v \tag{82}$$

Because the volume is constant, this is also the change in internal energy:

$$c_v = \frac{de}{dT}|_v \tag{83}$$

Joule's Law states that e only depends on temperature for an ideal gas, even if the volume changes. This implies that:

$$c_v = \frac{de}{dT} \tag{84}$$

So we get the first law:

$$dq = c_v dT + p\, d\alpha \tag{85}$$

We can also define a specific heat at constant pressure:

$$c_p \equiv \frac{dq}{dT}|_p \tag{86}$$

In this case, the volume expands in such a way to keep p constant. We require more heat to raise the temperature, because some heat must now be used for work. We can rewrite the first law thus:

$$dq = c_v dT + d(p\alpha) - \alpha dp \tag{87}$$

Recall that the ideal gas law is:

$$p = \rho RT = \alpha^{-1} RT \tag{88}$$

So

$$d(p\alpha) = RdT \tag{89}$$

Thus the first law can be written:

$$dq = (c_v + R)dT - \alpha dp \tag{90}$$

At constant pressure, $dp = 0$, so:

$$\frac{dq}{dT}|_p = c_p = c_v + R \tag{91}$$

So the specific heat at constant pressure is greater than that at constant volume. For dry air:

$$c_v = 717Jkg^{-1}K^{-1}, \qquad c_p = 1004Jkg^{-1}K^{-1} \tag{92}$$

So we find that:

$$R = 287Jkg^{-1}K^{-1} \tag{93}$$

Thus an alternate version of the first law, for a dry gas is:

$$dq = c_p dT - \alpha dp \tag{94}$$

This relation, or the alternate, constant volume version (85), can be used to relate the temperature, density and pressure and determine how they change with heating.

We often consider *adiabatic* processes, in which there is no heating at all. We can derive a useful relation applicable when $dq = 0$, which we will use later on. With zero heating:

$$c_v dT + p\, d\alpha = 0 \tag{95}$$

If we eliminate the dT term using the ideal gas law, we obtain:

$$\frac{c_v}{R} d(p\alpha) + p\,d\alpha = \frac{c_v}{R}(p\,d\alpha + \alpha\,dp) + p\,d\alpha = 0 \tag{96}$$

So:

$$c_v\,\alpha\,dp + (c_v + R)\,p\,d\alpha = c_v \alpha\,dp + c_p\,p\,d\alpha = 0 \tag{97}$$

Thus:

$$\frac{dp}{p} + \gamma \frac{d\alpha}{\alpha} = d\,lnp + \gamma d\,ln\alpha = 0 \tag{98}$$

where

$$\gamma \equiv \frac{c_p}{c_v} \tag{99}$$

So:

$$p\alpha^{\gamma} = p\rho^{-\gamma} = const. \tag{100}$$

for an adiabatic process. We will use this relation when discussing sound waves (sec. 2.2).

1.11 Potential temperature

Heating changes both the temperature and pressure of air. But temperature also changes with height (pressure) in the atmosphere. However, we can redefine the temperature to remove the pressure contribution. This "potential temperature" then only changes due to heating.

Imagine a parcel with a pressure p and temperature T. We can compress or expand the parcel adiabatically (without heating) to a standard (surface) pressure: $p_s = 100\ kPa\ = 1000\,mb$. The temperature of the parcel then is defined to be its potential temperature. For adiabatic motion we have:

$$c_p dT - \alpha dp = 0 \tag{101}$$

Using the ideal gas law:

$$\alpha = \frac{RT}{p} \tag{102}$$

we obtain:

$$c_p\, d\, lnT - R\, d\, lnp = 0 \tag{103}$$

So we can define the potential temperature, θ:

$$c_p\, lnT - R\, lnp = c_p\, ln\theta - R\, lnp_s \tag{104}$$

Rearranging and taking the exponential of both sides:

$$\theta = T\, (\frac{p_s}{p})^{R/c_p} \tag{105}$$

Thus θ depends on T and p. Substituting in:

$$\begin{aligned} c_p\, dlnT - R\, dlnp &= c_p\, dln\theta + R\, dln\frac{p}{p_s} - R\, dlnp \\ &= c_p\, dln\theta + const. \end{aligned} \tag{106}$$

So:

$$dq = c_p dln\theta + const. \tag{107}$$

As promised, the heating changes only the potential temperature. Thus a parcel on a surface with constant θ, called an *isentropic* surface, can move about the surface without being heated. In the language of thermodynamics, such parcels conserve their *entropy*.

2 Waves

Now we consider a class of dynamical phenomena in the atmosphere. These are the *wave* (oscillatory) modes of motion. Waves represent solutions to the *linearized* equations of motion. These waves are observable and are fundamentally important in the changing atmosphere.

2.1 General waves

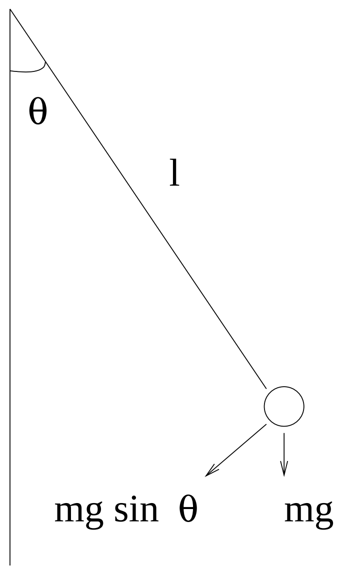

Figure 14: The oscillating pendulum.

A simple example of a wave is the oscillation of a pendulum. Gravity causes the pendulum mass to accelerate downward, but the rod prevents it from falling. When the mass is not moving, the rod's restoring force exactly balances gravity. When disturbed from the vertical however, there is a component of the rod's force which acts toward the resting position, causing the mass to accelerate that way.

We can write this mathematically as:

$$ma_\theta = ml\frac{\partial^2}{\partial t^2}\theta = -mgsin\theta \approx -mg\theta \qquad (108)$$

if the deflection angle, θ, is small. So:

$$\frac{\partial^2}{\partial t^2}\theta + \frac{g}{l}\theta = 0 \qquad (109)$$

The general solution is:

$$\theta = Re\{Ae^{i\nu t}\} \qquad (110)$$

Where ν is the *frequency*. Note A can be a complex number. Substituting the solution into the equation, we find:

$$-\nu^2 + \frac{g}{l} = 0 \qquad (111)$$

Thus:

$$\nu \equiv \sqrt{\frac{g}{l}}$$

Notice that the amplitude has dropped out. This is a typical occurrence in linear wave problems. If:

$$A = Be^{i\phi} \qquad (112)$$

then:

$$\theta = Bcos(\nu t + \phi) \qquad (113)$$

The *phase*, ϕ, permits us to match initial conditions (i.e. the initial position and velocity of the pendulum).

Wave equations generally take the form:

$$\frac{\partial^2}{\partial t^2}\psi - a^2\frac{\partial^2}{\partial x^2}\psi = 0 \tag{114}$$

and have a solution:

$$\psi = Re\{Ae^{ik(x-ct)}\} \tag{115}$$

Substituting the solution into the equation yields:

$$k^2c^2 = k^2a^2 \quad \rightarrow \quad c = \pm a \tag{116}$$

So we can write:

$$\psi = Bcos[k(x-at)+\phi_1] + Ccos[(k(x+at)+\phi_2] \tag{117}$$

The solution has a *wavelength*

$$\lambda = \frac{2\pi}{k} \tag{118}$$

and a frequency:

$$\nu = -kc = \pm ka \tag{119}$$

The frequency is what an observer "hears". In addition, the wave propagates with a *phase speed* (the speed of the crests and troughs) of

$$c = \frac{\nu}{k} = \pm a \tag{120}$$

Note that the first wave term propagates to the right, because x must increase as t increases. By the same reasoning, the second wave moves to the left, because x must decrease as t increases.

2.2 Sound waves

The first example we will consider is sound waves. For simplicity, assume the motion is purely one-dimensional (in the x direction). It turns out that the Coriolis term is not important for sound waves, because the spatial scales are so small and the time scales short. We can also ignore friction. Under these assumptions, the momentum equation is:

$$\frac{du}{dt} = -\frac{1}{\rho}\frac{\partial}{\partial x}p \tag{121}$$

while the continuity equation is:

$$\frac{d\rho}{dt} + \rho\frac{\partial}{\partial x}u = 0 \tag{122}$$

or:

$$\frac{d\,ln\,\rho}{dt} + \frac{\partial}{\partial x}u = 0 \tag{123}$$

Since we are at small scales, we use the z-coordinate forms of the equations instead of those with the geopotential.

Thus we have two equations but three unknowns. For the third relation, we use:

$$p\rho^{-\gamma} = const. \tag{124}$$

which we derived in sec. (1.10). With this we can remove density from the continuity equation:

$$\frac{1}{\gamma}\frac{d\,ln\,p}{dt}+\frac{\partial}{\partial x}u=0 \tag{125}$$

or:

$$\frac{dp}{dt}+\gamma p\frac{\partial}{\partial x}u=0 \tag{126}$$

Now we have two equations with two unknowns. However, the equations are *nonlinear*, because they involve products of the unknowns, and thus are difficult to solve. To proceed, we *linearize* by assuming the waves represent a weak perturbation about a background state. Specifically, we separate the variables into constant *mean* and *perturbation* components:

$$u(x,y,z,t)=\overline{u}+u'(x,y,z,t), \quad p=\overline{p}+p', \quad \rho=\overline{\rho}+\rho'$$

To linearize, we assume the perturbations are much smaller than the means:

$$|u'|\ll\overline{u}$$

We ignore products of perturbations (as we did before when discussing the scaling of the perturbation pressure in sec. 1.7). Substituting into the equations:

$$(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x})\,u'+\frac{1}{\overline{\rho}}\frac{\partial}{\partial x}\rho'=0 \tag{127}$$

$$(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x})\,p'+\gamma\overline{p}\frac{\partial}{\partial x}u'=0 \tag{128}$$

Eliminating u' between the equations yields:

$$(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x})^2\,p'-\frac{\gamma\overline{p}}{\overline{\rho}}\frac{\partial^2}{\partial x^2}p'=0 \tag{129}$$

We now substitute a wave-like solution:

$$p' = Re\{Ae^{ik(x-ct)}\} \tag{130}$$

to get:

$$(-ikc + i\overline{u}k)^2 + \frac{\gamma\overline{p}}{\overline{\rho}}k^2 = 0 \tag{131}$$

which yields a phase speed:

$$c = \overline{u} \pm \sqrt{\frac{\gamma\overline{p}}{\overline{\rho}}} \tag{132}$$

What does this mean? If the background flow, $\overline{u}$, is absent, then waves can propagate to the left or right with a phase speed of $\sqrt{\gamma\overline{p}/\overline{\rho}}$. This speed then is determined by the background pressure and density, as well as γ. Increasing the pressure or decreasing the density yields faster sound waves.

The mean flow causes a *Doppler shift* of this phase speed. The waves moving in the same direction as the mean are faster while those in the opposite direction are slower. This directly affects the wave frequency, as $\nu = kc$. So the sound from an approaching train's whistle is higher when the train is approaching us and lower when it is moving away.

If $\overline{u} = \sqrt{\gamma\overline{p}/\overline{\rho}}$, the left-going wave is *stationary*. This occurs when a plane reaches the so-called "sound barrier".

2.3 Gravity waves

Sound waves are *longitudinal waves* because the restoring force is in the same direction as the phase velocity, as with waves moving along a coiled spring. Gravity waves are an example of a *transverse wave*, where the restoring force, gravity, is not parallel to the wave motion. The most familiar gravity waves are those at the surface of the ocean. These travel perpendicular to gravity. Gravity waves also exist in the atmosphere, and these can propagate in nearly any direction.

We begin by making a few assumptions. First, we consider two-dimensional waves, in (x, z) plane. Second, we assume the spatial scales are small, so that we can ignore the Coriolis force. We also use the z-coordinate versions of the equations. Third, we make the *Boussinesq approximation*. In this, we replace the density by a constant value, $\rho = \rho_0$, except in the hydrostatic relation. Under the Boussinesq approximation, the flow is incompressible (see eqn. 9, with a constant density), and this greatly simplifies the derivation. We also assume the motion is adiabatic, i.e. that there is no heating.

Under these approximations, the equations are:

$$\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u + w\frac{\partial}{\partial z}u = -\frac{1}{\rho}\frac{\partial}{\partial x}p \tag{133}$$

$$\frac{\partial}{\partial t}w + u\frac{\partial}{\partial x}w + w\frac{\partial}{\partial z}w = -\frac{1}{\rho}\frac{\partial}{\partial z}p - g \tag{134}$$

$$\frac{\partial}{\partial x}u + \frac{\partial}{\partial z}w = 0 \tag{135}$$

$$\frac{\partial}{\partial t}\theta + u\frac{\partial}{\partial x}\theta + w\frac{\partial}{\partial z}\theta = 0 \tag{136}$$

The last equation signifies that the potential temperature, θ, doesn't change following an air parcel.

As with the sound waves, we will reduce this set to one equation with one unknown. We first linearize by assuming perturbations:

$$u = \overline{u} + u', \quad w = w', \quad \rho = \rho_0 + \rho', \quad p = \overline{p}(z) + p', \quad \theta = \overline{\theta}(z) + \theta'$$

Thus we assume a constant mean velocity which has no vertical component (which would otherwise require a force to counteract gravity). We also have background temperature and pressure gradients which vary only in the vertical direction. The background pressure must satisfy the hydrostatic relation:

$$\frac{\partial}{\partial z}\overline{p} = -\rho_0 g \tag{137}$$

From the definition of the potential temperature, we have:

$$\theta = T\,(\frac{p_s}{p})^{R/c_p} = \frac{p}{\rho R}\,(\frac{p_s}{p})^{R/c_p} \tag{138}$$

Taking the log of both sides:

$$ln\,\theta = ln\,p^{1-R/c_p} - ln\,\rho + const. = \gamma^{-1} ln\,p - ln\,\rho + const. \tag{139}$$

The background state must satisfy this, so:

$$ln\,\overline{\theta} = \gamma^{-1} ln\,\overline{p} - ln\,\rho_0 + const. \tag{140}$$

From before we showed:

$$\frac{1}{\rho}\frac{\partial}{\partial z}p + g = \frac{1}{\rho_0 + \rho'}\frac{\partial}{\partial z}(\overline{p} + p') + g \approx \frac{1}{\rho_0}\frac{\partial}{\partial z}p' - \frac{\rho' g}{\rho_0} \tag{141}$$

We can eliminate ρ' by using the definition of the potential temperature:

$$ln(\theta) = ln\,\overline{\theta}(1+\frac{\theta'}{\overline{\theta}}) = \gamma^{-1} ln\,[\overline{p}(1+\frac{p'}{\overline{p}})] - ln\,[\rho_0(1+\frac{\rho'}{\rho_0})] + const. \tag{142}$$

Canceling the part due to the basic state, we get:

$$ln(1+\frac{\theta'}{\overline{\theta}}) = \gamma^{-1} ln\,(1+\frac{p'}{\overline{p}}) - ln\,(1+\frac{\rho'}{\rho_0}) + const. \tag{143}$$

Because $ln(1+\epsilon) \approx \epsilon$, we get:

$$\frac{\theta'}{\overline{\theta}} \approx \frac{p'}{\gamma\overline{p}} - \frac{\rho'}{\rho_0} \tag{144}$$

or

$$\rho' = \frac{\rho_0}{\gamma\overline{p}}p' - \frac{\rho_0\theta'}{\overline{\theta}} = \frac{p'}{c^2} - \frac{\rho_0\theta'}{\overline{\theta}} \tag{145}$$

In the atmosphere, density fluctuations are dominated by temperature changes, so we can neglect the first term on the RHS. So we have, simply:

$$\frac{\rho'}{\rho_0} \approx -\frac{\theta'}{\overline{\theta}} \tag{146}$$

Using these approximations, the equations become:

$$\frac{\partial}{\partial t}u' + \overline{u}\frac{\partial}{\partial x}u' = -\frac{1}{\rho_0}\frac{\partial}{\partial x}p' \tag{147}$$

$$\frac{\partial}{\partial t}w' + \overline{u}\frac{\partial}{\partial x}w' = -\frac{1}{\rho_0}\frac{\partial}{\partial z}p' - \frac{\theta' g}{\overline{\theta}} \tag{148}$$

$$\frac{\partial}{\partial x}u' + \frac{\partial}{\partial z}w' = 0 \tag{149}$$

$$\frac{\partial}{\partial t}\theta' + \overline{u}\frac{\partial}{\partial x}\theta + w'\frac{\partial}{\partial z}\overline{\theta} = 0 \tag{150}$$

Now we take the x-derivative of the vertical momentum equation and subtract the z-derivative of the horizontal momentum equation:

$$(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x})(\frac{\partial}{\partial x}w' - \frac{\partial}{\partial z}u') + \frac{g}{\overline{\theta}}\frac{\partial}{\partial x}\theta' = 0 \tag{151}$$

This removes p' from the equations. Then we use the other two equations to eliminate u' and θ'. This leaves an equation in terms of w':

$$(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x})^2(\frac{\partial^2}{\partial x^2}w' + \frac{\partial^2}{\partial z^2}w') + N^2\frac{\partial^2}{\partial x^2}w' = 0 \tag{152}$$

where:

$$N^2 = g\frac{\partial}{\partial z}ln\,\overline{\theta}$$

is the square of the *Brunt-Vaisala frequency*. This is a fundamental measure of a stably-stratified environment (in which the density decreases going upward).[2] An air parcel displaced vertically under stable stratification will oscillate up and down. The frequency of oscillation is the Brunt-Vaisala frequency, N.

Now that we have a single linear equation with one unknown, we can substitute in a wave solution:

$$w' = Re\{\hat{w}'e^{i(kx+mz-\nu t)}\} \tag{153}$$

to get:

$$(\nu - \overline{u}k)^2(k^2 + m^2) - N^2k^2 = 0 \tag{154}$$

Note that the amplitude, $\hat{w}'$, has dropped out again. So we have:

$$\nu = \overline{u}k \pm \frac{Nk}{(k^2+m^2)^{1/2}} \equiv \overline{u}k \pm \frac{Nk}{\kappa} \tag{155}$$

[2] An unstable stratification has denser fluid over lighter fluid. This tends to *overturn*, resulting in *convection*.

This is the *dispersion relation*, which relates the frequency to the wavenumber. The waves are called *internal waves*.

Notice that the frequency depends on the angle of propagation. If we define:

$$cos\chi \equiv \frac{k}{\kappa} \tag{156}$$

then:

$$\nu = \overline{u}k \pm Ncos\chi \tag{157}$$

If the phase lines are parallel to the ground, then $k = 0$ and $\chi = \pi/2$. Then the frequency is zero (there is no phase propagation). If on the other hand the phase lines are perpendicular to the ground, then $m = 0$ and $\chi = 0$; these have the maximum frequency. The more horizontal the phase lines are, the lower the frequency. In addition, the frequency is *always less than N* without a mean flow.

The phase velocities are:

$$c_x = \frac{\nu}{k} = \overline{u} \pm \frac{N}{\kappa}, \quad c_z = \frac{\nu}{m} = \overline{u}\frac{k}{m} \pm \frac{Nk}{m\kappa} \tag{158}$$

Notice that longer waves (waves with smaller κ) have faster phase speeds. Thus if there is an initial disturbance which generates a range of different size waves, the longest waves will separate out and propagate more rapidly away. Thus gravity waves are *dispersive*.

We can also obtain the *group velocities*:

$$c_{gx} = \frac{\partial \nu}{\partial k} = \overline{u} \pm \frac{N(l^2 - k^2)}{\kappa^{3/2}}, \quad c_{gz} = \frac{\partial \nu}{\partial m} = \mp \frac{Nm}{\kappa^{3/2}} \tag{159}$$

One can show that the group velocity pertains to the *energy propagation*. For example, gravity waves generated by the wind blowing over mountains carry energy upward, away from the mountains. Interestingly, the group velocity of internal waves need not be parallel to the phase velocity. In particular, one can show that the group velocity in the vertical direction is in the opposite direction to that of the phase velocity. So the waves carrying energy upward away from the mountains have their crests moving *downward*, yielding a false impression of the movement of energy.

In addition, wind blowing over a mountain can excite stationary waves downwind of the mountain. These are called *lee waves*. They have $c_x = 0$, so:

$$\overline{u} = \frac{N}{\kappa} \tag{160}$$

Solving for m, we get:

$$m^2 = \frac{N^2}{\overline{u}^2} - k^2 \tag{161}$$

If the RHS is *positive*, then m is real. This implies that the waves have a sinusoidal structure in z. Then the waves can transport energy high into the atmosphere. Such lee waves are important in the momentum budget in the upper atmosphere (above 75 km).

If the RHS of (161) is *negative*, then m is imaginary, so that $m = im_i$. Then the wave's vertical structure varies as:

$$exp(imz) \propto exp(-m_i z) \tag{162}$$

Thus the waves have an exponential structure in z rather than a sinusoidal one. The sign of the exponential must be negative, as the waves must decay going up rather than grow. In such cases, the waves, and thus the energy, are trapped at the surface and cannot influence the upper atmosphere.

From (161), we see that weaker mean winds favor upward propagation. So too do broader mountain ranges, which have a smaller value of k. So the Himalayas and Rockies have a greater effect on the atmosphere than do smaller ranges.

Lee waves often appear as bands of clouds, because the upward vertical motion associated with the waves (the positive w') cause condensation. Upward-propagating waves can also *break* further up, just as ocean gravity waves break on a beach. This breaking generates turbulence (and bumpy airplane rides).

2.4 The vorticity equation

Sound waves have very small spatial scales, while gravity waves have scales of tens of meters. There are still larger waves though, which have a scale comparable to the earth's radius. These *planetary* or *Rossby* waves are important for weather and large scale adjustment. To see how they come about, we will need the *vorticity equation*.

The vorticity is the curl of the velocity. It has components in x, y and z. But at synoptic scales, the vertical component is by far the most important. This is:

$$\zeta \equiv \frac{\partial}{\partial x}v - \frac{\partial}{\partial y}u \qquad (163)$$

To obtain an equation for the vorticity, we cross-differentiate the momentum equations (we take $\frac{\partial}{\partial x}$ of the equation for v and subtract $\frac{\partial}{\partial y}$ of the equation for u). Since we are at large scales, we use the pressure coordinate equations, (53) and (54) After some rearranging, we obtain:

$$(\frac{\partial}{\partial t} + \vec{u} \cdot \nabla)\,\zeta + v\frac{\partial f}{\partial y} = -(\zeta + f)\,(\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v) \tag{164}$$

Notice that the process of cross-differentiating has removed the geopotential. Because $f = f(y)$, we can rewrite the equation this way:

$$(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y})\zeta_a = -\zeta_a\,(\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v) \tag{165}$$

The quantity $\zeta_a = \zeta + f$ is referred to as the *absolute vorticity*. It is the sum of the *relative vorticity*, ζ, and the *planetary vorticity*, f. We can determine the relative sizes of these two terms by scaling:

$$\zeta \propto \frac{U}{L}, \quad \rightarrow \quad \frac{|\zeta|}{f} \propto \frac{U}{2\Omega L} = \epsilon$$

So if the Rossby number, ϵ, is small, the relative vorticity is smaller than the planetary vorticity. Then we can further approximate the equation as:

$$(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y})(\zeta + f) \approx -f\,(\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v) \tag{166}$$

2.4.1 Example: Constant divergence

Equations (165) and (166) state that the absolute vorticity of an air parcel changes in response to horizontal divergence. This process is different for the formation of cyclones and anticyclones. Consider a region where the divergence is constant:

$$\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = D = const. \tag{167}$$

First let $D > 0$, which corresponds to a divergent flow (for example, below a downdraft at the surface; Fig. 15). Then the vorticity equation (165) is:

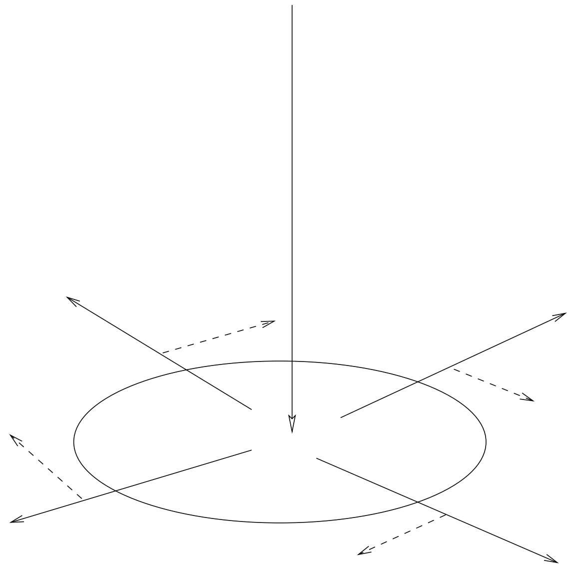

Figure 15: Divergent flow at the surface below a downdraft, producing anticyclonic circulation.

$$\frac{d}{dt}\zeta_a = -\zeta_a D \tag{168}$$

This implies:

$$\zeta_a = \zeta_a(0)\, e^{-Dt} \tag{169}$$

This implies that the absolute vorticity decays to zero, or that:

$$lim_{t\to\infty}\zeta \to -f \tag{170}$$

This is true regardless of the initial vorticity of the air parcel. Cyclonic and anticyclonic anomalies both become anticyclones with a vorticity approaching $-f$.

Physically, the outward flow associated with the divergence is diverted to the right by the Coriolis force (Fig. 15). This produces anticyclonic (clockwise) circulation.

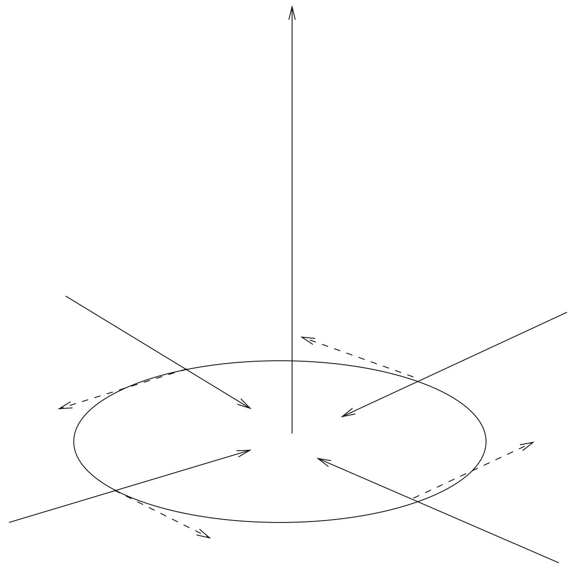

Figure 16: Convergent flow at the surface feeding an updraft.

Now consider convergent flow, with $D < 0$ (Fig. 16). Now we have:

$$\zeta_a = \zeta_a(0)\, e^{Dt} \tag{171}$$

So the vorticity increases without bound. But which sign obtains? If the Rossby number is small, then:

$$\zeta_a \approx f > 0 \tag{172}$$

So convergent flow produces intense *cyclones* rather than anticyclones. The inward flow in a convergence is steered to the right, generating cyclonic flow (Fig. 16). This is a reason why intense storms are usually cyclonic in the atmosphere.

2.4.2 Kelvin's theorem

Kelvin's theorem is a famous result due to the English scientist, Lord Kelvin. It concerns the change in "circulation" of an area of fluid. We can derive it in a straightforward way from the vorticity equation.

Consider an air parcel, with sides δx and δy. The time change in the area is:

$$\frac{\delta A}{\delta t} = \delta y \frac{\delta x}{\delta t} + \delta x \frac{\delta y}{\delta t} = \delta y \, \delta u + \delta x \, \delta v \tag{173}$$

The *relative* change in area is the divergence:

$$\frac{1}{\delta A} \frac{\delta A}{\delta t} = \frac{\delta u}{\delta x} + \frac{\delta v}{\delta y} \, . \tag{174}$$

So we can rewrite the divergence term (165) thus:

$$-(\zeta + f)(\frac{\partial}{\partial x} u + \frac{\partial}{\partial y} v) = -\frac{\zeta + f}{A} \frac{dA}{dt} \tag{175}$$

Putting this in the Lagrangian version of the vorticity equation, we get:

$$\frac{d}{dt}(\zeta + f) + \frac{\zeta + f}{A} \frac{dA}{dt} = 0 \tag{176}$$

or

$$\frac{d}{dt}[(\zeta + f)\, A] = 0 \tag{177}$$

The area times the absolute vorticity is the *absolute circulation*. This is conserved for inviscid motion, meaning the quantity doesn't change, following the air parcel. This is Kelvin's theorem. If the area changes, the vorticity must also change.

2.5 Barotropic potential vorticity

We can exploit the vorticity equation further under certain idealized conditions. Consider for example that the flow is in a layer where the temperature, T, is constant. Because of this, there is *no vertical shear* in the velocity field. To see this, take the p-derivative of the momentum equations:

$$\frac{d}{dt}\frac{\partial u}{\partial p} + \frac{\partial \overline{u}}{\partial p} \cdot \nabla u - f\frac{\partial v}{\partial p} = -\frac{\partial}{\partial x}\frac{\partial \Phi}{\partial p} = -\frac{R}{p}\frac{\partial}{\partial x}T \qquad (178)$$

$$\frac{d}{dt}\frac{\partial v}{\partial p} + \frac{\partial \overline{u}}{\partial p} \cdot \nabla v + f\frac{\partial u}{\partial p} = -\frac{\partial}{\partial y}\frac{\partial \Phi}{\partial p} = -\frac{R}{p}\frac{\partial}{\partial y}T \qquad (179)$$

after using the relation (65). With $T = const.$, the RHS of both equations is zero. Thus:

$$\frac{d}{dt}\frac{\partial u}{\partial p} = -\frac{\partial \overline{u}}{\partial p} \cdot \nabla u + f\frac{\partial v}{\partial p} \qquad (180)$$

$$\frac{d}{dt}\frac{\partial v}{\partial p} = -\frac{\partial \overline{u}}{\partial p} \cdot \nabla v - f\frac{\partial u}{\partial p} \qquad (181)$$

This implies that if:

$$\frac{\partial u}{\partial p} = \frac{\partial v}{\partial p} = 0 \qquad (182)$$

initially, there can be no shear later on. This is known as the *Taylor-Proudman* constraint.

A simpler way to see this is from the geostrophic relations. For example, if we take the p-derivatives of (55), we obtain:

$$f\frac{\partial}{\partial p}v = \frac{\partial}{\partial x}\frac{\partial \Phi}{\partial p} = -\frac{R}{p}\frac{\partial}{\partial x}T = 0 \qquad (183)$$

The same result obtains for $\frac{\partial}{\partial p}u$. Thus there is no vertical shear.

From the continuity equation (59), we have:

$$\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = -\frac{\partial}{\partial p}\omega \qquad (184)$$

We can use this to replace the divergence in the vorticity equation:

$$(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y})\,(\zeta + f) = (\zeta + f)\,\frac{\partial \omega}{\partial p} \tag{185}$$

Because there is no shear, the relative vorticity is also constant in z. So we can easily integrate the equation. Say that the constant temperature layer lies between p_2 and p_1. Then

$$h(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y})\,(\zeta + f) = (\zeta + f)\,(\omega(p_2) - \omega(p_1)) \tag{186}$$

where $h = p_2 - p_1$ is the layer thickness. Because $\omega = dp/dt$, we have:

$$h\,\frac{d}{dt}(\zeta + f) = (\zeta + f)\,\frac{d}{dt}(p_2 - p_1) = (\zeta + f)\,\frac{dh}{dt} \tag{187}$$

Then:

$$\frac{1}{\zeta + f}\frac{d}{dt}(\zeta + f) - \frac{1}{h}\frac{dh}{dt} = 0 \tag{188}$$

or:

$$\frac{d}{dt}[ln(\zeta + f) - ln(h)] = 0 \tag{189}$$

So we obtain the barotropic potential vorticity (PV) equation:

$$\frac{d}{dt}(\frac{\zeta + f}{h}) = 0 \tag{190}$$

The barotropic PV is conserved on a parcel in the layer. The PV takes into account changes in layer *thickness*. Consider the case shown in Fig. (f:vortstretch). The vortex moving to the right is stretched as the layer thickens, and this changes its vorticity. As h increases, the numerator in (190) must also increase, meaning the vorticity must become greater. So an initially cyclonic vortex intensifies as it stretches.

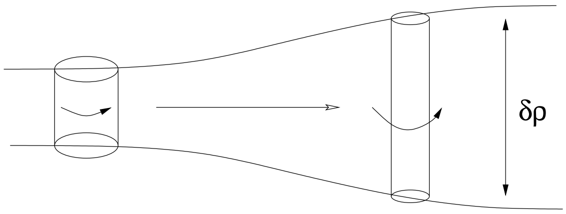

Figure 17: A vortex moving between two density surfaces. The vortex strengthens as the layer thickness increases.

2.6 Barotropic Rossby waves

Now we can derive the relation for planetary waves. In the simplest version of this, we assume the troposphere has constant density and a constant depth, h. Also, we will confine our attention to a limited range of latitudes, centered about a latitude θ_0. We can simplify the Coriolis parameter by making a Taylor expansion about the center latitude:

$$f(\theta) = f(\theta_0) + \frac{df}{d\theta}(\theta_0)\ (\theta - \theta_0) + \frac{1}{2}\frac{d^2 f}{d\theta^2}(\theta_0)\ (\theta - \theta_0)^2 + ... \qquad (191)$$

If range of latitudes is small, we can neglect the higher order terms, so that:

$$f \approx f(\theta_0) + \frac{df}{d\theta}(\theta_0)\ (\theta - \theta_0) \equiv f_0 + \beta y \qquad (192)$$

where

$$\beta = \frac{1}{R}\frac{df}{d\theta}(\theta_0) = \frac{2\Omega}{R}cos(\theta_0)$$

and

$$y = R(\theta - \theta_0)$$

Relation (192) is known as the "β-plane approximation". It simplifies life by shifting from spherical to cartesian coordinates.

Under these assumptions, the barotropic PV equation is:

$$\frac{d}{dt}(\zeta + f_0 + \beta y) = \frac{d}{dt}\zeta + \beta v = 0 \quad (193)$$

We linearize the equation by assuming a constant mean zonal velocity:

$$u = \overline{u} + u', \quad v = v'$$

Then:

$$(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x})\zeta' + \beta v' = 0 \quad (194)$$

With a constant density and layer depth, the continuity equation is:

$$\frac{\partial}{\partial x}u' + \frac{\partial}{\partial y}v' = 0 \quad (195)$$

This allows us to write the velocity in terms of a *streamfunction*:

$$u = -\frac{\partial}{\partial y}\psi, \quad v = \frac{\partial}{\partial x}\psi, \quad \zeta = (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})\psi \quad (196)$$

So the vorticity equation is:

$$(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x})\nabla^2\psi + \beta\frac{\partial}{\partial x}\psi = 0 \quad (197)$$

Now we have a single linear equation with one unknown, as desired. We substitute a wave-like solution:

$$\psi = Re\{\hat{\psi}e^{i(kx+ly-\nu t)}\} \quad (198)$$

and get:

$$(-i\nu + i\overline{u}k)(-k^2 - l^2)\hat{\psi} + i\beta k\hat{\psi} = 0 \quad (199)$$

or:

$$\nu = k\overline{u} - \frac{\beta k}{k^2 + l^2} \tag{200}$$

The corresponding zonal phase speed is:

$$c = \frac{\nu}{k} = \overline{u} - \frac{\beta}{k^2 + l^2} \tag{201}$$

This is the Rossby wave dispersion relation.

The dispersion relation has a number of interesting points. The phase speed depends on the wavenumber, so the waves are dispersive, as with gravity waves. The largest speeds occur with the largest waves, which have small values of k and l. But unlike the gravity or sound waves, which could propagate in any direction in the absence of a mean flow, Rossby waves propagate only *westward*. This is a signature of the waves. Satellite observations of Rossby waves in the Pacific Ocean show that the waves, originating off of California and Mexico, sweep westward toward Asia.

With a mean flow, the waves can be swept eastward, producing the appearance of eastward propagation. But the waves are still propagating westward relative to the mean flow. If:

$$\overline{u} = \frac{\beta}{k^2 + l^2} \tag{202}$$

the wave is stationary. So for a particular mean flow, we can define a stationary wavenumber:

$$\kappa \equiv (k^2 + l^2)^{1/2} = (\beta/\overline{u})^{1/2} \tag{203}$$

which determines the scale (wavelength) of the standing wave. Note that stationary waves occur only if the mean flow is eastward.

Despite that the phase velocity is only westward, the group velocity can be in any direction:

$$c_{gx} = \overline{u} + \frac{\beta(k^2 - l^2)}{\kappa^4}, \quad c_{gy} = \frac{2\beta kl}{\kappa^4}, \tag{204}$$

An interesting case is when long waves, with small values of k, encounter a western boundary, like Asia in the Pacific ocean. Assume $\overline{u} = 0$ in the Pacific. The long waves transport energy westward, because $c_{gx} < 0$ if $k < l$. But the reflected waves must carry energy eastward, or else energy would accumulate at the wall. For this to happen, the reflected wave must have $k > l$. So the reflected waves are *short*. This effect is clearly seen in numerical simulations.

2.7 Quasi-geostrophic potential vorticity

To consider the more realistic case with density stratification, we need to take thermodynamics into account. With several approximations, we can derive a single vorticity equation which does this. We return to our approximate vorticity equation:

$$(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y})\,(\zeta + f) = -f\,(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) \tag{205}$$

With the continuity equation, we have:

$$(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y})\,(\zeta + f) - f\,\frac{\partial}{\partial p}\omega = 0 \tag{206}$$

We linearize the equation using a constant mean zonal velocity and we also invoke the β-plane approximation:

$$(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x})\,\zeta' + \beta v' - f_0\,\frac{\partial}{\partial p}\omega' = 0 \tag{207}$$

We neglect the term $\beta y \frac{\partial}{\partial p}\omega'$, which is assumed to be small.

As we have seen, the Rossby number at synoptic scales is small, which implies that the velocities are nearly geostrophic. We exploit this by making the *quasi-geostrophic* approximation:

$$u' \approx -\frac{1}{f_0}\frac{\partial}{\partial y}\Phi, \quad v' \approx \frac{1}{f_0}\frac{\partial}{\partial x}\Phi, \tag{208}$$

and thus that:

$$\zeta' \approx \frac{1}{f_0}\nabla^2\Phi \tag{209}$$

So the vorticity equation is:

$$(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x})\,\nabla^2\Phi + \beta\frac{\partial\Phi}{\partial x} - f_0^2\,\frac{\partial}{\partial p}\omega' = 0 \tag{210}$$

We have two unknowns, Φ and ω. To close the system, we use the thermodynamic equation. As before, we assume there is no heating, so that the potential temperature is conserved. Thus

$$\frac{d}{dt}\theta = 0 \tag{211}$$

We linearize this, assuming a weak perturbation on a stable background stratification:

$$\theta = \overline{\theta}(p) + \theta'(x, y, p, t) \tag{212}$$

Then we obtain:

$$\frac{\partial}{\partial t}\theta' + \overline{u}\frac{\partial}{\partial x}\theta' + \omega'\frac{\partial}{\partial p}\overline{\theta} = 0 \tag{213}$$

Now, from (105), the potential temperature is proportional to the temperature on a surface of constant pressure. So we can write the equation in terms of temperature:

$$\frac{\partial}{\partial t}T' + \overline{u}\frac{\partial}{\partial x}T' + \omega'\frac{\partial}{\partial p}\overline{T} = 0 \tag{214}$$

From the hydrostatic relation (65), we have:

$$T = -\frac{p}{R}\frac{\partial \Phi}{\partial p} \tag{215}$$

we can write:

$$(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x})\,\frac{\partial \Phi}{\partial p} + \sigma\omega = 0 \tag{216}$$

where:

$$\sigma \equiv -\frac{R}{p}\frac{\partial}{\partial p}\overline{T} \tag{217}$$

is a measure of the stratification (like the Brunt-Vaisala frequency).

We can then combine the thermodynamic and vorticity equations to eliminate ω'. The result is:

$$(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x})\,[\nabla^2\Phi + \frac{\partial}{\partial z}(\frac{f_0^2}{\sigma}\frac{\partial}{\partial z}\Phi)] + \beta\frac{\partial}{\partial x}\Phi = 0 \tag{218}$$

This is the *quasi-geostrophic potential vorticity equation*. This takes into account changes in vorticity due to both the Coriolis term and to changes in temperature. This is a useful and powerful relation.

2.8 Baroclinic Rossby waves

Now we can examine baroclinic Rossby waves, or Rossby waves in the presence of stratification. We substitute a wave solution:

$$\Phi = \hat{\Phi} e^{i(kx+ly+mz-\nu t)} \tag{219}$$

to get:

$$\nu = \overline{u}k - \frac{\beta k}{k^2 + l^2 + f_0^2 m^2/\sigma} \tag{220}$$

with a zonal phase speed of:

$$c_x = \overline{u} - \frac{\beta}{k^2 + l^2 + f_0^2 m^2/\sigma} \tag{221}$$

We see that the relations are nearly the same as those we obtained with barotropic waves. The difference is that there is an additional term in the denominator of the term multiplied by β. Like the barotropic waves, the baroclinic waves propagate to the west relative to the mean flow and larger waves propagate faster than smaller waves. But the additional term in the denominator means the waves propagate *slower* than their barotropic counterparts.

Of central interest for the large scale atmospheric circulation are the stationary waves, generated by flow over mountains. Assuming $c_x = 0$, we can solve for m:

$$m^2 = \frac{\sigma}{f_0}\left[\frac{\beta}{\overline{u}} - (k^2 + l^2)\right] \tag{222}$$

As with the gravity waves, we have two possibilities. If $m^2 < 0$, then m is imaginary and the waves decay exponentially in the vertical. These are trapped at the lower boundary. But if $m^2 > 0$, or:

$$(k^2 + l^2) < \frac{\beta}{\overline{u}} \tag{223}$$

the waves can propagate upward, even up into the stratosphere. The relation implies that long waves (small k, l) are more likely to propagate upward than short waves. Also, because the LHS is positive, the relation cannot be satisfied unless the mean flow is eastward ($\overline{u} > 0$).

In fact, Rossby waves are found in the stratosphere, and can dramatically alter the circulation (even changing the equator-to-pole temperature difference—a so-called "stratospheric warming" event). They are observed there during the winter months. But they disappear during the summer, because the mean wind reverses direction ($\overline{u} < 0$). Thus this (relatively) simple theory (due to Charney and Drazin, 1961) can account for this disappearance.