Atmosphere-Ocean Dynamics

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1 Equations

1.1 Primitive equations

The primitive equations are the full equations which express how the important variables, the velocities, density, etc., change in time. To write them, we require derivatives. Consider a scalar variable, for example the density, ρ, which varies in both time and space. By the chain rule, the total change in the ρ is:

$$d\rho = \frac{\partial \rho}{\partial t}\,dt + \frac{\partial \rho}{\partial x}\,dx + \frac{\partial \rho}{\partial y}\,dy + \frac{\partial \rho}{\partial z}\,dz \tag{1}$$

So:

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + u\,\frac{\partial \rho}{\partial x} + v\,\frac{\partial \rho}{\partial y} + w\,\frac{\partial \rho}{\partial z} = \frac{\partial \rho}{\partial t} + \vec{u}\cdot\nabla\rho \tag{2}$$

We refer to the left side as the *Lagrangian* derivative and the RHS is the *Eulerian* derivative. The Lagrangian formulation applies to moving measurements, like balloons or drifters, while the Eulerian applies to fixed measurements, like weather stations or current meters. In the Lagrangian frame, the density is only a function of time, so the derivative is a total one. But in the Eulerian form, the density is a function of space and time. So the time derivative on the RHS is a partial one (and is called the *local* derivative).

In both the atmosphere and ocean, the velocities are governed by the Navier-Stokes equations, or the *momentum* equations. Consider that we are in a planar region on the earth's surface, centered at latitude θ. The equations are:[1]

[1]There are several additional terms, called *curvature* terms, which stem from using spherical coordinates. But these terms are negligible at the scales of interest and so are left out here.

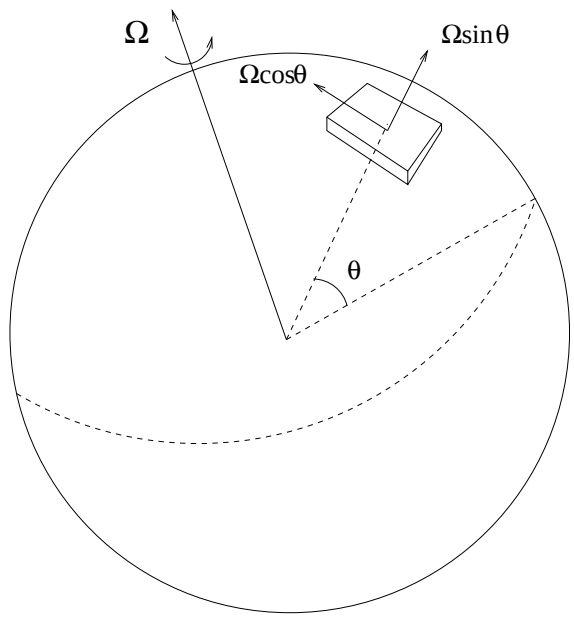

Figure 1: A region of the atmosphere at latitude θ. Note the rotation vector projects onto the z and y axes in the local coordinates.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + 2\Omega w\, cos\theta - 2\Omega v\, sin\theta = -\frac{1}{\rho}\frac{\partial p}{\partial x} + F_x \quad (3)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + 2\Omega u\, sin\theta = -\frac{1}{\rho}\frac{\partial p}{\partial y} + F_y \quad (4)$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} - 2\Omega u\, cos\theta = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + F_z \quad (5)$$

Here x, y and z are the local east-west, north-south and vertical directions, and (u, v, w) are the corresponding velocities. The density is ρ, p is the pressure and $g = 9.8$ m/sec is the gravitational constant, and:

$$\Omega = \frac{2\pi}{86400}\, sec^{-1}$$

is the Earth's rotation rate.

These equations derive from Newton's second law, $F = ma$. There are two types of force, *real* and *apparent*. The real forces are due to gradients in the pressure (p), to gravity (g) and to friction (F_i). These forces are quite familiar.

The apparent forces are less familiar and come about because the earth is rotating. Consider a person on earth, at a position $\vec{r}$, and a fixed observer looking at him from space. One can show that the latter sees:

$$\vec{u}_F = \vec{u}_R + \vec{\Omega} \times \vec{r} \tag{6}$$

This states the person's velocity, measured from space (the fixed frame), is equal to his velocity (in the rotating frame) plus the rotational velocity of the earth. Even if the person stands still, the fixed observer perceives he is moving, as he rotates around the pole. Similarly, one can show[2]:

$$(\frac{d\vec{u}_F}{dt})_F = (\frac{d\vec{u}_R}{dt})_R + 2\vec{\Omega} \times \vec{u}_R + \vec{\Omega} \times \vec{\Omega} \times \vec{r}$$

Thus the acceleration in the fixed frame has two additional terms: the Coriolis acceleration and the centrifugal acceleration. The centrifugal acceleration (the last term) acts perpendicular to the earth's rotation axis and is constant in time. It is possible to absorb this into the gravity term and then neglect it thereafter. The Coriolis force on the other hand depends on the velocity. It acts perpendicular to the velocity, causing a change in velocity direction but not the speed.[3] Written out for our planar region, the acceleration is:

$$2\vec{\Omega} \times \vec{u} = (0, 2\Omega cos\theta, 2\Omega sin\theta) \times (u, v, w) =$$

$$2\Omega(w\, cos\theta - v\, sin\theta,\ u\, sin\theta, -u\, cos\theta) \tag{7}$$

[2]See, e.g., my notes from GEF2220.

[3]Because of this, the Coriolis force does *no work*.

For the friction terms, we can assume molecular damping occurring at small scales. Then we would write:

$$\vec{F} = (F_x, F_y, F_z) = \nu \nabla^2 \vec{u} = \nu \nabla^2 (u, v, w) \tag{8}$$

where ν is the molecular viscosity. This has a value on the order of 10^{-5} m^2/sec.

Then there is the *continuity* equation, which expresses the conservation of mass:

$$\frac{\partial}{\partial t}\rho + \vec{u} \cdot \nabla \rho + \rho \nabla \cdot \vec{u} = \frac{d}{dt}\rho + \rho \nabla \cdot \vec{u} = 0 \tag{9}$$

This can be derived by considering the flux of mass through an infinitesimal Eulerian volume, or by writing the conservation of mass for a Lagrangian volume (e.g. sec. 1.4.3). The Lagrangian form of the equation expresses that the density changes if the volume changes, and the latter occurs if the flow is divergent.

In addition to these four equations, we have an "equation of state" which relates the density to the temperature and, for the ocean, the salinity. In the atmosphere, the density and temperature are linked via the *Ideal Gas Law*:

$$p = \rho R T \tag{10}$$

where $R = 287\, Jkg^{-1}K^{-1}$ is the gas constant for dry air. The law is thus applicable for a dry gas, i.e. one without moisture. But a similar equation applies in the presence of moisture if one replaces the temperature with the so-called "virtual temperature" (Holton, *An Introduction to Dynamic Meteorology*).

In the ocean, both salinity and temperature affect the density. The dependence is expressed:

$$\rho = \rho(T, S) = \rho_c(1 - \alpha_T(T - T_{ref}) + \alpha_S(S - S_{ref})) + h.o.t. \qquad (11)$$

where ρ_c is a constant, T_{ref} and S_{ref} are reference values for temperature and salinity and where $h.o.t.$ means "higher order terms". Increasing the temperature or decreasing the salinity reduces the density (makes lighter water). An important point is that the temperature and salinity corrections are much less than one, so that the density is dominated by the first term, ρ_c, which is constant. We exploit this in section (1.3) in making the so-called Boussinesq approximation.

We require one additional equation for the atmosphere, and this expresses how the system responds to heating. This is the *thermodynamic energy* equation:

$$c_v \frac{dT}{dt} + p\frac{d}{dt}(\frac{1}{\rho}) = c_p \frac{dT}{dt} - (\frac{1}{\rho})\frac{dp}{dt} = J \qquad (12)$$

The equation derives from the First Law of Thermodynamics, which states that the heat added to a volume minus the work done by the volume equals the change in its internal energy. Here c_v and c_p are the specific heats at constant volume and pressure, respectively, and J represents the heating. So heating changes the temperature and also the pressure and density of air.

However, we will find it convenient to use a different, though related, equation, pertaining to the *potential temperature*. The potential temperature is defined

$$\theta = T\,(\frac{p_s}{p})^{R/c_p} \tag{13}$$

This is the temperature a parcel would have if it were moved *adiabatically* (with zero heating) to a reference pressure, usually taken to be the pressure at the earth's surface. The advantage is that we can write the thermodynamic energy equation in terms of only one variable:

$$c_p \frac{d(ln\theta)}{dt} = \frac{1}{T} J \tag{14}$$

This relation is simpler than (12) because it doesn't involve the pressure. It implies that the potential temperature is conserved on an air parcel if there is no heating ($J = 0$), i.e.:

$$\frac{d\theta}{dt} = 0 \tag{15}$$

These equations can be used to model either the atmosphere or ocean. However, the equations are coupled and nonlinear and have never been solved analytically! Without analytical solutions, it is very difficult to understand exactly how they behave.

Our goal is to reduce the equations to a simpler set. The new equations, which apply at the *synoptic* or weather scales, can be obtained by a systematic scaling of the above equations. These are the *quasi-geostrophic equations*. Due to their simplicity, they are much easier to manipulate and understand.

1.2 The Geostrophic Relations

Not all the terms in the horizontal momentum equations are equally important. To see which ones dominate, we scale the equations. Take the

x-momentum equation:

$$\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u + w\frac{\partial}{\partial z}u + 2\Omega w\, cos\theta - 2\Omega v\, sin\theta = -\frac{1}{\rho}\frac{\partial}{\partial x}p + \nu\nabla^2 u$$

$$\frac{U}{T} \quad \frac{U^2}{L} \quad \frac{U^2}{L} \quad \frac{UW}{D} \quad 2\Omega W \quad 2\Omega U \quad \frac{\triangle_H p}{\rho L} \quad \frac{\nu U}{L^2}$$

$$\frac{1}{2\Omega T} \quad \frac{U}{2\Omega L} \quad \frac{U}{2\Omega L} \quad \frac{W}{2\Omega D} \quad \frac{W}{U} \quad 1 \quad \frac{\triangle_H p}{2\Omega\rho U L} \quad \frac{\nu}{2\Omega L^2}$$

In the second line we have *scaled* the equation by assuming typical values for the variables. In the third line, we have divided through by the scaling of the second Coriolis acceleration, $2\Omega U$ (which we have assumed will be important). The resulting parameters are all *dimensionless*, i.e. they have no units.

To estimate these parameters, we use values typical of weather systems:

$$U \approx 10\, m/sec, \quad 2\Omega = \frac{4\pi}{86400\, sec} \approx 10^{-4} sec^{-1},$$

$$L \approx 10^6\, m, \quad D \approx 10^4\, m, \quad T = L/U \approx 10^5\, sec$$

$$\triangle_H P/\rho \approx 10^3\, m^2/sec^2, \quad W \approx 1\, cm/sec, \tag{16}$$

The horizontal scale, 1000 km, is the *synoptic scale*. Notice that we assume the scale is the same in the x and y directions. Similarly we use a single velocity scale for both u and v; the vertical velocity though has a different scale, as vertical motion is much weaker at these horizontal scales.

The time scale, proportional to the length scale divided by the velocity scale, is the *advective* time scale. With an advective time scale, we have:

$$\frac{1}{2\Omega T} = \frac{U}{2\Omega L} \equiv \epsilon$$

So the first term is the same size as the second and third terms. This parameter is the *Rossby number*. At synoptic scales it is approximately:

$$\frac{U}{2\Omega L} = 0.1$$

So the first three terms are smaller than the second Coriolis term.

However, the other terms are even smaller:

$$\frac{W}{2\Omega D} = 0.01, \quad \frac{W}{U} = .001$$

and so can be neglected. The friction term:

$$\frac{\nu U}{2\Omega U L^2} \approx 10^{-13}$$

is miniscule. Lastly, the pressure gradient term scales as:

$$\frac{\triangle p_H}{2\Omega \rho U L} = 1$$

and thus is comparable in size to the second Coriolis term.

The scalings given above are applicable to the atmosphere, but using values relevant to the ocean yields similar results. Furthermore, the scaling of the y-momentum equation is identical to that of the x-momentum equation. The dominant balances are thus:

$$-fv = -\frac{1}{\rho}\frac{\partial}{\partial x}p \tag{17}$$

$$fu = -\frac{1}{\rho}\frac{\partial}{\partial y}p \tag{18}$$

where:

$$f \equiv 2\Omega sin\theta$$

is the vertical component of the Coriolis parameter. These are the *geostrophic relations*, the primary balance in the horizontal direction at synoptic scales. Thus if we know the pressure field, we can deduce the velocities.

Consider the flow in Fig. (2). The pressure is high to the south and low to the north. Left alone, this would force the air to move north. Because $\frac{\partial}{\partial y}p < 0$, we have that $u > 0$ (eastward), from (18). The Coriolis force is acting to the right of the motion, exactly balancing the pressure gradient force. Because the two forces balance, the motion is constant in time.

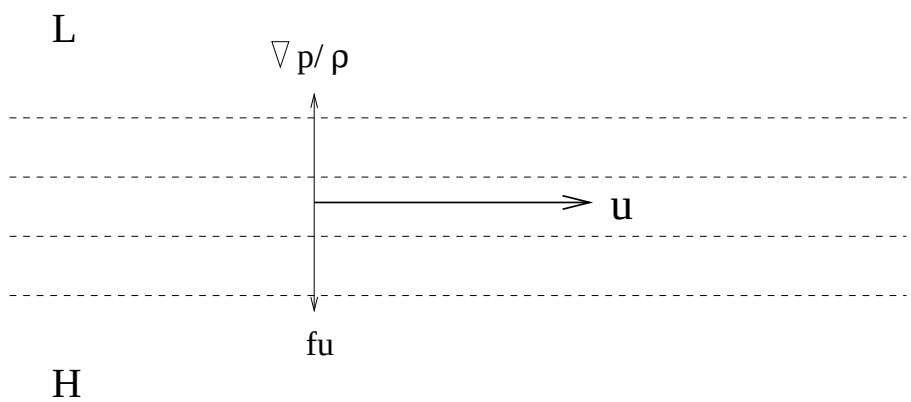

Figure 2: The geostrophic balance.

Note that since $f = 2\Omega sin\theta$, it is *negative* in the southern hemisphere. So the flow in Fig. (2) would be westward, with the Coriolis force acting to the left. In addition, the Coriolis force is identically *zero* at the equator. So the geostrophic balance cannot hold there.

1.3 The Hydrostatic Balance

Now we scale the vertical momentum equation. For this, we need an estimate of the vertical variation in pressure, which is different than the hori-

zontal variation:

$$\triangle_V P/\rho \approx 10^5 m^2/sec^2$$

Thus we have:

$$\frac{\partial}{\partial t}w + u\frac{\partial}{\partial x}w + v\frac{\partial}{\partial y}w + w\frac{\partial}{\partial z}w - 2\Omega u cos\theta = -\frac{1}{\rho}\frac{\partial}{\partial z}p - g + \nu\nabla^2 w \quad (19)$$

$$\begin{array}{cccccccc}
\frac{WU}{L} & \frac{UW}{L} & \frac{UW}{L} & \frac{W^2}{D} & 2\Omega U & \frac{\triangle_V P}{\rho D} & g & \frac{\nu W}{L^2} \\
\frac{UW}{gL} & \frac{UW}{gL} & \frac{UW}{gL} & \frac{W^2}{gD} & \frac{2\Omega U}{g} & \frac{\triangle_V P}{g\rho D} & 1 & \frac{\nu W}{gL^2} \\
10^{-8} & 10^{-8} & 10^{-8} & 10^{-9} & 10^{-4} & 1 & 1 & 10^{-20}
\end{array}$$

Notice that we divided through by g, assuming that term will be large. We see that the vertical pressure gradient and gravity terms are much larger than any of the others. So the vertical momentum equation can be replaced by:

$$\frac{\partial}{\partial z}p = -\rho g \quad (20)$$

This is the *hydrostatic* relation. This is a tremendous simplification over the full vertical momentum equation. However, notice that the same balance applies if there is *no motion at all*. If we set $u = v = w = 0$ in the vertical momentum equation, we obtain the same balance. Thus the balance may not be that relevant for the dynamic (moving) part of the flow.

But it is. Let's separate the pressure and density into static and dynamic components:

$$p(x,y,z,t) = p_0(z) + p'(x,y,z,t)$$

$$\rho(x,y,z,t) = \rho_0(z) + \rho'(x,y,z,t) \tag{21}$$

The dynamic components are usually much smaller than the static components, so that:

$$|p'| \ll |p_0|, \quad |\rho'| \ll |\rho_0|, \tag{22}$$

Thus we can write:

$$-\frac{1}{\rho}\frac{\partial}{\partial z}p - g = -\frac{1}{\rho_0+\rho'}\frac{\partial}{\partial z}(p_0+p') - g \approx -\frac{1}{\rho_0}\left(1-\frac{\rho'}{\rho_0}\right)\frac{\partial}{\partial z}(p_0+p') - g$$

$$\approx -\frac{1}{\rho_0}\frac{\partial}{\partial z}p' + \left(\frac{\rho'}{\rho_0^2}\right)\frac{\partial}{\partial z}p_0 = -\frac{1}{\rho_0}\frac{\partial}{\partial z}p' - \frac{\rho'}{\rho_0}g \tag{23}$$

Note we neglect terms proportional to the product of the dynamical variables, like $p'\rho'$.

How do we scale these dynamical pressure terms? Measurements suggest the vertical variation of p' is comparable to the horizontal variation:

$$\frac{1}{\rho_0}\frac{\partial}{\partial z}p' \propto \frac{\triangle_H P}{\rho_0 D} \approx 10^{-1} m/sec^2 \ .$$

The perturbation density, ρ', is roughly $1/100$ as large as the static density, so:

$$\frac{\rho'}{\rho_0}g \approx 10^{-1} m/sec^2 \ .$$

To scale these, we again divide by g, to that both terms are of order 10^{-2}. Thus while they are smaller than the static terms, they are still *two orders of magnitude larger* than the next largest term in (19). Thus the approximate

vertical momentum equation is still the hydrostatic balance, except for the perturbation pressure and density:

$$\frac{\partial}{\partial z}p' = -\rho' g \tag{24}$$

The hydrostatic approximation is so good that it is used in most numerical models instead of the full vertical momentum equation. Models which use the latter are rarer and are called "non-hydrostatic" models.

While the values given above are for the atmosphere, a scaling using oceanic values produces the same result. The hydrostatic balance is thus an excellent approximation, in either system.

1.4 Approximations

Here we describe a few approximations which will allow us to further simplify the equations.

1.4.1 The β-plane approximation

After scaling, we see that the horizontal component of the Coriolis term, $2\Omega cos\theta$, vanishes from the momentum equations. The term which remains is the vertical component, $2\Omega sin\theta$. We call this f. Note though that while all the other terms in the momentum equations are in Cartesian coordinates, f is a function of latitude.

To remedy this, we focus on a limited range of latitudes. Then we can Taylor-expand f about the central latitude, θ_0:

$$f(\theta) = f(\theta_0) + \frac{df}{d\theta}(\theta_0)\ (\theta - \theta_0) + \frac{1}{2}\frac{d^2 f}{d\theta^2}(\theta_0)\ (\theta - \theta_0)^2 + ... \tag{25}$$

We will neglect the higher order terms, so that:

$$f \approx f(\theta_0) + \frac{df}{d\theta}(\theta_0)\ (\theta - \theta_0) \equiv f_0 + \beta y \tag{26}$$

where

$$f_0 = 2\Omega sin(\theta_0)$$

$$\beta = \frac{1}{a}\frac{df}{d\theta}(\theta_0) = \frac{2\Omega}{a} cos(\theta_0)$$

and

$$y = a(\theta - \theta_0)$$

where a is the radius of the earth. We call (26) the β-plane approximation. Thus f is only a function of y, i.e. in the North-South direction. Following the Taylor expansion, the linear term must be much smaller than f_0, so that:

$$\frac{\beta L}{f_0} \ll 1$$

This constrains the latitude range, L, since:

$$L \ll \frac{f_0}{\beta} = \frac{2\Omega sin(\theta)}{2\Omega cos(\theta)/a} = a\, tan(\theta_0) \approx a \tag{27}$$

So L must be smaller than the earth's radius, which is roughly 6400 km.

Because the β term is much smaller than the f_0 term, we can ignore it in the geostrophic relations. Specifically, we can write:

$$v = \frac{1}{f\rho}\frac{\partial \phi}{\partial x} \approx \frac{1}{f_0 \rho}\frac{\partial \phi}{\partial x} \tag{28}$$

and similarly

$$u \approx -\frac{1}{f_0 \rho}\frac{\partial \phi}{\partial y} \tag{29}$$

However, the relations are still non-linear, as the terms on the right hand side involve the product of the density and the pressure. We remedy that in the following two sections.

1.4.2 The Boussinesq approximation

In the atmosphere, the background density ρ_0 varies significantly with height. In the ocean however, the density barely changes at all. This allows us to make the *Boussinesq* approximation. In this, we take the density to be constant, except in the "buoyancy term" on the RHS of the hydrostatic relation in (24).

Making this approximation, the geostrophic relations become:

$$-f_0 v_g = -\frac{1}{\rho_c}\frac{\partial}{\partial x}p \tag{30}$$

$$f_0 u_g = -\frac{1}{\rho_c}\frac{\partial}{\partial y}p \tag{31}$$

where ρ_c is the constant density in (11). Now the terms on the RHS are linear.

This simplification has an important effect because it is this which makes the geostrophic velocities horizontally non-divergent. In particular:

$$\frac{\partial}{\partial x}u_g + \frac{\partial}{\partial y}v_g = -\frac{1}{\rho_c f_0}\frac{\partial^2 p}{\partial y \partial x} + \frac{1}{\rho_c f_0}\frac{\partial^2 p}{\partial x \partial y} = 0 \tag{32}$$

This non-divergence comes about because the geostrophic velocities, which are horizontal, are much greater than the vertical velocities.

Under the Boussinesq approximation, the continuity equation is also somewhat simpler. In particular, if we set $\rho_0 = \rho_c$ in (**??**), we obtain:

$$\nabla \cdot \vec{u} = 0 \tag{33}$$

So the total velocities are (3-D) non-divergent, i.e. the flow is *incompressible*. This assumption is frequently made in oceanography.

1.4.3 Pressure coordinates

We cannot responsibly apply the Boussinesq approximation to the atmosphere, except possibly in the planetary boundary layer (this is often done, for example, when considering the surface Ekman layer). But it is possible to achieve the same simplifications if we change the vertical coordinate to pressure instead of height.

We do this by exploiting the hydrostatic balance. Consider a pressure surface in two dimensions, (x, z). Applying the chain rule, we have:

$$\triangle p = \frac{\partial p}{\partial x} \triangle x + \frac{\partial p}{\partial z} \triangle z = 0 \tag{34}$$

on the surface. Substituting the hydrostatic relation, we get:

$$\frac{\partial p}{\partial x} \triangle x - \rho g \triangle z = 0 \tag{35}$$

so that:

$$\frac{\partial p}{\partial x}|_z = \rho g \frac{\triangle z}{\triangle x}|_p \equiv \rho \frac{\partial \Phi}{\partial x}|_p \tag{36}$$

where the subscripts indicate derivatives taken in vertical (z) and pressure (p) coordinates and where Φ is the *geopotential*:

$$\Phi \equiv \int_0^z g\, dz \approx gz \tag{37}$$

(Note that g varies somewhat with height above the ground). Making this alteration removes the density from momentum equation because:

$$-\frac{1}{\rho}\nabla p|_z \quad \rightarrow \quad -\nabla\Phi|_p$$

So the geostrophic balance in pressure coordinates is simply:

$$f_0 v_g = \frac{\partial}{\partial x}\Phi, \qquad f_0 u_g = -\frac{\partial}{\partial y}\Phi \tag{38}$$

As with the Boussinesq approximation, the terms on the RHS are now linear. Thus in pressure coordinates, the horizontal velocities are also horizontally non-divergent.

In addition, the coordinate change also simplifies the continuity equation. We could show this by applying a coordinate transformation directly to (9), but it is even simpler to do it as follows. Consider a Lagrangian box with a volume:

$$\delta V = \delta x\,\delta y\,\delta z = -\delta x\,\delta y\,\frac{\delta p}{\rho g} \tag{39}$$

after substituting from the hydrostatic balance. The mass of the box is:

$$\delta M = \rho\,\delta V = -\frac{1}{g}\delta x\,\delta y\,\delta p$$

Conservation of mass implies:

$$\frac{1}{\delta M}\,\frac{d}{dt}\,\delta M = \frac{-g}{\delta x\delta y\delta p}\,\frac{d}{dt}\,(-\frac{\delta x\delta y\delta p}{g}) = 0 \tag{40}$$

Rearranging:

$$\frac{1}{\delta x}\,\delta(\frac{dx}{dt}) + \frac{1}{\delta y}\,\delta(\frac{dy}{dt}) + \frac{1}{\delta p}\,\delta(\frac{dp}{dt}) = 0 \tag{41}$$

If we let $\delta \rightarrow 0$, we get:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0 \tag{42}$$

where ω (called "omega" in meteorology) is the velocity perpendicular to the pressure surface (like w is perpendicular to a z-surface). As with the Boussinesq approximation, the flow is incompressible in pressure coordinates.

The hydrostatic equation also takes a different form under pressure coordinates. Now we have that:

$$dp = -\rho g dz = -\rho d\Phi \tag{43}$$

So:

$$\frac{d\Phi}{dp} = -\frac{1}{\rho} = -\frac{RT}{p} \tag{44}$$

after invoking the Ideal Gas Law (10).

Pressure coordinates simplifies the equations considerably, but they are nonetheless awkward to work with in theoretical models. The lower boundary in the atmosphere (the earth's surface) is most naturally represented in z-coordinates, e.g. as $z = 0$. As the pressure varies at the earth surface, it is less obvious what boundary value to use for p. So we will use z-coodinates primarily when we begin looking at solution. But the solutions in p-coordinates are often very similar.

1.5 Thermal wind

If we combine the geostrophic and hydrostatic relations, we get the thermal wind relations. These tell us about the velocity shear. Take, for instance, the p-derivative of the geostrophic balance for v:

$$\frac{\partial v}{\partial p} = \frac{1}{f}\frac{\partial}{\partial x}\frac{\partial \Phi}{\partial p} = -\frac{R}{pf}\frac{\partial T}{\partial x} \tag{45}$$

after using (44). Note that the p passes through the x-derivative because it is constant on an isobaric (p) surface, i.e. they are independent variables. Likewise:

$$\frac{\partial u}{\partial p} = \frac{R}{pf}\frac{\partial T}{\partial y} \tag{46}$$

after using the hydrostatic relation (44). Thus the vertical shear is proportional to the lateral gradients in the temperature.

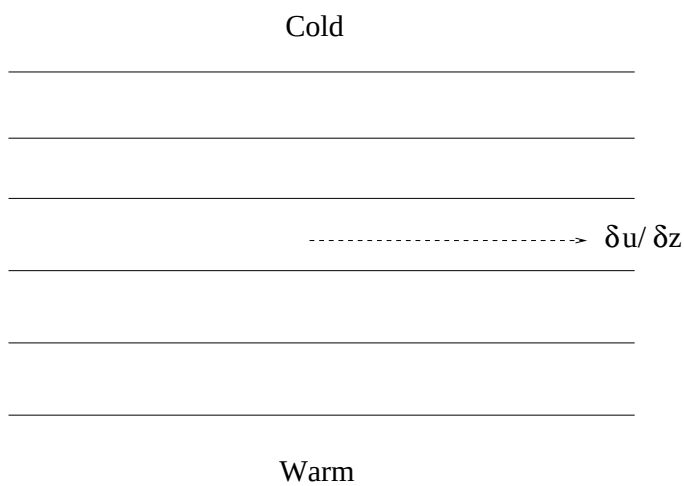

Figure 3: The thermal wind shear associated with a temperature gradient in the y-direction.

The thermal wind is parallel to the temperature contours, with the warm wind on the right. To see this, consider Fig. (3). There is a temperature gradient in y, meaning the shear is purely in the x-direction. The temperature is decreasing to the north, so the gradient is negative. From (46) we have then that $\partial u/\partial p$ is also negative. This implies that $\partial u/\partial z$ is positive (because pressure decreases going up. So the zonal velocity is increasing going up, i.e. with the warm air to the right.

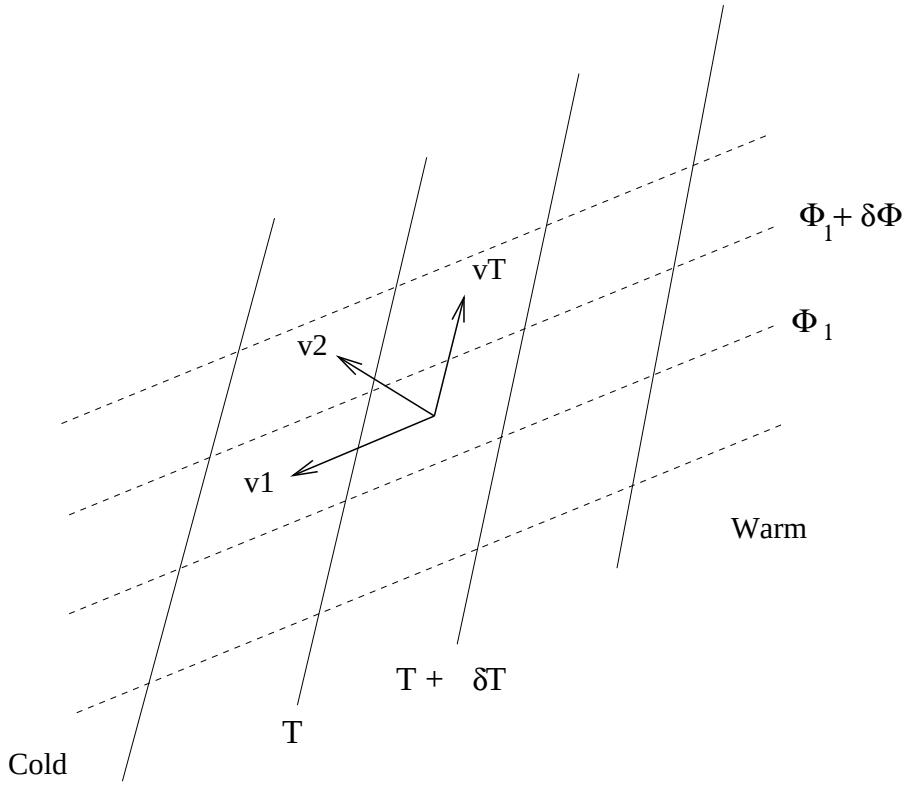

Figure 4: Thermal wind between two layers (1 and 2). The geopotential height contours for the lower layer, Φ_1, are the dashed lines and the temperature contours are the solid lines.

Using thermal wind, we can derive the geostrophic velocities on a nearby pressure surface, if we know the velocities on an adjacent surface and the temperature in the layer between the two levels. Consider the case shown in Fig. (4). The geopotential lines for the lower surface of the layer are indicated by dashed lines. The wind at this level is parallel to these lines, with the larger values of Φ_1 to the right. The temperature contours are the solid lines, with the temperature increasing to the right. The thermal wind vector is parallel to these contours, with the larger temperatures on the right. We add the vectors v_1 and v_T to obtain the vector v_2, which is the wind at the upper surface. This is to the northwest, advecting the warm air towards the cold.

Notice that the wind vector turns clockwise with height. This is called *veering* and is typical of warm advection. Cold advection produces counter-clockwise turning, called *backing*.

Thus the geostrophic wind is parallel to the geopotential contours with larger values to the right of the wind. The thermal wind on the other hand is parallel to the mean temperature contours, with larger values to the right. Recall though that the thermal wind is not an actual wind, but the *difference* between the lower and upper level winds.

The thermal wind relations for the ocean derive from taking z-derivatives of the Boussinesq geostrophic relations (30-31), and then invoking the hydrostatic relation. The result is:

$$\frac{\partial v}{\partial z} = -\frac{g}{\rho_c f}\frac{\partial \rho}{\partial x} \tag{47}$$

$$\frac{\partial u}{\partial z} = \frac{g}{\rho_c f}\frac{\partial \rho}{\partial y} \tag{48}$$

Thus the shear in the ocean depends on lateral gradients in *density*, which can result from changes in either temperature or salinity.

Relations (47) and (48) are routinely used to estimate ocean currents from density measurement made from ships. Ships collect *hydrographic* measurements of temperature and salinity, and these are then used to determine $\rho(x, y, z, t)$, from the equation of state (11). Then the thermal wind relations are integrated upward from chosen level to determine (u, v) above the level, for example:

$$u(x, y, z) - u(x, y, z_0) = \int_{z0}^{z} \frac{1}{\rho_c f}\frac{\partial \rho(x, y, z)}{\partial y}\, dz \tag{49}$$

If (u, v, z_0) is set to zero at the lower level, it is known as a "level of no

motion".

1.6 The vorticity equation

A central quantity in dynamics is the vorticity, which is the curl of the velocity:

$$\vec{\zeta} \equiv \nabla \times \vec{u} = (\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}) \tag{50}$$

The vorticity resembles angular momentum in that it pertains to "spinning" motion. A tornado has significant vorticity, with its strong, counterclockwise swirling motion.

The rotation of the earth alters the vorticity because the earth itself is rotating. As noted in sec. (1.1), the velocity seen by a fixed observer is the sum of the velocity seen in the rotating frame (earth) and a rotational term:

$$\vec{u}_F = \vec{u}_R + \vec{\Omega} \times \vec{r} \tag{51}$$

The vorticity is altered as well:

$$\vec{\zeta}_a = \nabla \times (\vec{u} + \vec{\Omega} \times \vec{r}) = \vec{\zeta} + 2\vec{\Omega} \tag{52}$$

We call $\vec{\zeta}_a$ the *absolute vorticity*. It is the sum of the *relative vorticity*, $\vec{\zeta} = \nabla \times \vec{u}$, and the *planetary vorticity*, $2\vec{\Omega}$.

Because synoptic scale motion is dominated by the horizontal velocities, the most important component of the vorticity is the vertical component:

$$\zeta_a \cdot \hat{k} = (\frac{\partial}{\partial x} v - \frac{\partial}{\partial y} u) + 2\Omega sin(\theta) \equiv \zeta + f \tag{53}$$

This is the only component we will be considering.

We can derive an equation for ζ from the horizontal momentum equations. For this, we use the approximate equations that we obtained after scaling, retaining the terms to order Rossby number—the geostrophic terms, plus the time derivative and advective terms. We will use the Boussinesq equations; the exact same equation obtains if one uses pressure coordinates.

The equations are:

$$\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u - fv = -\frac{1}{\rho_c}\frac{\partial}{\partial x}p$$

$$\frac{\partial}{\partial t}v + u\frac{\partial}{\partial x}v + v\frac{\partial}{\partial y}v + fu = -\frac{1}{\rho_c}\frac{\partial}{\partial y}p \tag{54}$$

where

$$f = f_0 + \beta y$$

To obtain the vorticity equation, we *cross-differentiate* the equations: we take the x derivative of the second equation and subtract the y derivative of the first. The result, after some re-arranging, is:

$$\frac{\partial}{\partial t}\zeta + u\frac{\partial}{\partial x}\zeta + v\frac{\partial}{\partial y}\zeta + v\frac{df}{dy} + (\zeta + f)(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) = 0 \tag{55}$$

or, alternately:

$$\frac{d_H}{dt}(\zeta + f) = -(\zeta + f)(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) \tag{56}$$

where:

$$\frac{d_H}{dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} \tag{57}$$

is the Lagrangian derivative based on the horizontal velocities. Note that we can write the equation this way because f is only a function of y.

A useful feature of the vorticity equation is that the pressure term has dropped out. This follows from the Boussinesq approximation—if we hadn't made that, then there would be terms involving derivatives of the density. Likewise, the geopotential drops out when using pressure coordinates. This is left for an exercise.

1.6.1 Kelvin's theorem

The vorticity equation is based on a result known as *Kelvin's theorem*, derived in Appendix A. This is of fundamental importance in rotating fluid dynamics. It concerns how the vorticity and the latitude of a fluid parcel is related to its area.

To see this, consider a small area of fluid:

$$A = \delta x \, \delta y \tag{58}$$

Using the horizontal time derivative, we have:

$$\frac{dA}{dt} = \delta y \frac{d}{dt}\delta x + \delta x \frac{d}{dt}\delta y = \delta y \, \delta u + \delta x \, \delta v \tag{59}$$

Dividing by A, we have:

$$\frac{1}{A}\frac{dA}{\delta t} = \frac{\delta u}{\delta x} + \frac{\delta v}{\delta y} \rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \tag{60}$$

in the limit as $\delta \rightarrow 0$. So the *relative* change in the area is equal to the horizontal divergence.

As such, the vorticity equation (56) can be written:

$$\frac{d_H}{dt}(\zeta + f) = -\frac{(\zeta + f)}{A}\frac{d_H A}{dt} \tag{61}$$

(we can write the dA/dt term in this way since A is only a function of (x, y)). Combining the left and right hand sides:

$$\frac{d_H}{dt}(\zeta + f)A = 0 \tag{62}$$

So the product of the absolute vorticity and the fluid area is conserved by the motion. This is Kelvin's theorem, due to Lord Kelvin (1869). Thus we have:

$$(\zeta + f)A = const. \tag{63}$$

So if a parcel's area or latitude changes, it's vorticity must change to compensate. An example is given in the problems.

1.6.2 Quasi-geostrophic vorticity equation

In sec. (1.2), we saw that the horizontal velocities are predominantly in geostrophic balance. We can exploit this to write a slightly simpler version of the vorticity equation. First, we can replace the horizontal velocities with their geostrophic equivalents in the Lagrangian derivative:

$$\frac{d_H}{dt} \rightarrow \frac{d_g}{dt} \equiv \frac{\partial}{\partial t} + u_g\frac{\partial}{\partial x} + v_g\frac{\partial}{\partial y} \tag{64}$$

Similarly, we can replace the vorticity with its geostrophic version:

$$\zeta \rightarrow \zeta_g = \frac{\partial}{\partial x}v_g - \frac{\partial}{\partial y}u_g \tag{65}$$

So the vorticity equation is, approximately:

$$\frac{d_g}{dt}(\zeta_g + \beta y) = -(\zeta_g + f_0 + \beta y)(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) \tag{66}$$

We have substituted in the β-plane version of f, and have left out the f_0 term on the LHS because it is constant (so its derivative is zero).

We can simplify the RHS a bit as well. In the β-plane approximation, the βy term is much less than f_0. But it turns out the vorticity is also much smaller. Their ratio scales as:

$$\frac{|\zeta_g|}{f_0} \propto \frac{U}{f_0 L} = \epsilon$$

since the vorticity scales as U/L. Since the Rossby number, ϵ, is on the order of 1/10 for synoptic scale motion, the vorticity is ten times smaller than f_0. So the vorticity equation is well-approximated by:

$$\frac{d_g}{dt}(\zeta_g + \beta y) = -f_0(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) \tag{67}$$

Now consider the right hand side. If we replaced the velocities by their geostrophic versions the term would vanish, because the geostrophic velocities are horizontally non-divergent (sec. 1.4.2). So we have to look more closely at the incompressibility condition. First, let's write the velocities this way:

$$u = u_g + u_a, \quad v = v_g + v_a \tag{68}$$

Here (u_a, v_a) are the *ageostrophic velocities*. These are much smaller than the geostrophic velocities. In particular:

$$\frac{|u_a|}{|u_g|} = O|\epsilon| \tag{69}$$

and the same for v_a/v_g. Thus if the Rossby number is 0.1, the ageostrophic velocities are ten times smaller.

Substituting these into the incompressibility condition (33) yields:

$$\frac{\partial}{\partial x}(u_g + u_a) + \frac{\partial}{\partial y}(v_g + v_a) + \frac{\partial}{\partial z}w =$$

$$\frac{\partial}{\partial x}u_a + \frac{\partial}{\partial y}v_a + \frac{\partial}{\partial z}w = 0 \tag{70}$$

because, again, the geostrophic velocities are horizontally non-divergent. Now if all three terms are equally important, than w must be similar in size to the ageostrophic velocities— it too is order Rossby number.

Going back to the vorticity equation, we can write:

$$\frac{d_g}{dt}(\zeta_g + \beta y) = f_0 \frac{\partial}{\partial z}w \tag{71}$$

The RHS, despite being small, is important. It causes changes in the absolute vorticity for parcels advected by the geostrophic flow.

We will write the variables in terms of a *streamfunction*, defined as:

$$\psi = \frac{p}{\rho_c f_0} \tag{72}$$

Then the geostrophic relations are simply:

$$u = -\frac{\partial}{\partial y}\psi, \qquad v = \frac{\partial}{\partial x}\psi \tag{73}$$

and the vorticity is:

$$\zeta_g = \frac{\partial}{\partial x}v_g - \frac{\partial}{\partial y}u_g = \nabla^2\psi \tag{74}$$

Using these, the vorticity equation is:

$$(\frac{\partial}{\partial t} - \frac{\partial \psi}{\partial y}\frac{\partial}{\partial x} + \frac{\partial \psi}{\partial y}\frac{\partial}{\partial x})(\nabla^2\psi + f) = f_0\frac{\partial}{\partial z}w \tag{75}$$

This is the *quasi-geostrophic vorticity equation*. The beauty of this is that it has only two unknowns—the streamfunction (representing pressure) and the vertical velocity. This equation will be the basis of much of the work that follows.

There are two points to mention here. One is that essentially the same equation is obtained using pressure coordinates, if one instead defines the streamfunction to be $\psi = \Phi/f_0$. Second is that the vertical velocity— though typically small at synoptic scales— is nevertheless an important forcing term, coupling the interior motion with that occurring in the boundary layers. An example of this is given in the next section.

1.7 Boundary layers

Kelvin's theorem applies in the absence of friction, which we've seen is weak at synoptic scales. However, without friction there would be nothing to remove energy supplied by the sun (to the atmosphere) and by the winds (to the ocean), and the velocities would accelerate to infinity. Where friction *is* important is in boundary layers at the earth's surface in the atmosphere, and at the surface and bottom of the ocean. How do these layers affect the interior motion?

Here we consider perhaps the simplest representation of a boundary layer in a rotating frame. We assume the layer has a constant density and we use z-coordinates. Having a constant density is like the Boussinesq approximation, except that we also set $\rho = \rho_c$ in the hydrostatic relation. Thus, with constant density, there is no vertical shear. We will also assume that the Coriolis parameter is constant and write it f_0.

A central feature of the boundary layer is that the geostrophic balance is broken by friction. Thus, instead of (30) and (31), we have:

$$-f_0 v = -\frac{1}{\rho_c}\frac{\partial}{\partial x}p \; + \; \frac{\partial}{\partial z}\frac{\tau_x}{\rho_c} \tag{76}$$

$$f_0 u = -\frac{1}{\rho_c}\frac{\partial}{\partial y}p \; + \; \frac{\partial}{\partial z}\frac{\tau_y}{\rho_c} \tag{77}$$

where τ_x and τ_y are stresses acting in the x and y directions. We rewrite these, as follows:

$$-f_0(v - v_g) = -f_0 v_a = \frac{\partial}{\partial z}\frac{\tau_x}{\rho_c} \tag{78}$$

$$f_0(u - u_g) = f_0 u_a = \frac{\partial}{\partial z}\frac{\tau_y}{\rho_c} \tag{79}$$

where (u_a, v_a) again are ageostrophic velocities. Thus the ageostrophic velocities in the boundary layer are directly proportional to the stresses; if we know the stresses, we can find these velocities.

We will be mostly concerned with how the boundary layer affects the motion in the interior. As hinted in sec. (1.6), the key ingredient is the vertical velocity, which acts as a forcing term in the vorticity equation (75). If there is vertical flow into the boundary layer, it must come from the interior and that in turn is associated with vorticity. So it is important to determine what the vertical velocities are in the boundary layers.

Consider the layer at the surface of the ocean first. To obtain w, we can integrate the incompressibility condition (33) over the depth of the layer:

$$\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v + \frac{\partial}{\partial z}w = \frac{\partial}{\partial x}u_a + \frac{\partial}{\partial y}v_a + \frac{\partial}{\partial z}w = 0 \tag{80}$$

Recall that the divergence involves only the ageostrophic velocities (because the geostrophic velocities are horizontally non-divergent). Thus:

$$w(0) - w(\delta) = -\int_{-\delta}^{0} (\frac{\partial}{\partial x} u_a + \frac{\partial}{\partial y} v_a)\, dz \tag{81}$$

where δ is the thickness of the layer. Since there is no flow out of the ocean surface, we can write $w(0) = 0$. Then we have, at the base of the layer:

$$w(\delta) = \frac{\partial}{\partial x} U + \frac{\partial}{\partial y} V \tag{82}$$

where (U, V) are the horizontal *transports* in the layer:

$$U \equiv \int_{-\delta}^{0} u_a\, dz, \qquad V \equiv \int_{-\delta}^{0} v_a\, dz \tag{83}$$

We find these by integrating (78) and (79) vertically. For the ocean surface layer, the stress at the surface is due to the wind, τ^w. We assume the stress at the base of the layer is zero (because the stress only acts in the layer itself). So we obtain:

$$U = \frac{\tau_y^w}{\rho_c f_0}, \qquad V = -\frac{\tau_x^w}{\rho_c f_0}$$

Thus the ageostrophic transport in the layer is *90 degrees to the right of the wind stress*. If the wind is blowing to the north, the transport is to the east. This is Ekman's (1905) famous result. Nansen had noticed that icebergs don't move downwind, but drift to the right of the wind. Ekman's model explains why this happens.

To get the vertical velocity, we take the divergence of these transports:

$$w(\delta) = \frac{\partial}{\partial x} \frac{\tau_y^w}{\rho_c f_0} - \frac{\partial}{\partial y} \frac{\tau_x^w}{\rho_c f_0} = \frac{1}{\rho_c f_0} \nabla \times \tau^w \tag{84}$$

So the vertical velocity is *proportional to the curl of the wind stress*. It is the curl, not the stress itself, which is most important for the interior flow in the ocean at synoptic scales.

An important point here is that we made no assumptions about the stress in the surface layer to obtain this result. By integrating over the layer, we only need to know the stress at the surface. So the result (84) is *independent* of the stress distribution, $\tau(z)/\rho$, in the layer.

Then there is the bottom boundary layer, which exists in both the ocean and atmosphere. Assuming the bottom is flat, the integral of the continuity equation is:

$$w(\delta) - w(0) = w(\delta) = -(\frac{\partial}{\partial x}U + \frac{\partial}{\partial y}V) \tag{85}$$

This time, we assume the vertical velocity vanishes at the top of the layer. Again we can integrate (78) and (79) to find the transports. However, we don't know the stress at the bottom. All we know is that the bottom boundary isn't moving.

Thus we must specify the stress in the layer. To do this, we *parametrize* the stress in terms of the velocity. The simplest way to do this is to write:

$$\frac{\vec{\tau}}{\rho_c} = A_z \frac{\partial}{\partial z}\vec{u} \tag{86}$$

where A_z, is known as a *mixing coefficient*. Thus, if the vertical shear is large in the layer the stress is great, and vice versa. Generally, A_z varies with height, often in a non-trivial way. In such cases, it can be difficult to find analytical solutions.

So we will for the present assume that A_z is constant. This follows Ekman's (1905) original formulation, and the solutions is now referred to

as an *Ekman* boundary layer. In this, we assume that there is geostrophic flow in the fluid interior, i.e. above the boundary layer, with velocities (u_g, v_g). The boundary layer's role is to bring the velocities to rest at the lower boundary. Using these stresses, we can solve for the ageostrophic velocities in the layer. The details are given in Appendix B. Integrating the velocities with height, one finds:

$$U = -\frac{\delta_e}{2}(u_g + v_g), \quad V = \frac{\delta_e}{2}(u_g - v_g)$$

where (u_g, v_g) are the velocities in the interior. In the solutions, the depth of the Ekman layer, δ, is determined by the mixing coefficient, A_z. This is:

$$\delta_e = \sqrt{\frac{2A_z}{f_0}} \tag{87}$$

So we have:

$$w(\delta_e) = \frac{\delta_e}{2}(\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial x}) + \frac{\delta_e}{2}(-\frac{\partial u_g}{\partial y} + \frac{\partial v_g}{\partial y})$$

$$= \frac{\delta_e}{2}(-\frac{\partial v_g}{\partial y} + \frac{\partial v_g}{\partial x}) + \frac{\delta_e}{2}(-\frac{\partial u_g}{\partial y} + \frac{\partial v_g}{\partial y}) = \frac{\delta_e}{2}(\frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y})$$

$$= \frac{\delta_e}{2}\nabla \times \vec{u}_g = \frac{\delta_e}{2}\zeta_g \tag{88}$$

We have used the fact that $\frac{\partial}{\partial x}u_g + \frac{\partial}{\partial y}v_g = 0$, as noted before. Thus the vertical velocity from the bottom Ekman layer is *proportional to the relative vorticity in the interior*.

So we can include the boundary layers without actually worrying about what is actually happening in the layers themselves. We will see that the bottom layers cause relative vorticity to decay in time (sec. 2.6), and the stress at the ocean surface forces the ocean (e.g. sec. 2.7). We can include these two effects and then neglect explicit friction hereafter.

1.8 ***Problems***

1.1 *Scaling*

Scale the y-momentum equation. Assume:

$$U = 1m/sec, \quad W = 1cm/sec, \quad L = 100m, \quad D = 0.5km$$

Also use the advective time scale, $T \propto L/U$.

Which are the dominant terms? Imagine that we can't measure the pressure drop, $\triangle p/\rho$. Can you estimate what it is, given the above scaling? What if the pressure drop were actually much less than this—what could you say about the motion?

Finally, write the approximate equation.

1.2 *The vorticity equation*

Derive equation (56). Now derive it again, but using pressure coordinates instead of z-coordinates.

1.3: *Atmospheric pressure*

The surface pressure in the atmosphere is due to the weight of all the air in the atmospheric column above the surface. Use the hydrostatic relation to estimate how large the surface pressure is. Assume that the atmospheric density decays exponentially with height:

$$\rho(z) = \rho_0 \, exp(-z/H)$$

where $\rho_0 = 1.2 \, kg/m^3$ and the scale height, $H = 8.6$ km. Assume too that the pressure at $z = \infty$ is zero.

1.4: *Ageostrophic velocities*

Use scaling to figure out how big the ageostrophic velocities typically are. In particular, use the fact that the horizontal divergence of the ageostrophic velocities is the same size as the vertical derivative of the vertical velocity.

1.4: *Vertical velocities*

Consider the incompressibility condition. Break the horizontal velocities into ageostrophic and geostrophic parts and substitute them into the equation. What happens if

$$\frac{\partial w}{\partial z} = O|1|$$

What would the velocity field look like, in particular, if the lower boundary was a flat surface, with $w = 0$?

1.6: Problem 2.6 in Holton.

Derive an expression for the density ρ that results when a parcel of dry air initially at pressure p_s and density ρ_s expands adiabatically to pressure p.

1.7: *Southern Jet*

Say the temperature at the South Pole is -20C and it's 40C at the Equator. Assuming the average wind speed is zero at the Earth's surface (1000 hPa), what is the mean zonal speed at 250 hPa at 45S?

Assume the temperature gradient is constant with latitude and pressure. Derive the thermal wind relations in pressure coordinates (by taking the p-derivatives of the geostrophic relations). Then integrate the relations with

respect to pressure to find the velocity difference between the surface and 250 hPa.

1.8: *Conservation of vorticity*

A circular region of air at 30 ° N with a radius of 100 km expands to twice its original radius. If the air is initially at rest, what is the mean tangential velocity at the edge of the circle after expansion? Use Kelvin's theorem (sec. 1.6.1).

1.9: (Problem 4.2 in Holton)

A cylindrical column of air at 30 ° N with radius 100 km expands to twice its original radius. If the air is initially at rest, what is the mean tangential velocity at the perimeter after expansion?

2 Barotropic flows

Now we will examine some solutions to the vorticity equation. We assume the fluid is barotropic, so that there is no vertical shear in the horizontal velocities. While this may seem like a gross over-simplification, many of the phenomena seen in the barotropic case carry over to the more general baroclinic situation.

2.1 Barotropic PV equation

Assume we have a layer of fluid (atmosphere or ocean) which is bounded by two surfaces, the lower one at z_0 and the upper at z_1. So the depth $D = z_1 - z_0$. We will work in z-coordinates (although similar results obtain in pressure coordinates). The vorticity equation (75) is:

$$\frac{d_g}{dt}\left(\nabla^2\psi + \beta y\right) = f_0 \frac{\partial}{\partial z} w \tag{89}$$

Because the velocities don't vary in z, we can integrate the equation over the depth of the layer:

$$\int_{z0}^{z1} \frac{d_g}{dt}(\nabla^2\psi + \beta y)\, dz = D\frac{d_g}{dt}(\nabla^2\psi + \beta y) = f_0 w(z_1) - f_0 w(z_0) \tag{90}$$

or:

$$\frac{d_g}{dt}\left(\nabla^2\psi + \beta y\right) = \frac{f_0}{D}[w(z_1) - w(z_0)] \tag{91}$$

This states that the absolute vorticity:

$$\zeta_g + \beta y$$

changes in response to vertical motion at the top and/or bottom boundary.

We must evaluate w on the two bounding surfaces. There are two effects which can cause such vertical motion: Ekman layers and bottom topography. Take topography first. Consider a fluid parcel on the lower boundary, z_0. For this parcel:

$$z = z_0 = h(x, y) \tag{92}$$

where $h(x, y)$ is the height of the bottom topography. If we apply the Lagrangian derivative to both sides, we get:

$$\frac{d}{dt}z = w(z_0) = \frac{d}{dt}h = \frac{d_H}{dt}h \tag{93}$$

If we replace the horizontal velocities with geostrophic ones (in keeping with the vorticity equation), we have:

$$w(z_0) = \frac{d_g}{dt}h = \vec{u}_g \cdot \nabla h \tag{94}$$

This is the boundary condition on w with topography.

An important point here is that because the vertical velocity is order Rossby number, the topography must be too. If the topography were order one, the vertical velocities would be much larger and could hence violate the non-divergence of the geostrophic velocities. Thus we *require that the bottom topography be weak*. In particular, if we write the total depth as:

$$D = D_0 - h(x, y) \tag{95}$$

where D_0 is a constant depth, then QG requires that $h \ll D_0$ (Fig. 5).

Separating the vertical velocity due to the bottom topography out in (91), we get:

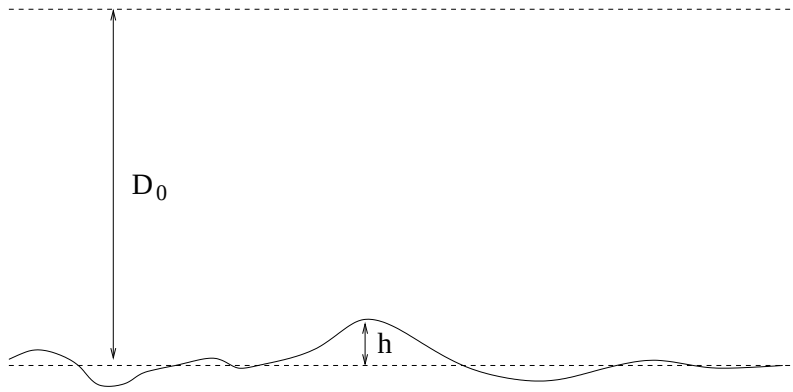

Figure 5: The geometry of our fluid layer. The topographic height, h, is much less than the depth of the layer.

$$\frac{d_g}{dt}\left(\nabla^2\psi + \beta y\right) = \frac{f_0}{D_0}[w_e(z_1) - w_e(z_0)] - \frac{f_0}{D_0}\vec{u}_g \cdot \nabla h \tag{96}$$

The "e" subscript on the vertical velocities indicate that these terms are associated with Ekman layers. Note too that we approximate the total depth by D_0 in the denominators, since the topography is an order Rossby number correction. Because the topography doesn't change in time, we can move the last term to the LHS:

$$\frac{d_g}{dt}\left(\nabla^2\psi + \beta y + \frac{f_0}{D_0}h\right) = \frac{f_0}{D_0}[w_e(z_1) - w_e(z_0)] \tag{97}$$

Now we add the Ekman layers. In the atmosphere, we would simply set the vertical velocity at the top boundary to zero. This would be like assuming the tropopause was a rigid lid. The ocean is different though, because the wind is causing divergence at the upper surface. So we include the wind stress term from (84):

$$w_e(z_1) = \frac{1}{\rho_0 f_0}\nabla \times \vec{\tau}_w \tag{98}$$

Remember that we take the vertical component of the wind stress curl.

Here the wind forces motion over the entire depth of the ocean, because there is no vertical shear.

The bottom Ekman layer exists in both the atmosphere and ocean. This exerts a drag proportional to the relative vorticity. From (88), we have:

$$w_e(z_0) = \frac{\delta_e}{2}\zeta_g \tag{99}$$

where $\delta_e = \sqrt{2A_z/f_0}$. The Ekman layers thus damp the motion in the interior when there is vorticity.

Combining all the terms, we arrive at the **barotropic PV equation**:

$$\frac{d_g}{dt}\left(\nabla^2\psi + \beta y + \frac{f_0}{D_0}h\right) = \frac{1}{\rho_0 D_0}\nabla \times \vec{\tau}_w - r\nabla^2\psi \tag{100}$$

The constant, r, is called the "Ekman drag coefficient" and is defined:

$$r = \frac{f_0\delta_e}{2D_0} = \sqrt{\frac{A_z f_0}{2D_0^2}}$$

Hereafter we consider solutions to equation (100). But first, let's consider some general properties.

2.2 Geostrophic contours

Consider what happens when there is no forcing, so that no Ekman layers are present. Then the barotropic PV:

$$q = \nabla^2\psi + \beta y + \frac{f_0}{D_0}h$$

is conserved on a fluid parcel. Thus the quantity is conserved on all parcels in a given flow—this is a strong constraint. Now the PV is comprised of a time-varying portion (the vorticity) and a time-independent part (due to β and the bottom topography). Thus we can rewrite equation (100) this way:

$$\frac{d_g}{dt}\zeta + \vec{u}_g \cdot \nabla q_s = 0 \tag{101}$$

where the function:

$$q_s \equiv \beta y + \frac{f_0}{D_0} h$$

defines the *geostrophic contours*, the stationary (unchanging) part of the potential vorticity.

In oceanography, q_s is often referred to as the "f/H" contours (where their H is our D). To see why, expand the ratio f/D:

$$\frac{f}{D} = \frac{f_0 + \beta y}{D_0 - h} = \frac{f_0}{D_0}\,(\frac{1 + \beta y/f_0}{1 - h/D_0}) \tag{102}$$

Because the β and topographic terms are small under QG, we can rewrite this thus:

$$\frac{f}{D} \approx \frac{f_0}{D_0}\,(1 + \frac{\beta}{f_0} y)(1 + \frac{h}{D_0}) \approx \frac{1}{D_0}(f_0 + \beta y + \frac{f_0}{D_0} h) \tag{103}$$

ignoring the product of the small terms. Thus "f/H" is the same as q_s, apart from an additive constant (f_0) and a constant factor (D_0).

One can also show that f/H is the stationary part of the potential vorticity under the *shallow water equations*, which are the equations which govern a constant density fluid with topography. Interestingly, the shallow water equations apply to flows with a fully varying Coriolis parameter and steep topography. They are the equations that we solve for predicting the global tides. Our term, q_s, is the QG approximation of the shallow water f/H.

If a parcel crosses the geostrophic contours, its relative vorticity changes, to conserve the total PV. Consider the example in figure (6). Here there is no topography, so the contours are just latitude lines ($q_s = \beta y$). Northward

motion is accompanied by a decrease in relative vorticity because as y increases, ζ_g must decrease to preserved the total PV. If the parcel has zero vorticity initially, it acquires negative vorticity (clockwise circulation) in the northern hemisphere. Southward motion likewise generates positive vorticity. Of course this is just Kelvin's theorem again.

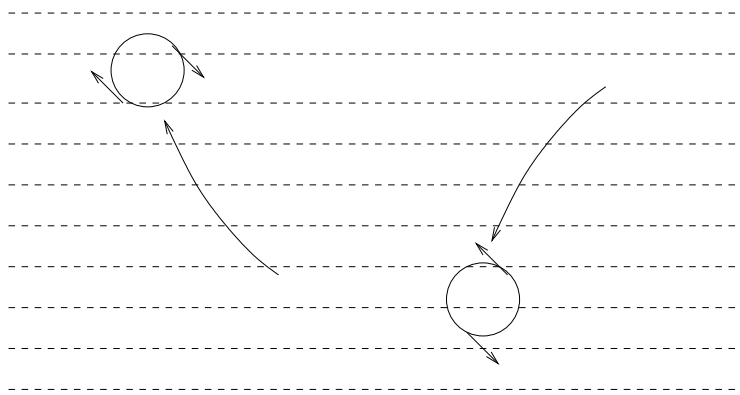

Figure 6: The change in relative vorticity due to northward or southward motion relative to βy.

Topography generally distorts the geostrophic contours. If it is large enough, it can overwhelm the βy term locally, even causing *closed* contours (near mountains or basins). But the same principle holds, as shown in Fig. (7). Motion towards larger values of q_s generates negative vorticity and motion to lower values of q_s generates positive vorticity.

If the flow is steady, then (100) is just:

$$\vec{u}_g \cdot \nabla(\zeta_g + q_s) = 0 \tag{104}$$

This implies that for a steady flow, the geostrophic flow is *parallel to the total PV contours*, $q = \zeta_g + q_s$. If the relative vorticity is weak, so that $\zeta_g \ll q_s$, then:

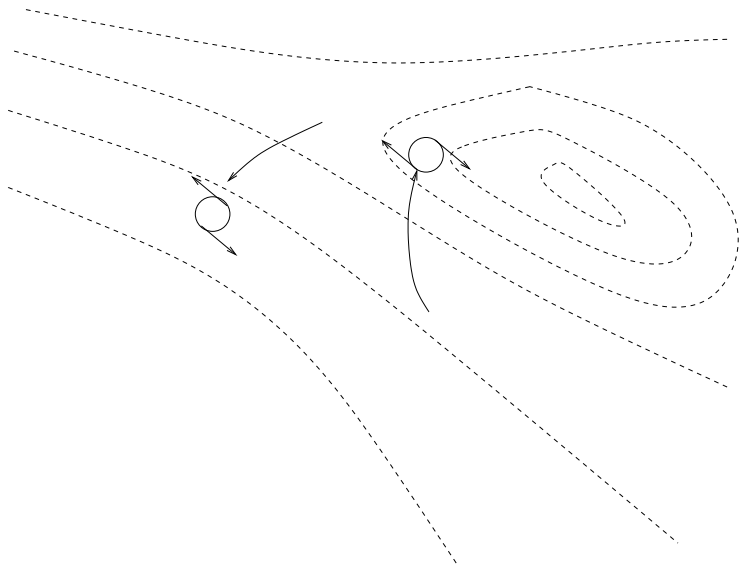

Figure 7: The change in relative vorticity due to motion across geostrophic contours with topography.

$$\vec{u}_g \cdot \nabla q_s = 0 \tag{105}$$

So the flow follows the geostrophic contours.

A simple example is one with no topography. Then we have:

$$\vec{u}_g \cdot \nabla \beta y = \beta v_g = 0 \tag{106}$$

So the steady flow is purely *zonal*. An example is the Jet Stream in the atmosphere. Because this is approximately zonal (it is nearly so in the Southern Hemisphere), it is nearly a steady solution of the PV equation.

Alternately, if the region is small enough that we can ignore changes in the Coriolis parameter, then:

$$\vec{u}_g \cdot \nabla h = 0 \tag{107}$$

(after dropping the constant f_0/D_0 factor). In this case, the flow follows

the topography. This is why many major currents in the ocean are often parallel to the isobaths.

Whether such steady flows exist depends on the boundary conditions. The atmosphere is a *re-entrant domain*, so a zonal wind can simply wrap around the earth (Fig. 8, left). But most ocean basins have lateral boundaries (continents), which block the flow. Thus steady, along-contour flows in a basin can occur *only where topography causes the contours to close* (Fig. 8, right). This can happen in basins.

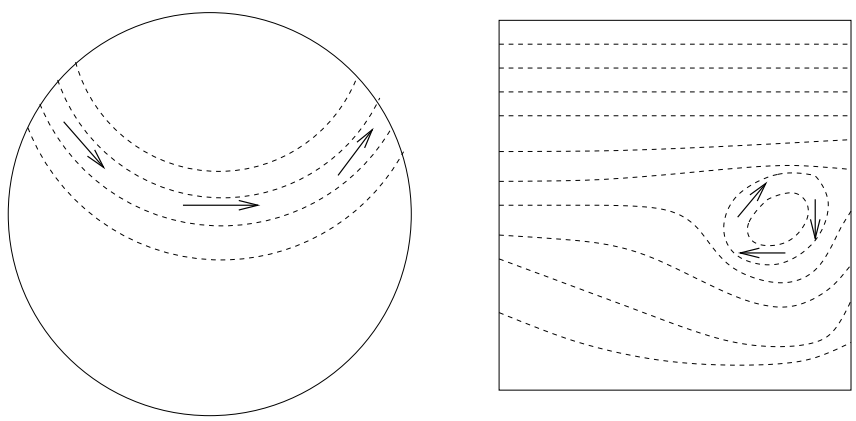

Figure 8: Steady, along-geostrophic contour flow in the atmosphere (left) and in the ocean (right).

For example, consider Fig. (9). This is a plot of the mean surface velocities near the Lofoten Basin off the west coast of Norway.[4]. The strong current on the right hand side is the Norwegian Atlantic Current, which flows in from the North Atlantic and proceeds toward Svalbard. Notice how this follows the continental slope (the steep topography between the continental shelf and deeper ocean). In the basin itself, the flow is more variable, but there is a strong, clockwise circulation in the deepest part of the basin, where the topographic contours are closed. Thus both closed

[4]The mean velocities were obtained by averaging velocities from freely-drifting surface buoys, deployed in the Nordic Seas under the POLEWARD project

and open geostrophic contour flows are seen here.

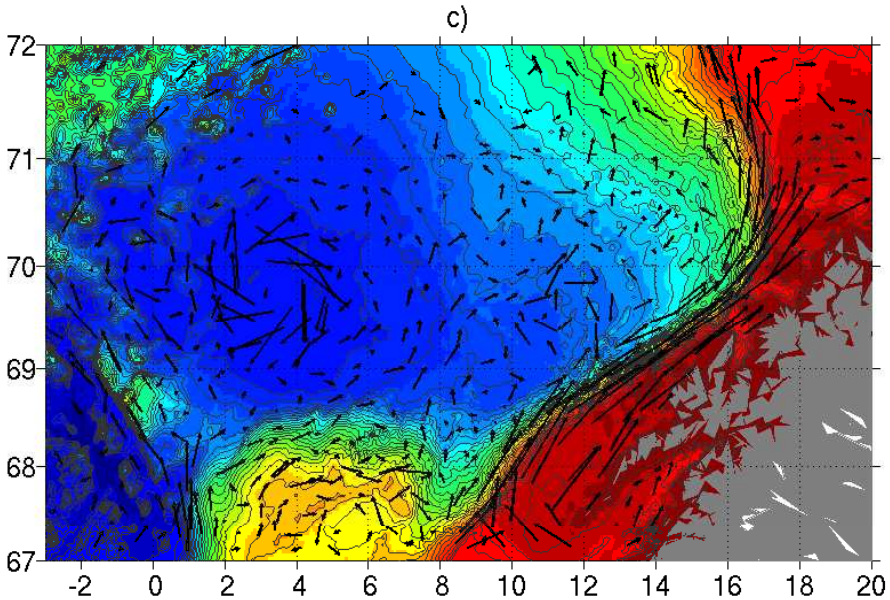

Figure 9: Mean velocities estimated from surface drifters in the Lofoten Basin west of Norway. The color contours indicate the water depth. Note the strong flow along the continental margin and the clockwise flow in the center of the basin, near 2° E. From Koszalka et al. (2010).

If the relative vorticity is not small compared to q_s, the flow will deviate from the latter contours. This can be seen for example with the Gulf Stream, which crosses f/H contours as it leaves the east coast of the U.S. Indeed, if the relative vorticity is much stronger than q_s, we have:

$$\vec{u}_g \cdot \nabla \zeta_g \approx 0 \tag{108}$$

as a condition for a steady flow. The flow follows contours of constant vorticity. An example is flow in a vortex. Then the vorticity contours are circular or ellipsoidal and the streamlines have the same shape. The vortex persists for long times precisely because it is near a steady state.

2.3 Barotropic Rossby waves

2.3.1 Linearization

The barotropic PV equation (100) is still a nonlinear equation, so analytical solutions are difficult to find. But we can make substantial progress by *linearizing* the equation.

Consider the case with no topography. As we found in the previous section, the only steady flow we could expect is a zonal one. So we could write:

$$u = U + u', \quad v = v'$$

Here, U is a constant zonal velocity which is assumed to be much greater than the primed velocities. In the atmosphere, U would represent the Jet Stream. Because U is constant, the relative vorticity is just:

$$\zeta = \frac{\partial}{\partial x} v' - \frac{\partial}{\partial y} u' = \zeta'$$

We substitute the velocities and vorticity into the PV equation to get:

$$\frac{\partial}{\partial t}\zeta' + (U + u')\frac{\partial}{\partial x}\zeta' + v'\frac{\partial}{\partial y}\zeta' + \beta v' = 0 \tag{109}$$

Because the primed variables are small, we neglect their products. That leaves an equation with only linear terms. Written in terms of the streamfunction (and dropping the primes), we have:

$$(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x})\nabla^2\psi + \beta\frac{\partial}{\partial x}\psi = 0 \tag{110}$$

This is the *barotropic Rossby wave equation*. It has only one unknown, the streamfunction, ψ.

2.3.2 Wave solutions

Equation (110) is a first order *wave equation*. There are standard methods to solve such equations. One of the most common is the *Fourier transform*, in which we write the solution as an infinite series of sinusoidal waves. Exactly which type of wave depends on the boundary condition. To illustrate the method, we assume an infinite plane. Although this is not very realistic for the atmosphere, the results are very similar to those in a east-west re-entrant channel.

Thus we will write:

$$\psi = Re\{\sum_k \sum_l A(k,l)e^{ikx+ily-i\omega t}\} \tag{111}$$

where:

$$e^{i\theta} = cos(\theta) + isin(\theta) \tag{112}$$

is a complex number. The amplitude, A, can also be complex, i.e.

$$A = A_r + iA_i \tag{113}$$

However, since the wavefunction, ψ, is real, we need to take the real part of the product of A and $e^{i\theta}$. This is signified by the $Re\{x\}$ operator.

Now because the Rossby wave equation is linear, we can consider the solution for a *single* wave. This is because with a linear equation, we can add individual wave solutions together to obtain the full solution. So we consider the following solution:

$$\psi = Re\{Ae^{ikx+ily-i\omega t}\} \tag{114}$$

A is the wave *amplitude*, k and l are *wavenumbers* in the x and y directions, and ω is the wave *frequency*.

Consider the simpler case of a one-dimensional wave (in x), with a unit amplitude:

$$\psi = Re\{e^{ikx-i\omega t}\} = cos(kx - \omega t) \tag{115}$$

The wave has a *wavelength* of $2\pi/k$. If $\omega > 0$, the wave propagates toward larger x (Fig. 10). This is because as t increases, $-\omega t$ decreases, so x must increase to preserve the phase of the wave (the argument of the cosine).

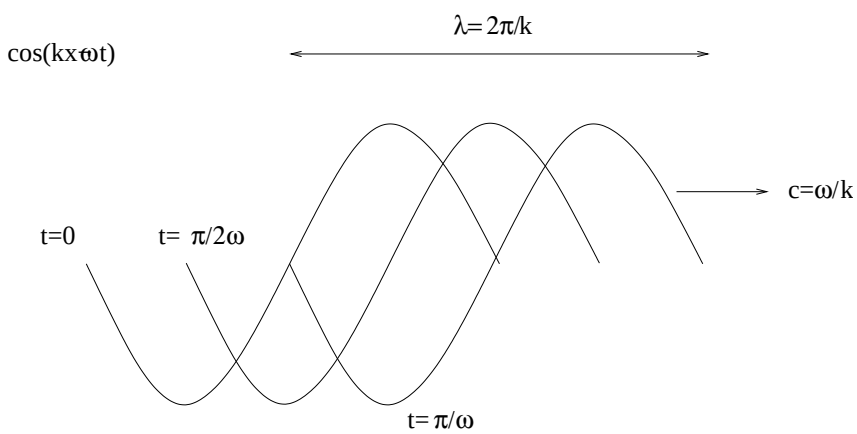

Figure 10: A one-dimensional wave, propagating toward the right.

The speed at which the crests move to the right is the wave's *phase speed*. This is clear if we write the wave this way:

$$\psi = cos\, k(x - ct) \tag{116}$$

where:

$$c = \frac{\omega}{k}$$

Notice that c has units of length over time, as expected for a velocity.

If the phase speed depends on the wavelength (wavenumber), we say the wave is *dispersive*. This is because different size waves will move at different speeds. Thus a packet of waves, originating from a localized region, will separate according to wavelength if they are dispersive. Waves that are *non-dispersive* move at the same speed regardless of wavelength. A packet of such waves would move away from their region of origin together.

2.3.3 Rossby wave phase speed

Now we return to the linearized barotropic PV equation (110) and substitute in our general wave solution in (114). We get:

$$(-i\omega - ikU)(-k^2 - l^2)\, Ae^{ikx+ily-i\omega t} - i\beta k\, Ae^{ikx+ily-i\omega t} = 0 \qquad (117)$$

(We will drop the $Re\{x\}$ operator, but remember that in the end, it is the real part we're interested in). Notice that both the wave amplitude and the sine term drop out. This is typical of linear wave problems: we get no information about the amplitude from the equation itself; that requires specifying initial conditions. Solving for ω, we get:

$$\omega = kU - \frac{\beta k}{k^2 + l^2} \qquad (118)$$

This is the *Rossby wave dispersion relation*. It relates the frequency of the wave to its wavenumbers. The corresponding zonal phase speed is:

$$c_x = \frac{\omega}{k} = U - \frac{\beta}{k^2 + l^2} \equiv U - \frac{\beta}{\kappa^2} \qquad (119)$$

where κ is the total wavenumber.

There are a number of interesting features about this. First, the phase speed depends on the wavenumbers, so the waves are dispersive. The largest speeds occur when k and l are small, corresponding to long wavelengths. Thus large waves move faster than small waves.

Second, all waves propagate *westward* relative to the mean velocity, U. If $U = 0$, $c < 0$ for all (k, l). This is a distinctive feature of Rossby waves. Satellite observations of Rossby waves in the Pacific Ocean show that the waves, originating off of California and Mexico, sweep westward toward Asia.

Third, the wave speed depends on the orientation of the wave crests. The most rapid westward propagation occurs when the crests are oriented north-south, with $k \neq 0$ and $l = 0$. If the wave crests are oriented east-west, so that $k = 0$, then the wave frequency is zero and there is no wave motion at all.

The phase speed also has a meridional component, and this can be either towards the north or south:

$$c_y = \frac{\omega}{l} = \frac{Uk}{l} - \frac{\beta k}{l(k^2 + l^2)} \tag{120}$$

The sign of c_y thus depends on the signs of k and l. So Rossby waves can propagate northwest, southwest or west—but not east.

With a mean flow, the waves can be swept eastward, producing the appearance of eastward propagation. The short waves are more susceptible to eastward propagation. In particular, if

$$\kappa > \kappa_s \equiv (\frac{\beta}{U})^{1/2}$$

the wave moves eastward. Longer waves move westward, opposite to the mean flow. If $\kappa = \kappa_s$, the wave is *stationary* and the crests don't move

at all—the wave is propagating west at exactly the same speed that the background flow is going east. Stationary waves can only occur if the mean flow is eastward, because the waves propagate westward.

Example: At what background wind velocity is a wave with a wavelength of 1000 km stationary? What about a wavelength of 5000 km? Assume we are at 45 degrees N and that $k = l$.

The wave has wavenumbers:

$$k = l = \frac{2\pi}{10^6} m^{-1} = 6.28 \times 10^{-6}\, m^{-1}$$

and:

$$\kappa^2 = k^2 + l^2 = 2k^2 = 7.90 \times 10^{-11}\, m^{-2}$$

At 45 N, we have:

$$\beta = \frac{1}{6.3 \times 10^6} \frac{4\pi}{86400} cos(45) = 1.63 \times 10^{-11}\, m^{-1} sec^{-1}$$

So:

$$\frac{\beta}{\kappa^2} = .21\, m/sec$$

So if $U = 0.21$ m/sec, the wave is stationary.

For $\lambda = 5000$ km, we find:

$$U_s = \frac{\beta\lambda^2}{2(4\pi^2)} = \frac{1.63 \times 10^{-11}(5 \times 10^6)^2}{8\pi^2} = 5.2 m/sec \tag{121}$$

2.3.4 Westward propagation: mechanism

We have discussed how motion across the mean PV contours, q_s, induces relative vorticity. The same is true with a Rossby wave. Fluid parcels

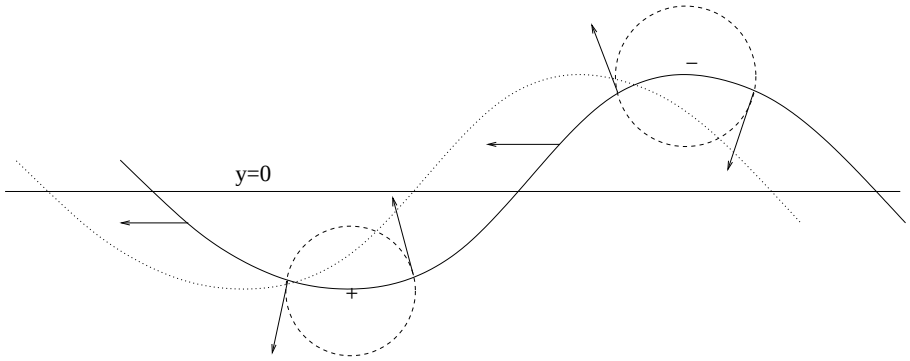

Figure 11: Relative vorticity induced in a Rossby wave. Fluid advected northwards acquires negative vorticity and fluid advected southwards positive vorticity.

which are advected north in the wave acquire negative vorticity, while those advected south acquire positive vorticity (Fig. 11). Thus one can think of a Rossby wave as a string of negative and positive vorticity anomalies (Fig. 12).

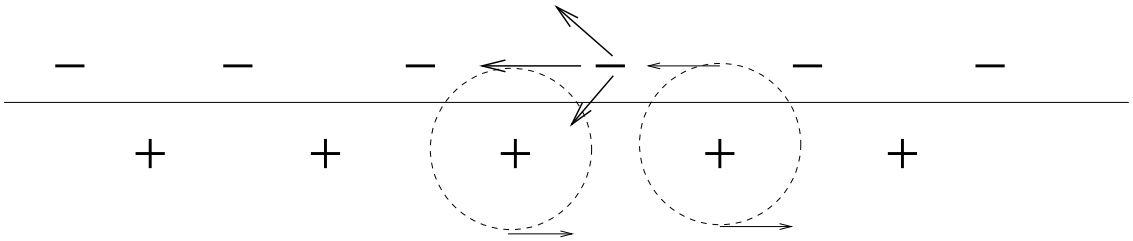

Figure 12: The Rossby wave as a string of vorticity anomalies. The cyclone in the right hand circle advects the negative anomaly to the southwest, while the left cyclone advects it toward the northwest. The net effect is westward motion.

Now the negative anomalies to the north will act on the positive anomalies to the south, and vice versa. Consider the two positive anomalies shown in Fig. (12). The right one advects the negative anomaly between them southwest, while the left one advects it northwest. Adding the two velocities together, the net effect is a westward drift for the anomaly. Similar reasoning suggests the positive anomalies are advected westward by the negative anomalies.

What does a Rossby wave look like? Recall that ψ is proportional to the geopotential, or the pressure in the ocean. So a sinusoidal wave is a sequence of high and low pressure anomalies. An example is shown in Fig. (13). This wave has the structure:

$$\psi = cos(x - \omega t)sin(y) \tag{122}$$

(which also is a solution to the wave equation, as you can confirm). This appears to be a grid of high and low pressure regions.

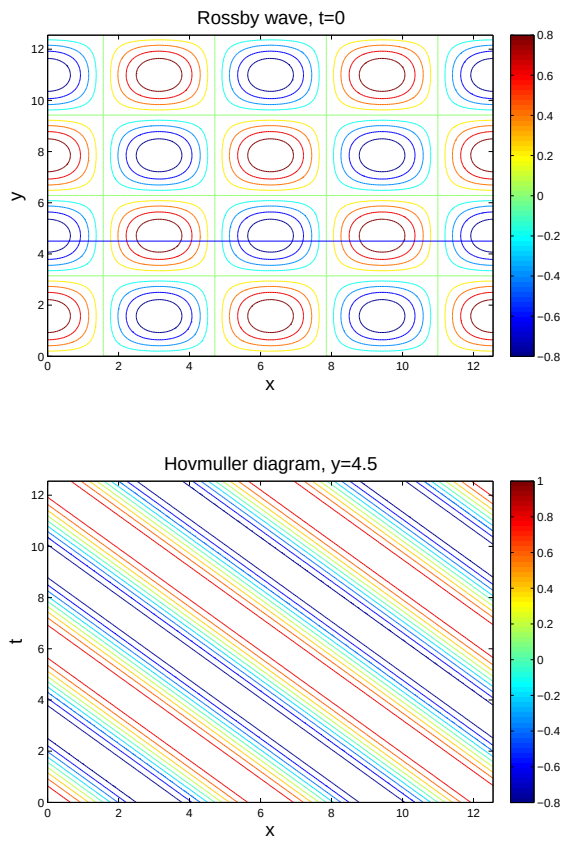

Figure 13: A Rossby wave, with $\psi = cos(x - \omega t)sin(y)$. The red corresponds to high pressure regions and the blue to low. The lower panel shows a "Hovmuller" diagram of the phases at $y = 4.5$ as a function of time.

The whole wave in this case is propagating westward. Thus if we take a cut at a certain latitude, here $y = 4.5$, and plot $\psi(x, 4.5, t)$, we get the plot

in the lower panel. This shows the crests and troughs moving westward at a constant speed (the phase speed). This is known as a "Hovmuller" diagram.

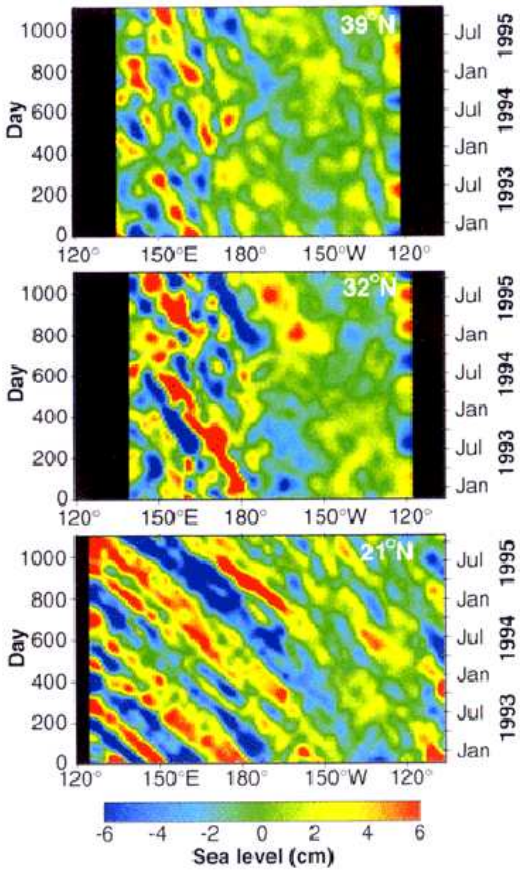

Figure 14: Three Hovmuller diagrams constructed from sea surface height in the North Pacific. From Chelton and Schlax (1996).

Three examples from the ocean are shown in Fig. (14). These are Hovmuller diagrams constructed from sea surface height in the Pacific, at three different latitudes. We see westward phase propagation in all three cases. Interestingly, the phase speed (proportional to the tilt of the lines) differs in the three cases. To explain this, we will need to take stratification into account, as discussed later on. In addition, the waves are more pronounced

west of 150-180 W. The reason for this however is still unknown.

2.3.5 Group Velocity

Thus Rossby waves propagate westward. But this actually poses a problem. Say we are in an ocean basin, with $U = 0$. If there is a disturbance on the eastern wall, Rossby waves will propagate westward into the interior, communicating to the rest of the basin the changes occurring on the eastern wall. But say the disturbance is on the west wall. If the waves can go only toward the wall, the energy would necessarily be trapped there. How do we reconcile this?

The answer is that the phase velocity tells us only about the motion of the crests and troughs—it does not tell us how the energy is moving. To see how energy moves, it helps to consider a *packet* of waves with different wavelengths. If the Rossby waves were initiated by a localized source, say a meteor crashing into the ocean, they would start out as a wave packet. Wave packets have both a phase velocity and a "group velocity". The latter tells us about the movement of packet itself, and this reflects how the energy is moving. It is possible to have a packet of Rossby waves which are moving eastwards, while the crests of the waves in the packet move westward.

Consider the simplest example, of two waves with different wavelengths and frequencies, but the same (unit) amplitude:

$$\psi = cos(k_1 x + l_1 y - \omega_1 t) + cos(k_2 x + l_2 y - \omega_2 t) \tag{123}$$

Imagine that k_1 and k_2 are almost equal to k, one slightly larger and the other slightly smaller. We'll suppose the same for l_1 and l_2 and ω_1 and ω_2. Then we can write:

$$\psi = cos[(k+\delta k)x + (l+\delta l)y - (\omega+\delta\omega)t]$$

$$+cos[(k-\delta k)x + (l-\delta l)y - (\omega-\delta\omega)t] \tag{124}$$

From the cosine identity:

$$cos(a \pm b) = cos(a)cos(b) \mp sin(a)sin(b) \tag{125}$$

So we can rewrite the streamfunction as:

$$\psi = 2\,cos(\delta kx + \delta ly - \delta\omega t)\,cos(kx + ly - \omega t) \tag{126}$$

The combination of waves has two components: a plane wave (like we considered before) multiplied by a *carrier wave*, which has a longer wavelength and lower frequency. The carrier wave has a phase speed of:

$$c_x = \frac{\delta\omega}{\delta k} \approx \frac{\partial\omega}{\partial k} \equiv c_{gx} \tag{127}$$

and

$$c_y = \frac{\delta\omega}{\delta l} \approx \frac{\partial\omega}{\partial l} \equiv c_{gy} \tag{128}$$

The phase speed of the carrier wave is the *group velocity*, because this is the speed at which the group (in this case two waves) moves. While the phase velocity of a wave is ratio of the frequency and the wavenumber, the group velocity is the *derivative* of the frequency by the wavenumber.

This is illustrated in Fig. (15). This shows two waves, $cos(1.05x)$ and $cos(0.095x)$. Their sum yields the wave packet in the lower panel. The

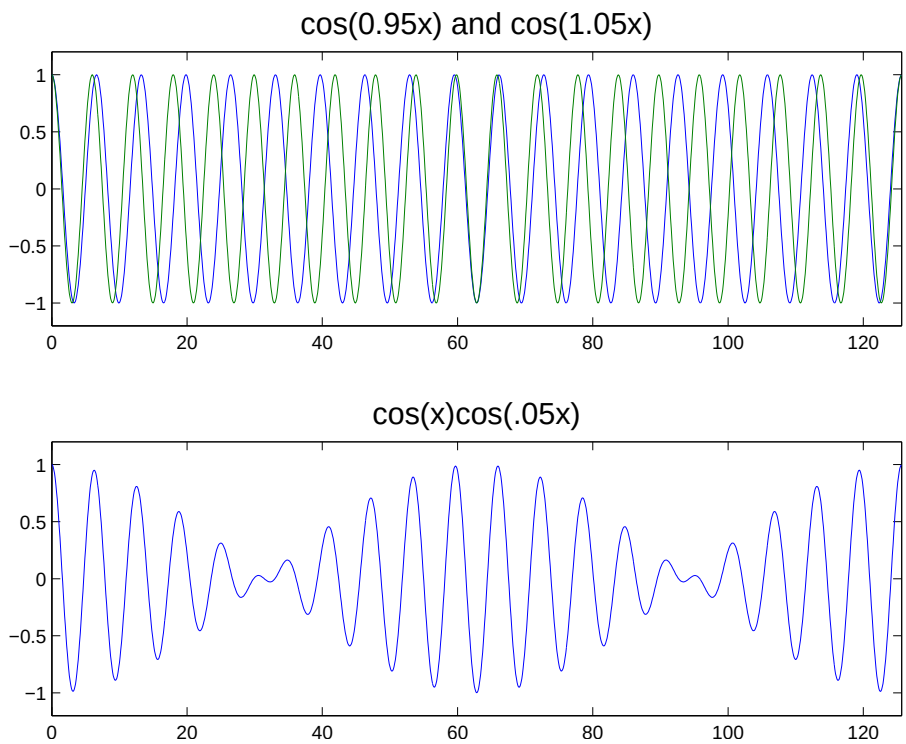

Figure 15: A wave packet of two waves with nearly the same wavelength.

smaller ripples propagate with the phase speed, $c = \omega/k = \omega/1$, westward. But the larger scale undulations move with the group velocity, and this can be either west *or* east.

The group velocity concept applies to any type of wave. For Rossby waves, we take derivatives of the Rossby wave dispersion relation for ω. This yields:

$$c_{gx} = \frac{\partial \omega}{\partial k} = \beta \frac{k^2 - l^2}{(k^2 + l^2)^2} \ , \quad c_{gy} = \frac{\partial \omega}{\partial l} = \frac{2\beta k l}{(k^2 + l^2)^2} \tag{129}$$

Consider for example the group velocity in the zonal direction, c_{gx}. The sign of this depends on the relative sizes of the zonal and meridional wavenumbers. If

$$k > l$$

the wave packet has a positive (eastward) zonal velocity. Then the energy is moving in the *opposite* direction to the phase speed. This answers the question about the disturbance on the west wall. Energy can indeed spread eastward into the interior, if the zonal wavelength is shorter than the meridional one. Note that for such waves, the phase speed is still westward. So the crests will move toward the west wall while energy is carried eastward!

Another interesting aspect is that the group velocity in the y-direction is *always* in the opposite direction to the phase speed in y, because:

$$\frac{c_{gy}}{c_y} = -\frac{2l^2}{k^2 + l^2} < 0\,. \tag{130}$$

So northward propagating waves have southward energy flux!

The group velocity can also be derived by considering the energy equation for the wave. This is shown in Appendix C.

2.4 Rossby wave reflection

A good example of these properties of Rossby waves is the case of a wave reflecting off a solid boundary. Consider what happens to a westward propagating plane Rossby wave which encounters a straight wall, oriented along $x = 0$. The incident wave can be written:

$$\psi_i = A_i\, e^{ik_i x + il_i y - i\omega_i t}$$

where:

$$\omega_i = \frac{-\beta k_i}{k_i^2 + l_i^2}$$

The incident wave has a westward group velocity, so that

$$k_i < l_i$$

Let's assume too that the group velocity has a northward component (so that the wave is generated somewhere to the south). So the phase velocity is oriented *southwest*.

The wall will produce a reflected wave. If this weren't the case, all the energy would have to be absorbed by the wall. We assume all the energy is reflected. The reflected wave is:

$$\psi_r = A_r \, e^{ik_r x + il_r y - i\omega_r t}$$

The total streamfunction is the sum of the incident and reflected waves:

$$\psi = \psi_i + \psi_r \tag{131}$$

In order for there to be no flow into the wall, we require that the zonal velocity vanish at $x = 0$, or:

$$u = -\frac{\partial}{\partial y}\psi = 0 \quad at \;\; x = 0 \tag{132}$$

This implies:

$$-il_i \, A_i \, e^{il_i y - i\omega_i t} - il_r \, A_r \, e^{il_r y - i\omega_r t} = 0 \tag{133}$$

In order for this condition to hold at all times, the frequencies must be equal:

$$\omega_i = \omega_r = \omega \tag{134}$$

Likewise, if it holds for all values of y along the wall, the meridional wavenumbers must also be equal:

$$l_i = l_r = l \tag{135}$$

Note that because the frequency and meridional wavenumbers are preserved on reflection, the meridional phase velocity, $c_y = \omega/l$, remains unchanged. Thus (133) becomes:

$$il\, A_i\, e^{ily-i\omega t} + il\, A_r\, e^{ily-i\omega t} = 0 \tag{136}$$

which implies:

$$A_i = -A_r \equiv A \tag{137}$$

So the amplitude of the wave is preserved, but the phase is changed by 180°.

Now let's go back to the dispersion relations. Because the frequencies are equal, we have:

$$\omega = \frac{-\beta k_i}{k_i^2 + l^2} = \frac{-\beta k_r}{k_r^2 + l^2}\ . \tag{138}$$

This is possible because the dispersion relation is quadratic in k and thus admits two different values of k. Solving the Rossby dispersion relation for k, we get:

$$k = -\frac{\beta}{2\omega} \pm \frac{\sqrt{\beta^2 - 4\omega^2 l^2}}{2\omega} \tag{139}$$

The incident wave has a smaller value of k because it has a westward group velocity; so it is the additive root. The reflected wave is thus the negative root. So the wavenumber *increases* on reflection, by an amount:

$$|k_r - k_i| = 2\sqrt{\frac{\beta^2}{4\omega^2} - l^2} \tag{140}$$

The incident waves are long but the reflected waves are *short*.

You can also show that the meridional velocity, v, *increases* upon reflection. One can also show that the mean energy (Appendix A) increases on reflection. The reflected wave is more energetic because the energy is squeezed into a shorter wave. However, the *flux* of energy is conserved; the amount of energy going in equals that going out. So energy does not accumulate at the wall.

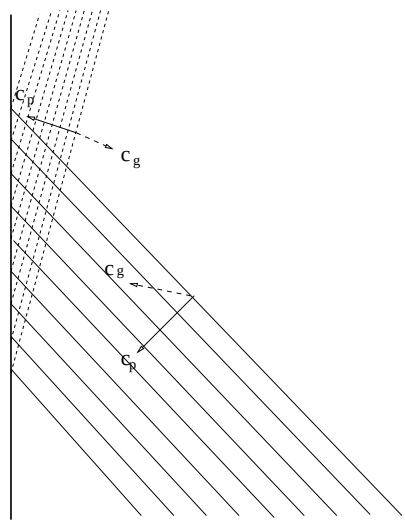

Figure 16: A plane Rossby wave reflecting at a western wall. The incident wave is shown by the solid lines and the reflected wave by the dashed lines. The phase velocities are indicated by the solid arrows and the group velocities by the dashed arrows. Note the wavelength in y doesn't change, but the reflected wavelength in x is much shorter. Note too the reflected wave has a phase speed directed toward the wall, but a group velocity away from the wall.

Thus Rossby waves change their character on reflection. Interestingly, the change depends on the *orientation* of the boundary. A tilted boundary (e.g. northwest) will produce different results. In fact, the case with a zonally-oriented boundary (lying, say, along $y = 0$) is *singular*; you must introduce other dynamics, like friction, to solve the problem. Rossby waves, in many ways, are strange.

2.5 Mountain waves

Barotropic Rossby waves have been used to study the mean surface pressure distribution in the atmosphere. This is the pressure field you get when averaging over long periods of time (e.g. years). The central idea is that the mean wind, U, blowing over topography can excite stationary waves ($c_x = 0$). As demonstrated by Charney and Eliassen (1949), one can find a reasonable first estimate of the observed distribution using the linear, barotropic vorticity equation.

We start with the vorticity equation, with zero forcing but including topography:

$$\frac{d_g}{dt}\left(\zeta + \beta y + \frac{f_0}{D}h\right) = 0 \tag{141}$$

We will linearize about a mean zonal flow:

$$u = U + u', \quad v = v', \quad \zeta = \zeta'$$

We will also assume the topography is weak:

$$h = h'$$

in keeping with QG. Then the Rossby wave equation becomes:

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\zeta' + \beta v' + U\frac{\partial}{\partial x}\frac{f_0}{D}h' = 0 \tag{142}$$

Substituting in the streamfunction, we have:

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\nabla^2\psi + \beta\frac{\partial}{\partial x}\psi = -\frac{f_0}{D}U\frac{\partial}{\partial x}h' \tag{143}$$

We put the topographic term on the RHS because it does not involve the streamfunction, and so acts instead like a forcing term. So the winds blowing over the mountains generate the response.

The homogeneous solution to this equation are just the Rossby waves we discussed earlier. These are called "free Rossby waves". If we were to suddenly "turn on" the wind, we would excite free waves. The particular solution, or the "forced wave", is the part generated by the topographic term on the RHS. This is the portion of the flow that will remain after the free waves have propagated away.

So the forced wave is the portion that will determine the time mean flow. To find that, we can ignore the time derivative:

$$U\frac{\partial}{\partial x}\nabla^2\psi + \beta\frac{\partial}{\partial x}\psi = -\frac{f_0}{D}U\frac{\partial}{\partial x}h' \tag{144}$$

All the terms involve a derivative in x, so we can simply integrate the equation once in x to get rid of that. We can ignore the constant of integration, which would amount to adding a constant to the streamfunction. The latter would have no effect on the velocity field (why?).

In line with our previous derivations, we write the topography as a sum of Fourier modes:

$$h'(x,y) = Re\{\sum_k \sum_l h(k,l)\, e^{ikx+ily}\} \tag{145}$$

For simplicity, we will focus on the response to a single wave mode:

$$h' = h_0 cos(kx)cos(ly) \tag{146}$$

We can always construct the response to more complicated topography by adding the solutions for different (k,l), because the Rossby wave equation is linear. Substituting this in yields:

$$U\nabla^2\psi + \beta\psi = -\frac{f_0 h_0}{D}U cos(kx)cos(ly) \tag{147}$$

For the reasons given, we focus on the particular solution. This has the general form:

$$\psi = Acos(kx)cos(ly) \tag{148}$$

Plugging in:

$$\left(U(-k^2 - l^2) + \beta\right) Acos(kx)cos(ly) = -\frac{f_0 h_0}{D} Ucos(kx)cos(ly) \tag{149}$$

or:

$$A = \frac{f_0 h_0}{D(\kappa^2 - \beta/U)} = \frac{f_0 h_0}{D(\kappa^2 - \kappa_s^2)} \tag{150}$$

where:

$$\kappa_s \equiv (\frac{\beta}{U})^{1/2}$$

is the wavenumber of the stationary Rossby wave with a background velocity, U (sec. 2.3.3). Notice with forcing that we obtain an expression for the amplitude, A—it doesn't drop out. So the forced solution is:

$$\psi = \frac{f_0 h_0}{D(\kappa^2 - \kappa_s^2)} cos(kx)cos(ly) \tag{151}$$

The pressure field thus resembles the topography. If the wavenumber of the topography, κ, is greater than the stationary wavenumber, the amplitude is positive. Then the forced wave is *in phase* with the topography. If the topographic wavenumber is smaller, the atmospheric wave is 180° out of phase with the topography. The latter case applies to large scale topography, for which the wavenumber is small. So we expect negative pressures over mountains and positive pressures over valleys. With small scale topography, the pressure over the mountains will instead be positive.

What happens though when $\kappa = \kappa_S$? Then the streamfunction is infinite! This is a typical situation with forced oscillations. If the forcing is at the natural frequency of the system, the response is infinite (we say the response is *resonant*). Having infinite winds is not realistic, so we must add additional dynamics. In particular, we can add friction.

We do this as follows. We must go back to the barotropic vorticity equation, but with a bottom Ekman layer:

$$\frac{d_g}{dt}\left(\zeta + \beta y + \frac{f_0}{D}h\right) = -r\zeta \tag{152}$$

Linearizing as before, we obtain:

$$U\frac{\partial}{\partial x}\nabla^2\psi + \beta\frac{\partial}{\partial x}\psi = -\frac{f_0}{D}U\frac{\partial}{\partial x}h' - r\nabla^2\psi \tag{153}$$

Using the same topography, we get:

$$(U\frac{\partial}{\partial x} + r)\nabla^2\psi + \beta\frac{\partial}{\partial x}\psi = \frac{kf_0h_0}{D}Usin(kx)cos(ly) \tag{154}$$

The equation is exactly as before, except that we have an additional factor in front of the relative vorticity. This prevents us from integrating the equation in x, like we did before. It also means that the cosine/cosine particular solution will no longer work. Instead, we use the following complex expression:

$$\psi = Re\{Ae^{ikx}\}cos(ly) \tag{155}$$

Remember that the amplitude, A, may also be a complex number. To be consistent, we write the topography in the same way, i.e.:

$$h' = Re\{h_0e^{ikx}\}cos(ly) \tag{156}$$

(even though we know that h_0 is real). So we have:

$$(U\frac{\partial}{\partial x}+r)\nabla^2\psi+\beta\frac{\partial}{\partial x}\psi=-\frac{ikf_0h_0}{D}Ue^{ikx}cos(ly) \tag{157}$$

Substituting in the wave solution, we get:

$$\left[(ikU+r)(-k^2-l^2)+ik\beta\right]A=-\frac{ikf_0h_0}{D}U \tag{158}$$

after canceling the sinusoidal terms. Solving for A, we get:

$$A=\frac{f_0h_0}{D(\kappa^2-\kappa_s^2-iR)} \tag{159}$$

where:

$$R\equiv\frac{r\kappa^2}{kU} \tag{160}$$

As promised, the amplitude is complex.

The amplitude is as before, except for the additional term in the denominator proportional to the Ekman drag, r. This term does two things. First, it removes the singularity. At $\kappa=\kappa_s$, we have:

$$A=i\frac{f_0h_0}{DR} \tag{161}$$

So the response is no longer infinite. However, the response is still greatest at this wavenumber. Having $\kappa\neq\kappa_s$ produces a weaker amplitude.

Second, friction causes a *phase shift* in the pressure field relative to the topography. Consider the response at $\kappa=\kappa_S$. Then the amplitude is purely imaginary, as seen above. Putting this into the full solution, we get:

$$\psi=Re\{Ae^{ikx}\}cos(ly)=-\frac{f_0h_0}{DR}sin(kx)cos(ly) \tag{162}$$

The topography on the other hand is proportional to $cos(kx)$. So the streamfunction is 90° out of phase with the mountains. In this case, the low pressure is downstream of the mountain. The extent of the phase shift depends on the difference between κ and κ_s. The larger the difference, the more aligned the pressure field is with the topography (either in phase, or 180° out of phase).

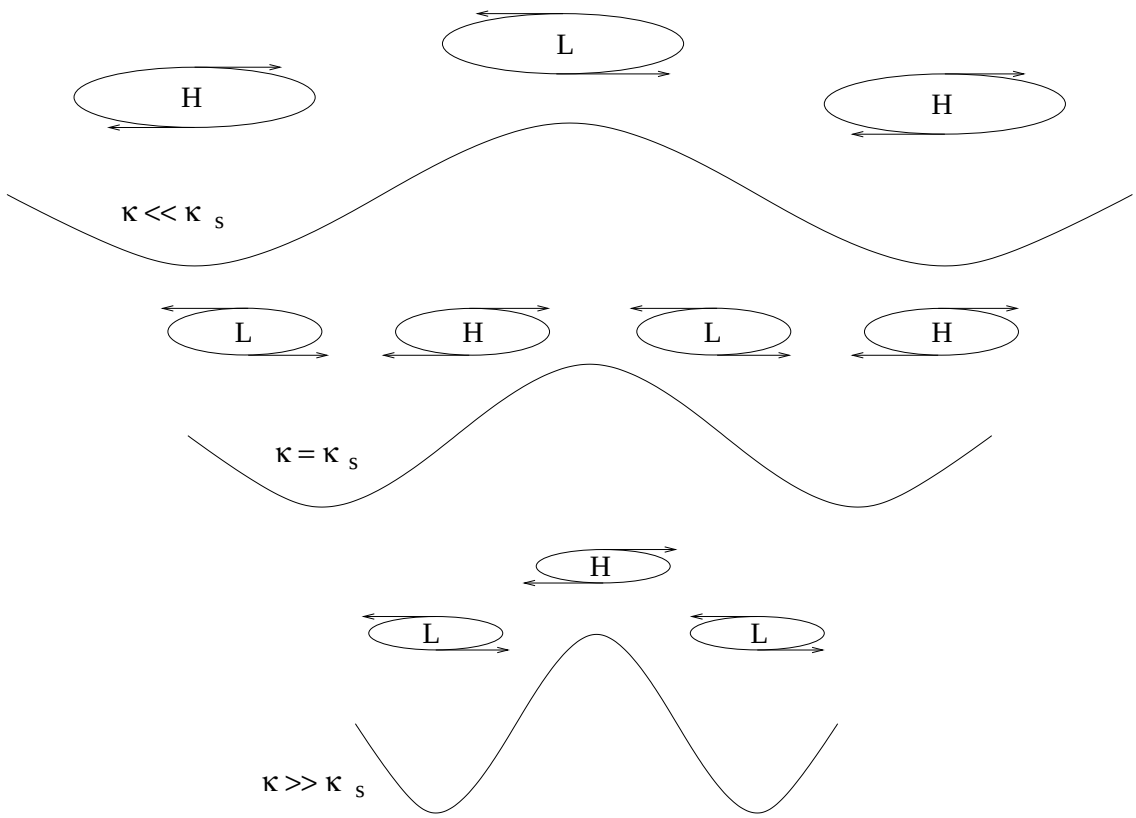

Figure 17: The mean pressure distribution over a sinusoidal mountain range. The topographic wavenumber is less than (upper), greater than (bottom) and equal to (middle) the stationary wavenumber.

We summarize the results with sinusoidal topography and Ekman friction graphically in Fig. (17). When the topographic wavenumber is much less than the stationary wavenumber for the velocity, U, the pressure field is aligned but anti-correlated with the topography. When the wavenum-

ber is much greater than κ_s, the pressure is aligned and correlated. When $\kappa = \kappa_s$, the pressure is 90 $^\circ$ out of phase with the mountains.

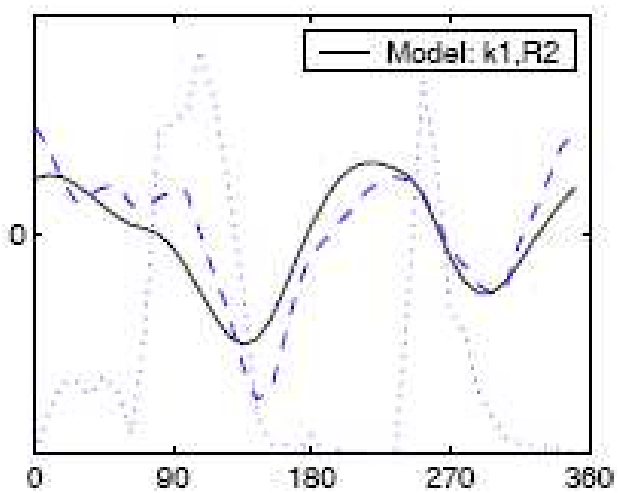

Figure 18: Charney and Eliassen's (1949) solution of the barotropic mountain wave problem at 45N. The dotted line indicates the topographic profile, the solid line is the model solution and the dashed line is the observed mean pressure at 500 mb. From Vallis (2007).

Charney and Eliassen (1949) applied the barotropic equation to the actual atmosphere. But instead of using a sinusoidal topography, they used the observed topographic profile at 45 N. The result of their calculation is shown in Fig. (18). The topography is indicated by the dotted lines. The two maxima come from the Himalayas and the Rocky Mountains. The solution, with U=17 m/sec and r=1/6 day^{-1}, is indicated by the solid line. The dashed line shows the observed mean pressure at 500 mb. We see the model exhibits much of the same structure as the observed pressure field. Both have low pressure regions down wind from the mountains, and a marked high pressure upwind of the Rockies.

The agreement between the model and observations is remarkably good,

given the simplicity of the model. In fact, it is probably too good. Charney and Eliassen used a meridional channel for their calculation (as one would do with a QG β-plane.), but if one redoes the calculation on a sphere, the Rossby waves can disperse meridionally and the amplitude is decreased (Held, 1983). Nevertheless, the relative success of the model demonstrates the utility of Rossby wave dynamics in understanding the low frequency atmospheric response.

2.6 Spin down

In the preceding example, bottom Ekman friction was used to make the response finite at the resonant wavenumber. What effect does bottom friction have on time-dependent motion?

Friction damps the velocities, causing the winds to slow. The simplest example of this is with no bottom topography and a constant f. Then the barotropic vorticity equation is:

$$\frac{d_g}{dt}\zeta = -r\zeta \tag{163}$$

This is a nonlinear equation. However it is easily solved in the Lagrangian frame. Following a parcel, we have that:

$$\zeta(t) = \zeta(0)e^{-rt} \tag{164}$$

So the vorticity decreases exponentially. The e-folding time scale is known as the Ekman *spin-down time*:

$$T_e = r^{-1} = (\frac{2}{A_z f_0})^{1/2} D \tag{165}$$

Typical atmospheric values are:

$$D = 10km, \;\; f = 10^{-4} sec^{-1}, \;\; A_z = 10\, m^2/sec$$

assuming the layer covers the entire troposphere. Then:

$$T_e \approx \; 4\; days$$

If all the forcing (including the sun) were suddenly switched off, the winds would slow down, over this time scale. After about a week or so, the winds would be weak.

If we assume that the barotropic layer does not extend all the way to the tropopause but lies nearer the ground, the spin-down time will be even shorter. This is actually what happens in the stratified atmosphere, with the winds near the ground spinning down but the winds aloft being less affected. So bottom friction favors flows intensified further up. The same is true in the ocean.

2.7 The Gulf Stream

The next example is one of the most famous in dynamical oceanography. It was known at least since the mid 1700's, when Benjamin Franklin mapped the principal currents of the North Atlantic (Fig. 19), that the Gulf Stream is an intense current which lies on the *western* side of the basin, near North America. The same is true of the Kuroshio Current, on the western side of the North Pacific, the Agulhas Current on the western side of the Indian Ocean, and numerous other examples. Why do these currents lie in the west? A plausible answer came from a work by Stommel (1948), based on the barotropic vorticity equation. We will consider this problem, which also illustrates the technique of *boundary layer analysis*.

Figure 19: Benjamin Franklin's map of the Gulf Stream.

We retain the β-effect and bottom Ekman drag, but neglect topography (the bottom is flat). We also include the surface Ekman layer, to allow for wind forcing. The result is:

$$\frac{d_g}{dt}(\zeta + \beta y) = \frac{d_g}{dt}\zeta + \beta v = \frac{1}{\rho_0 D}\nabla \times \vec{\tau}_w - r\zeta \tag{166}$$

We will search for steady solutions, as with the mountain waves. Moreover, we will not linearize about a mean flow—it is the mean flow itself we're after. So we neglect the first term in the equation entirely. Using the streamfunction, we get:

$$\beta \frac{\partial}{\partial x}\psi = \frac{1}{\rho_0 D}\nabla \times \vec{\tau}_w - r\nabla^2\psi \tag{167}$$

For our "ocean", we will assume a square basin. The dimensions of the basin aren't important, so we will just use the region $x = [0, L]$ and $y = [0, L]$ (L might be 5000 km).

It is important to consider the geostrophic contours in this case:

$$q_s = \beta y \tag{168}$$

which are just latitude lines. In this case, all the geostrophic contours intersect the basin walls. From the discussion in sec. (2.2), we know that there can be no steady flows without forcing, because such a flow would be purely zonal and would have to continue through the walls. However, with forcing there can be steady flow; we will see that this flow *crosses* the geostrophic contours.

Solutions to (167) can be obtained in a general form, once the wind stress is specified. But Stommel used a more elegant method. The main idea is as follows. Since the vorticity equation is linear, we can express the solution as the sum of two components:

$$\psi = \psi_I + \psi_B \tag{169}$$

The first part, ψ_I, is that driven by the wind forcing. We assume that this part is present in the whole domain. We assume moreover that the friction is weak, and does not affect this interior component. Then the interior component is governed by:

$$\beta \frac{\partial}{\partial x} \psi_I = \frac{1}{\rho_0 D} \nabla \times \vec{\tau} \tag{170}$$

This is the *Sverdrup relation*, after H. U. Sverdrup. It is perhaps the most important dynamical balance in oceanography. It states that vertical flow from the base of the surface Ekman layer, due to the wind stress curl, drives meridional motion. This is the motion across the geostrophic contours, mentioned above.

We can solve (170) if we know the wind stress and the boundary conditions. For the wind stress, Stommel assumed:

$$\vec{\tau} = -\frac{L}{\pi} cos(\frac{\pi y}{L})\,\hat{i}$$

The wind is purely zonal, with a cosine dependence. The winds in the northern half of the domain are eastward, and they are westward in the southern half. This roughly resembles the situation over the subtropical North Atlantic. Thus the wind stress curl is:

$$\nabla \times \vec{\tau} = -\frac{\partial}{\partial y}\tau^x = -sin(\frac{\pi y}{L})$$

Again, this is the vertical component of the curl. From the Sverdrup relation, this produces southward flow over the whole basin, with the largest velocities occurring at the mid-basin ($y = L/2$). We then integrate the Sverdrup relation (170) to obtain the streamfunction in the interior.

However, we can do this in two ways, either by integrating from the western wall or *to* the eastern wall (the reason why these produce different results will become clear). Let's do the latter case first. Then:

$$\int_x^L \frac{\partial}{\partial x}\psi_I\, dx = \psi_I(L,y) - \psi_I(x,y) = -\frac{1}{\beta\rho_0 D}\, sin(\frac{\pi y}{L})(L-x) \quad (171)$$

To evaluate this, we need to know the value of the streamfunction on the eastern wall, $\psi_I(L,y)$.

Now ψ_I must be a constant. If it weren't, there would be flow into the wall, because:

$$u(L,y) = -\frac{\partial}{\partial y}\psi_I(L,y) \quad (172)$$

If psi_I were constant, there would be flow into the wall. But what is the constant? We can simply take this to be zero, because using any other constant would not change the velocity field. So we have:

$$\psi_I(x,y) = \frac{1}{\beta\rho_0 D}\, sin(\frac{\pi y}{L})(L-x) \tag{173}$$

Notice though that this solution has flow *into* the western wall, because:

$$u_I(0,y) = -\frac{\partial}{\partial y}\psi_I(0,y) = -\frac{\pi}{\beta\rho_0 D} cos(\frac{\pi y}{L}) \neq 0 \tag{174}$$

This can't occur.

To fix the flow at the western wall, we use the second component of the flow, ψ_B. Let's go back to the vorticity equation, with the interior and boundary streamfunctions substituted in:

$$\beta\frac{\partial}{\partial x}\psi_I + \beta\frac{\partial}{\partial x}\psi_B = \frac{1}{\rho_0 D}\nabla\times\vec{\tau}_w - r\nabla^2\psi_B \tag{175}$$

We have ignored the term $r\nabla^2\psi_I$; specifically, we assume this term is much smaller than $r\nabla^2\psi_B$. The reason is that ψ_B has rapid variations near the wall, so the second derivative will be much larger than that of ψ_I, which has a large scale structure. Using (170), the vorticity equation reduces to:

$$\beta\frac{\partial}{\partial x}\psi_B = -r\nabla^2\psi_B \tag{176}$$

ψ_B is assumed to be vanishingly small in the interior. But it will not be small in a boundary layer. We expect that boundary layer to occur in a narrow region near the western wall, because ψ_B must cancel the zonal interior flow at the wall.

This boundary layer will be narrow in the x-direction. The changes in y on the other hand should be more gradual, as we expect the boundary

layer to cover the entire west wall. Thus the derivatives in x will be much greater than in y. So we have:

$$\beta \frac{\partial}{\partial x}\psi_B = -r\nabla^2\psi_B \approx -r\frac{\partial^2}{\partial x^2}\psi_B \tag{177}$$

This has a general solution:

$$\psi_B = Aexp(-\frac{\beta x}{r}) + B$$

In order for the boundary correction to vanish in the interior, the constant B must be zero. We then determine A by making the zonal flow vanish at the west wall (at $x = 0$). This again implies that the streamfunction is constant. That constant must be zero, because we took it to be zero on the east wall. If it were a different constant, then ψ would have to change along the northern and southern walls, meaning $v = \frac{\partial}{\partial x}\psi$ would be non-zero. Thus we demand:

$$\psi_I(0, y) + \psi_B(0, y) = 0 \tag{178}$$

Thus:

$$A = -\frac{L}{\beta\rho_0 D}\, sin(\frac{\pi y}{L}) \tag{179}$$

So the total solution is:

$$\psi = \frac{1}{\beta\rho_0 D}\, sin(\frac{\pi y}{L})\ [L - x - Lexp(-\frac{\beta x}{r})] \tag{180}$$

We examine the character of this solution below. But first let's see what would have happened if we integrated the Sverdrup relation (170) from the *western* wall instead of to the eastern. Then we would get:

$$\beta \int_0^x \frac{\partial}{\partial x}\psi \, dx = \beta\psi(x,y) - \beta\psi(0,y) = -x \, sin(\frac{\pi y}{L}) \tag{181}$$

Setting $\psi(0,y) = 0$, we get:

$$\psi(x,y) = -\frac{x}{\beta\rho_0 D} \, sin(\frac{\pi y}{L}) \tag{182}$$

This solution has flow into the eastern wall, implying we must have a boundary layer there. Again the boundary layer should have more rapid variation in x than in y, so the appropriate boundary layer equation is (177), with a solution:

$$\psi_B = Aexp(-\frac{\beta x}{r}) + B$$

We take B to be zero again, so the solution vanishes in the interior.

But does it? To satisfy the zero flow condition at $x = L$, we have:

$$\psi_I(L,y) + \psi_B(L,y) = 0 \tag{183}$$

or:

$$-\frac{L}{\beta\rho_0 D} \, sin(\frac{\pi y}{L}) + Aexp(-\frac{\beta L}{r}) = 0 \tag{184}$$

Solving for A, we get:

$$A = \frac{L}{\beta\rho_0 D} \, exp(\frac{\beta L}{r}) \, sin(\frac{\pi y}{L}) \tag{185}$$

So the total solution in this case is:

$$\psi = \frac{1}{\beta\rho_0 D} \, sin(\frac{\pi y}{L}) \, [-x + Lexp(\frac{\beta(L-x)}{r})] \tag{186}$$

Now there is a problem. The exponential term in this case does not decrease moving away from the eastern wall. Rather, it grows exponentially. So the boundary layer solution *isn't confined* to the eastern wall! Thus we reject the possibility of an eastern boundary layer. The boundary layer must lie on the western wall. This is why, Stommel concluded, the Gulf Stream lies on the western boundary of the North Atlantic.

Another explanation for the western intensification was proposed by Pedlosky (1965). Recall that Rossby waves propagate to the west as long waves, and reflect off the western wall as short waves. The short waves move more slowly, with the result that the energy is intensified in the region near the west wall (sec. 2.4). Pedlosky showed that in the limit of low frequencies (long period waves), the Rossby wave solution converges to the Stommel solution. So western intensification occurs because Rossby waves propagate to the west.

Let's look at the (correct) Stommel solution. Shown in figure (20) is the Sverdrup solution (upper panel) and two full solutions with different r (lower panels). The Sverdrup solution has southward flow over the whole basin. So the mean flow crosses the geostrophic contours, as suggested earlier. There is, in addition, an eastward drift in the north and a westward drift in the south.

With the larger friction coefficient, the Stommel solution has a broad, northward-flowing western boundary current. With the friction coefficient 10 times smaller, the boundary current is ten times narrower and the northward flow is roughly ten times stronger. This is the Stommel analogue of the Gulf Stream.

Consider what is happening to a fluid parcel in this solution. The parcel's potential vorticity decreases in the interior, due to the negative wind

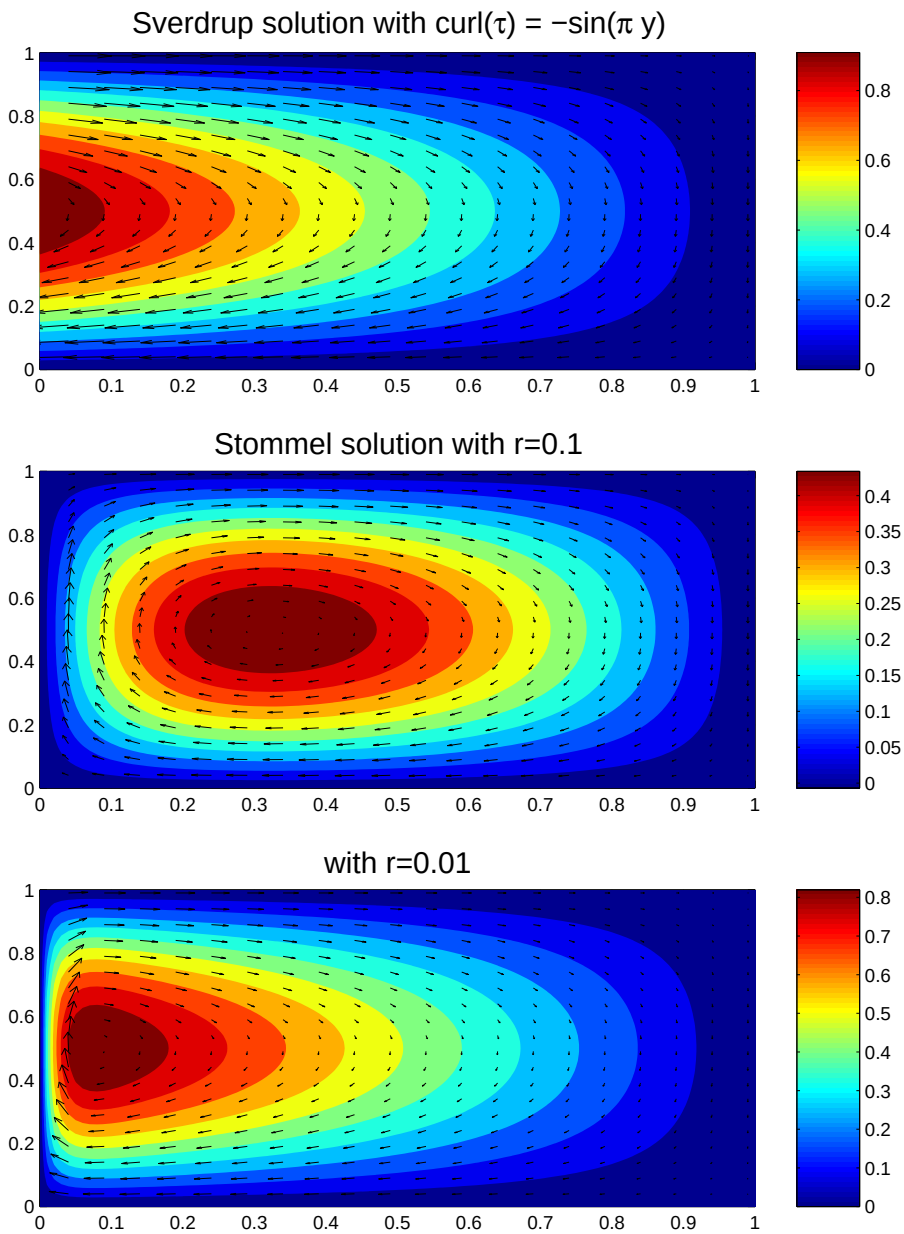

Figure 20: Solutions of Stommel's model for two different values of the friction coefficient, r.

stress curl, which causes the parcel to drift southward. We know the parcel needs to return to the north to complete its circuit, but to do that it must somehow acquire vorticity. Bottom friction permits the parcel to acquire vorticity in the western layer. You can show that if the parcel were in an eastern boundary layer, it's vorticity would *decrease* going northward. So the parcel would not be able to re-enter the northern interior.

The Stommel boundary layer is like the bottom Ekman layer (sec. 1.7), in several ways. In the Ekman layer, friction, which acts only in a boundary layer, brings the velocity to zero to satisfy the no-slip condition. This yields a strong vertical shear in the velocities. In the Stommel layer, friction acts to satisfy the no-normal flow condition and causes strong *lateral* shear. Both types of boundary layer also are passive, in that they do not force the interior motion; they simply modify the behavior near the boundaries.

2.8 Closed ocean basins

Next we consider an example with bottom topography. As discussed in sec. (2.2), topography can cause the geostrophic contours to close on themselves. This is an entirely different situation because mean flows can exist on the closed contours (they do not encounter boundaries; Fig. 8). Such mean flows can be excited by wind-forcing and can be very strong.

There are several regions with closed geostrophic contours in the Nordic Seas (Fig. 21), specifically in three basins: the Norwegian, Lofoten and Greenland gyres. The topography is thus steep enough here as to overwhelm the β-effect. Isachsen et al. (2003) examined how wind-forcing could excite flow in these gyres.

This time we take equation (100) with wind forcing and bottom topog-

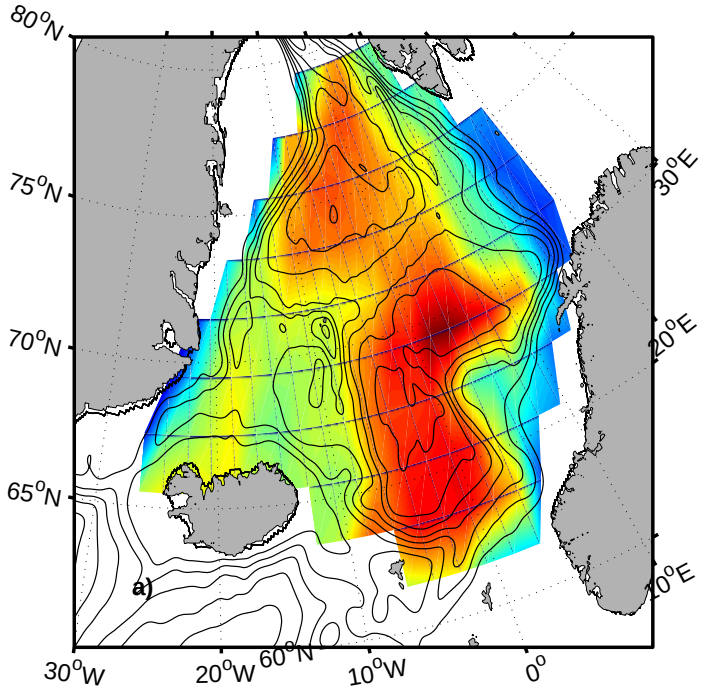

Figure 21: Geostrophic contours (solid lines) in the Nordic seas. Superimposed are contours showing the first EOF of sea surface height derived from satellite measurements. The latter shows strong variability localized in regions of closed q_s contours. From Isachsen et al. (2003).

raphy:

$$\frac{d_g}{dt}(\zeta + \beta y + \frac{f_0}{D}h) = \frac{1}{\rho_0 D}\nabla \times \vec{\tau} - r\zeta \tag{187}$$

We will linearize the equation, without a mean flow. We can write the result this way:

$$\frac{\partial}{\partial t}\zeta + \vec{u} \cdot \nabla q_s = \frac{1}{\rho_0 D}\nabla \times \vec{\tau} - r\zeta \tag{188}$$

where

$$q_s \equiv \beta y + \frac{f_0}{D}h$$

defines the geostrophic contours (sec.2.2). Recall that these are the so-

called “f/H” contours in the shallow water system. As noted, the q_s contours can close on themselves if the topography is strong enough to overwhelm the βy contribution to q_s (Fig. 8). This is the case in the Nordic Seas (Fig. 21).

As in the Gulf Stream model, we will assume the bottom friction coefficient, r, is small. In addition, we will assume that the wind forcing and the time derivative terms are as small as the bottom friction term (of order r). Thus the first, third and fourth terms in equation (188) are of comparable size. We can indicate this by writing the equation this way:

$$r\frac{\partial}{\partial t'}\zeta + \vec{u}\cdot\nabla q_s = r\frac{1}{\rho_0 D}\nabla\times\vec{\tau}' - r\zeta \tag{189}$$

where $t' = rt$ and $\tau' = \tau/r$ are the small variables normalized by r, so that they are order one.

Now we use a *perturbation expansion* and expand the variables in r. For example, the vorticity is:

$$\zeta = \zeta_0 + r\zeta_1 + r^2\zeta_2 + ...$$

Likewise, the velocity is:

$$\vec{u} = \vec{u}_0 + r\vec{u}_1 + r^2\vec{u}_2 + ...$$

We plug this into the vorticity equation and then collect terms which are multiplied by the same factor of r. The largest terms are those multiplied by one. These are just:

$$\vec{u}_0\cdot\nabla q_s = 0 \tag{190}$$

So the first order component *follows the q_s contours*. In other words, the

first order streamfunction is everywhere parallel to the q_s contours. Once we plot the q_s contours, we know what the flow looks like.

But this only tells us the *direction* of $\vec{u}_o$, not its strength or structure (how it varies from contour to contour). To find that out, we go to the next order in r:

$$\frac{\partial}{\partial t'}\zeta_0 + \vec{u}_1 \cdot \nabla q_s = \frac{1}{\rho_0 D}\nabla \times \vec{\tau}' - \zeta_0 \tag{191}$$

This equation tells us how the zeroth order field changes in time. However, there is a problem. In order to solve for the zeroth order field, we need to know the first order field because of the term with u_1. But it is possible to eliminate this, as follows. First, we can rewrite the advective term thus:

$$\vec{u}_1 \cdot \nabla q_s = \nabla \cdot (\vec{u}_1 q_s) - q_s(\nabla \cdot \vec{u}_1) \tag{192}$$

The second term on the RHS vanishes by incompressibility. In particular:

$$\nabla \cdot \vec{u} = 0 \tag{193}$$

This implies that the velocity is incompressible at each order. So the vorticity equation becomes:

$$\frac{\partial}{\partial t'}\zeta_0 + \nabla \cdot (\vec{u}_1 q_s) = \frac{1}{\rho_0 D}\nabla \times \vec{\tau}' - r\zeta_0 \tag{194}$$

Now, we can eliminate the second term if we integrate the equation over an area bounded by a closed q_s contour. This follows from Gauss's Law, which states:

$$\iint \nabla \cdot \vec{A}\, dx\, dy = \oint \vec{A} \cdot \hat{n}\, dl \tag{195}$$

Thus:

$$\iint \nabla \cdot (\vec{u} q_s)\, dA = \oint q_s \vec{u} \cdot \hat{n}\, dl = q_s \oint \vec{u} \cdot \hat{n}\, dl = 0 \tag{196}$$

We can take the q_s outside the line integral because q_s is constant on the bounding contour. The closed integral of $\vec{u} \cdot \hat{n}$ vanishes because of incompressibility:

$$\oint \vec{u} \cdot \hat{n}\, dl = \iint \nabla \cdot \vec{u}\, dA = 0$$

Thus the integral of (197) in a region bounded by a q_s contour is:

$$\frac{\partial}{\partial t'} \iint \zeta_0\, dxdy = \frac{1}{\rho_0 D} \iint \nabla \times \vec{\tau}'\, dxdy - \iint \zeta_0\, dxdy \tag{197}$$

Notice this contains only zeroth order terms. We can rewrite (197) by exploiting Stoke's Law, which states:

$$\iint \nabla \times \vec{A}\, dx\, dy = \oint \vec{A} \cdot\, \vec{dl} \tag{198}$$

So (197) can be rewritten:

$$\frac{\partial}{\partial t'} \oint \vec{u} \cdot \vec{dl} = \frac{1}{\rho_0 D} \oint \vec{\tau}' \cdot \vec{dl} - \oint \vec{u} \cdot \vec{dl} \tag{199}$$

We have dropped the zero subscripts, since this is the only component we will consider. In terms of the real time and wind stress, this is:

$$\frac{\partial}{\partial t} \oint \vec{u} \cdot \vec{dl} = \frac{1}{\rho_0 D} \oint \vec{\tau} \cdot \vec{dl} - r \oint \vec{u} \cdot \vec{dl} \tag{200}$$

Isachsen et al. (2003) solved (200) by decomposing the velocity into Fourier components in time:

$$\vec{u}(x, y, t) = \sum \tilde{u}(x, y, \omega)\, e^{i\omega t}$$

Then it is easy to solve (200) for the velocity integrated around the contour:

$$\oint \vec{u} \cdot \vec{dl} = \frac{1}{r + i\omega} \frac{1}{\rho_0 D} \oint \vec{\tau} \cdot \vec{dl} \tag{201}$$

Note the solution is actually for the integral of the velocity around the contour (rather than the velocity at every point). We can divide by the length of the contour to get the average velocity on the contour:

$$< u > \equiv \frac{\oint \vec{u} \cdot \vec{dl}}{\oint dl} = \frac{1}{r + i\omega} \frac{1}{\rho_0 D} \frac{\oint \vec{\tau} \cdot \vec{dl}}{\oint dl} \tag{202}$$

Isachsen et al. (2003) derived a similar relation using the shallow water equations. Their expression is somewhat more complicated but has the same meaning. They tested this prediction using various types of data from the Nordic Seas. One example is shown in figure (21). This shows the principal Empirical Orthogonal Function (EOF) of the sea surface height variability measured from satellite. The EOF shows that there are regions with spatially coherent upward and downward sea surface motion. These regions are exactly where the q_s contours are closed. This height variability reflects strong gyres which are aligned with the q_s contours.

Isachsen et al. took wind data, the actual bottom topography and an approximate value of the bottom drag to predict the transport in the three gyres (corresponding to the Norwegian, Lofoten and Greenland basins). The results are shown in figure (22). The simple model does astonishingly well, predicting the intensification and weakening of the gyres in all three basins.

2.9 Barotropic instability

Many of the "mean" flows in the atmosphere and ocean, like the Jet and Gulf Streams, are not steady at all. Instead, they meander and generate

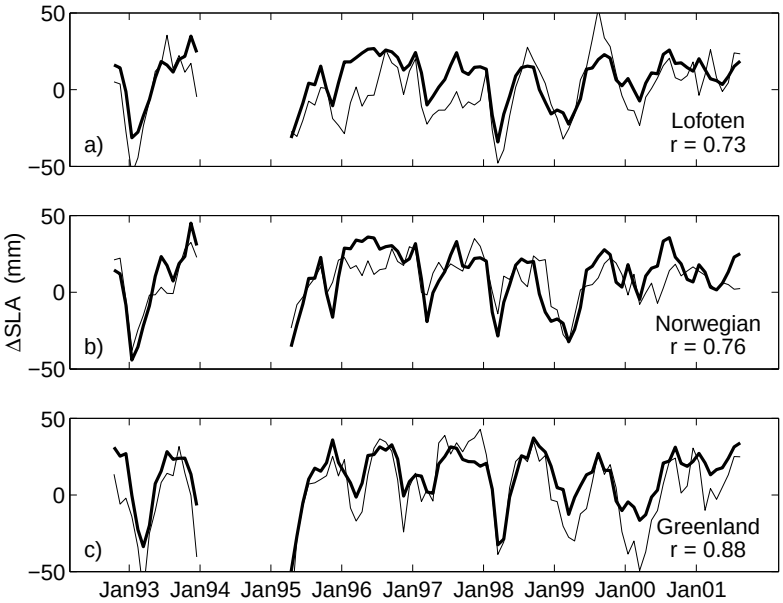

Figure 22: Time series of observed (thin line) and predicted (thick line) sea surface height displacements between the outer rim and the center of each of the principal gyres in the Nordic seas. The linear bottom drag coefficient was $R = 5 \times 10^{-4}$ m/sec. From Isachsen et al. (2003).

eddies (storms). The reason is that these flows are *unstable*. That means that if the flow is perturbed slightly, for instance by a slight change in heating or wind forcing, that perturbation will grow, extracting energy from the mean flow. These perturbations then develop into fully formed storms, both in the atmosphere and ocean.

We will first study instability in the barotropic context. We will ignore forcing and dissipation, and focus exclusively on the interaction between the mean flow and the perturbations. A constant mean flow, like we used when deriving the dispersion relation for free Rossby waves, is stable. But a mean flow which is *sheared* can be unstable. To illustrate this, we will examine a mean flow which varies in y. We will see that wave solutions exist in this case too, but that they can grow in time.

The barotropic vorticity equation with a flat bottom and no forcing or bottom drag is:

$$\frac{d_g}{dt}(\zeta + \beta y) = 0 \tag{203}$$

We again linearize the equation assuming a zonal flow, but now this can vary in y, i.e. $U = U(y)$. As a result, the mean flow has an associated vorticity:

$$\overline{\zeta} = -\frac{\partial}{\partial y}U \tag{204}$$

Like the mean flow, this is also time independent. So the PV equation is now:

$$\frac{d_g}{dt}(\zeta' - \frac{\partial}{\partial y}U + \beta y) = 0 \tag{205}$$

Thus the mean vorticity *alters the geostrophic contours*. In particular, we have:

$$q_s = \beta y - \frac{\partial}{\partial y}U \tag{206}$$

This suggests that the mean flow will affect the way Rossby waves propagate in the system.

The linearized version of the vorticity equation is:

$$(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x})\zeta' + v'\frac{\partial}{\partial y}q_s = 0 \tag{207}$$

Written in terms of the streamfunction, this is:

$$(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x})\nabla^2\psi + (\frac{\partial}{\partial y}q_s)\frac{\partial \psi}{\partial x} = 0 \tag{208}$$

Because the mean flow varies in y, we have to be careful about our choice of wave solutions. We can however assume a sinusoidal dependence in x and t. The form we will use is:

$$\psi = Re\{\hat{\psi}(y)\, e^{ik(x-ct)}\} \tag{209}$$

As we know, the amplitude can be complex, i.e.:

$$\hat{\psi} = \hat{\psi}_r + i\hat{\psi}_i$$

But the phase speed, c, *also* can be complex:

$$c = c_r + ic_i \tag{210}$$

As such, we have that:

$$e^{ik(x-ct)} = e^{ik(x-(c_r+ic_i)\,t)} = e^{ik(x-c_r t)+kc_i t} \tag{211}$$

The argument of the exponential has both real and imaginary parts. The real part determines how the phases change, as before. But the imaginary part can change the amplitude of the wave. In particular, if $c_i > 0$, the wave amplitude will *grow exponentially in time*. If this happens, we say the flow is *barotropically unstable*, because the wave solution grows in time, eventually becoming as strong as the background flow itself.

If we substitute the wave solution into (208), we get:

$$(-ikc + ikU)(-k^2\hat{\psi} + \frac{\partial^2}{\partial y^2}\hat{\psi}) + ik\hat{\psi}\frac{\partial}{\partial y}q_s = 0 \tag{212}$$

Canceling the ik yields:

$$(U - c)\,(\frac{\partial^2}{\partial y^2}\hat{\psi} - k^2\hat{\psi}) + \hat{\psi}\frac{\partial}{\partial y}q_s = 0 \tag{213}$$

This is known as the "Rayleigh equation". The solution of this determines which waves are unstable. However, because U and q_s are functions of y, this is generally not easy to solve.

2.9.1 Rayleigh-Kuo criterion

Remarkably though, it is possible to derive stability conditions for flows without actually finding a solution! We do this as follows. First we divide (213) by $U - c$:

$$(\frac{\partial^2}{\partial y^2}\hat{\psi} - k^2\hat{\psi}) + \frac{\hat{\psi}}{U-c}\frac{\partial}{\partial y}q_s = 0 \tag{214}$$

Then we multiply by the complex conjugate of the streamfunction:

$$\hat{\psi}^* = \hat{\psi}_r - i\hat{\psi}_i$$

This yields:

$$(\hat{\psi}_r\frac{\partial^2}{\partial y^2}\hat{\psi}_r + \hat{\psi}_i\frac{\partial^2}{\partial y^2}\hat{\psi}_i) + i(\hat{\psi}_r\frac{\partial^2}{\partial y^2}\hat{\psi}_i - \hat{\psi}_i\frac{\partial^2}{\partial y^2}\hat{\psi}_r) - k^2|\hat{\psi}|^2$$

$$+\frac{|\hat{\psi}|^2}{U-c}\frac{\partial}{\partial y}q_s = 0 \tag{215}$$

The denominator in the last term is complex. We write it in a more convenient form this way:

$$\frac{1}{U-c} = \frac{1}{U-c_r-ic_i} = \frac{U-c_r+ic_i}{|U-c|^2}$$

Now the denominator is purely real. So we have:

$$(\hat{\psi}_r\frac{\partial^2}{\partial y^2}\hat{\psi}_r + \hat{\psi}_i\frac{\partial^2}{\partial y^2}\hat{\psi}_i) + i(\hat{\psi}_r\frac{\partial^2}{\partial y^2}\hat{\psi}_i - \hat{\psi}_i\frac{\partial^2}{\partial y^2}\hat{\psi}_r) - k^2|\hat{\psi}|^2$$

$$+(U-c_r+ic_i)\frac{|\hat{\psi}|^2}{|U-c|^2}\frac{\partial}{\partial y}q_s = 0 \tag{216}$$

This equation has both real and imaginary parts, and each must separately equal zero. This gives us two equations. Consider the imaginary part of (216):

$$(\hat{\psi}_r \frac{\partial^2}{\partial y^2}\hat{\psi}_i - \hat{\psi}_i \frac{\partial^2}{\partial y^2}\hat{\psi}_r) + c_i \frac{|\hat{\psi}|^2}{|U-c|^2}\frac{\partial}{\partial y} q_s = 0 \tag{217}$$

Let's integrate this in y, over a region from $y = [0, L]$:

$$\int_0^L (\hat{\psi}_i \frac{\partial^2}{\partial y^2}\hat{\psi}_r - \hat{\psi}_r \frac{\partial^2}{\partial y^2}\hat{\psi}_i)\, dy = c_i \int_0^L \frac{|\hat{\psi}|^2}{|U-c|^2}\frac{\partial}{\partial y} q_s \, dy \tag{218}$$

We can rewrite the first terms by noting:

$$\begin{aligned}\hat{\psi}_i \frac{\partial^2}{\partial y^2}\hat{\psi}_r - \hat{\psi}_r \frac{\partial^2}{\partial y^2}\hat{\psi}_i &= \frac{\partial}{\partial y}(\hat{\psi}_i \frac{\partial}{\partial y}\hat{\psi}_r - \hat{\psi}_r \frac{\partial}{\partial y}\hat{\psi}_i) - \frac{\partial}{\partial y}\hat{\psi}_i \frac{\partial}{\partial y}\hat{\psi}_r + \frac{\partial}{\partial y}\hat{\psi}_r \frac{\partial}{\partial y}\hat{\psi}_i \\ &= \frac{\partial}{\partial y}(\hat{\psi}_i \frac{\partial}{\partial y}\hat{\psi}_r - \hat{\psi}_r \frac{\partial}{\partial y}\hat{\psi}_i) \end{aligned} \tag{219}$$

Substituting this into the LHS of (218), we get:

$$\int_0^L \frac{\partial}{\partial y}(\hat{\psi}_i \frac{\partial}{\partial y}\hat{\psi}_r - \hat{\psi}_r \frac{\partial}{\partial y}\hat{\psi}_i)\, dy = (\hat{\psi}_i \frac{\partial}{\partial y}\hat{\psi}_r - \hat{\psi}_r \frac{\partial}{\partial y}\hat{\psi}_i)\,|_0^L \tag{220}$$

Now to evaluate this, we need the boundary conditions on ψ.

Let's imagine the flow is confined to a channel. Then the normal flow vanishes at the northern and southern walls. This implies that the streamfunction is constant on those walls, and we take the constant to be zero. Thus:

$$\psi(y=0) = \psi(y=L) = 0$$

Then (220) vanishes. We obtain the same result if we simply pick $y = 0$ and $y = L$ to be latitudes where the perturbation vanishes (i.e. far away

from the mean flow). Either way, the equation for the imaginary part reduces to:

$$c_i \int_0^L \frac{|\hat{\psi}|^2}{|U-c|^2} \frac{\partial}{\partial y} q_s \, dy = 0 \tag{221}$$

In order for this to be true, either c_i or the integral must be zero. If $c_i = 0$, the wave amplitude is not growing and the wave is stable. For unstable waves, $c_i > 0$, meaning the integral must vanish. The squared terms in the integrand are always greater than zero, so a necessary condition for instability is that:

$$\frac{\partial}{\partial y} q_s = 0 \tag{222}$$

This implies *the meridional gradient of the background PV must change sign somewhere in the domain*. This is the *Rayleigh-Kuo criterion*. Under the β-plane approximation, we have:

$$\frac{\partial}{\partial y} q_s \equiv \beta - \frac{\partial^2}{\partial y^2} U \tag{223}$$

Thus instability requires $\beta = \frac{\partial^2}{\partial y^2} U$ somewhere in the domain.

Think about what this means. If $U = 0$, then $q_s = \beta y$. Then we have Rossby waves, all of which propagate westward. With a background flow, the waves need not propagate westward. If $\beta - \frac{\partial^2}{\partial y^2} U = 0$ somewhere, the mean PV gradient vanishes and the Rossby waves are *stationary*. So the wave holds its position in the mean flow, extracting energy from it. In this way, the wave grows in time.

The Rayleigh-Kuo criterion is a *necessary condition* for instability. That means that instability requires that this condition be met. But it is not a *suf-*

ficient condition—it doesn't guarantee that a jet will be unstable. However, the opposite case is a sufficient condition; if the gradient does *not* change sign, the jet must be stable.

As noted, the Rayleigh-Kuo condition is useful because we don't actually need to solve for the unstable waves to see if the jet is unstable. Such a solution is often very involved.

We can derive another stability criterion, following Fjørtoft (1950), by taking the real part of (216). The result is similar to the Rayleigh-Kuo criterion, but a little more specific. Some flows which are unstable by the Rayleigh criterion may be stable by Fjørtoft's. However this is fairly rare. Details are given in Appendix D.

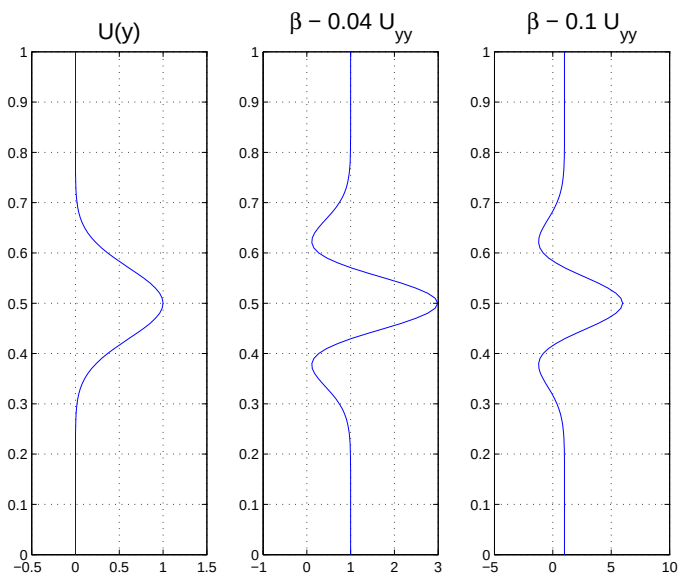

Figure 23: A westerly Gaussian jet (left panel). The middle and right panels show $\beta - \frac{\partial^2}{\partial y^2} u$ for the jet with amplitudes of 0.04 and 0.1, respectively. Only the latter satisfies Rayleigh's criterion for instability.

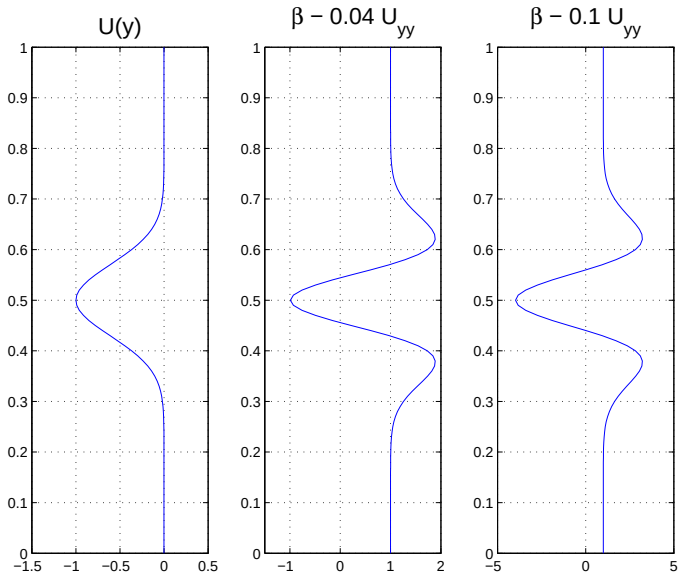

Figure 24: An easterly Gaussian jet (left panel). The middle and right panels show $\beta - \frac{\partial^2}{\partial y^2} u$ for the jet, with amplitudes of 0.04 and 0.1. Note that both satisfy Rayleigh's criterion for instability.

2.9.2 Examples

Let's consider some examples of barotropically unstable flows. Consider a westerly jet with a Gaussian profile (Kuo, 1949):

$$U = U_0 \, exp[-(\frac{y - y_0}{L})^2] \tag{224}$$

Shown in the two right panels of Fig. (23) is $\beta - \frac{\partial^2}{\partial y^2} U$ for two jet amplitudes, U_0. We take $\beta = L = 1$, for simplicity. With $U_0 = 0.04$, the PV gradient is positive everywhere, so the jet is stable. With $U_0 = 0.1$, the PV gradient changes sign both to the north and south of the jet maximum. So this jet *may* be unstable.

Now consider an *easterly* jet (Fig. 24), with $U_0 < 0$. With both amplitudes, $\beta - \frac{\partial^2}{\partial y^2} U$ is negative at the centers of the jets. So the jet is unstable with both amplitudes. This is a general result: easterly jets are more un-

stable than westerly jets.

An example of an evolving barotropic instability is shown in Fig. (25). This derives from a numerical simulation of a jet with a Gaussian profile of relative vorticity. So:

$$\zeta = -\frac{\partial}{\partial y} U = A e^{-y^2/L^2} \tag{225}$$

In this simulation, $\beta = 0$, so the PV gradient is:

$$\frac{\partial}{\partial y} q_s = -\frac{\partial^2}{\partial y^2} U = -\frac{2y}{L^2} A e^{-y^2/L^2} \tag{226}$$

This is zero at $y = 0$ and so satisfies Rayleigh's criterion. We see in the simulation that the jet is unstable, wrapping up into vortices. These have positive vorticity, like the jet itself.

An example of barotropic instability in the atmosphere is seen in Fig. (26). This shows three infrared satellite images of water vapor above the US. Note in particular the dark band which stretches over the western US in into Canada. This is a filament of air, near the tropopause. We see that the filament is rolling up into vortices, much like in the numerical simulation in (25).

Barotropic instability also occurs in the ocean. Consider the following example, from the southern Indian and Atlantic Oceans (Figs. 27-29). Shown in (27) is a Stommel-like solution for the region. Africa is represented by a barrier attached to the northern wall, and the island to its east represents Madagascar. The wind stress curl is indicated in the right panel; this is negative in the north, positive in the middle and negative in the south.

In the southern part of the domain, the flow is eastward. This repre-

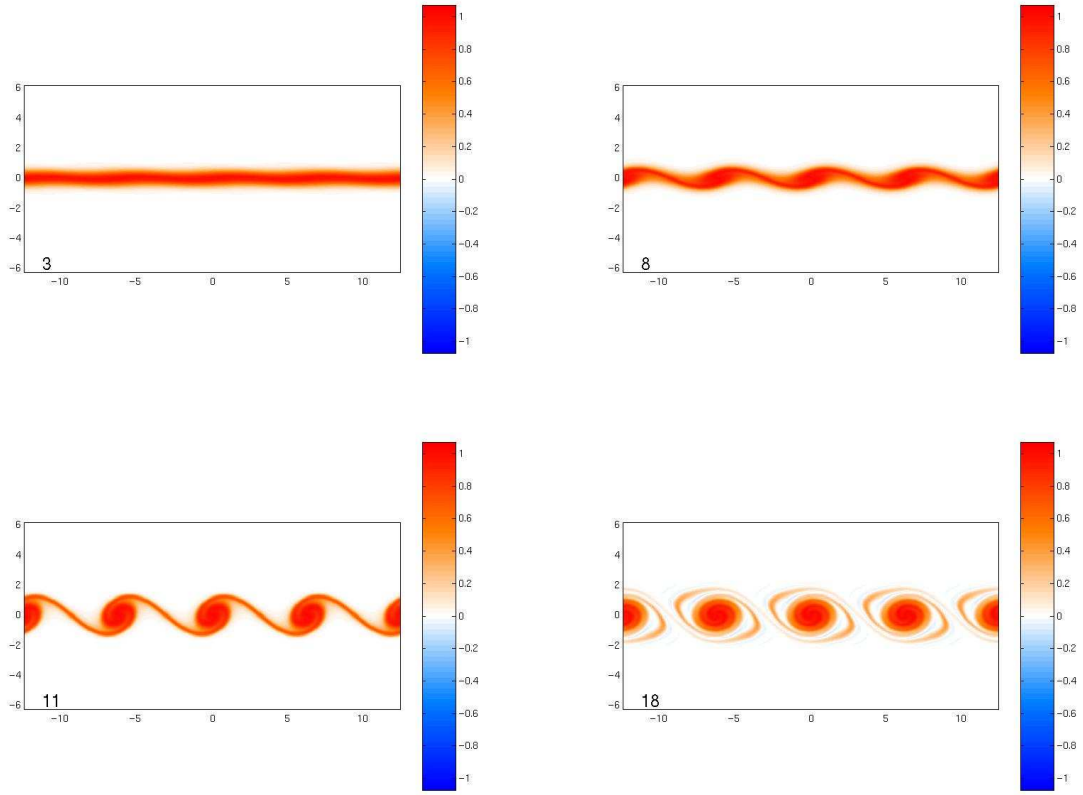

Figure 25: Barotropic instability of a jet with a Gaussian profile in relative vorticity. Courtesy of G. Hakim, Univ. of Washington.

sents the Antarctic Circumpolar Current (the largest ocean current in the world). In the "Indian ocean", the flow is to the west, towards Madagascar. This corresponds to the South Equatorial Current, which impinges on Madagascar. There are western boundary currents to the east of Africa and Madagascar. The boundary currents east of Madagascar flow westward toward Africa in two jets, to the north and south of the Island. Similarly, the western boundary current leaves South Africa to flow west and join the flow in the South Atlantic.

Shown in Fig. (28) is the PV gradient for this solution, in the region near South Africa and Madagascar. Clearly the gradient is dominated by the separated jets. Moreover, the gradient changes sign several times in

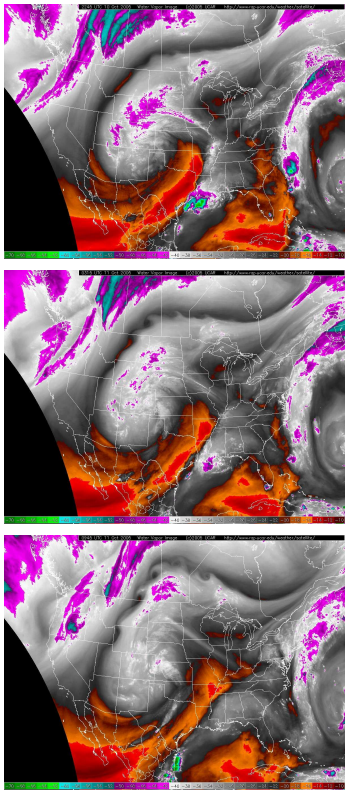

Figure 26: Barotropic instability of filaments on the tropopause, observed from water vapor infrared satellite imagery. The images were taken on the 11th of October, 2005, at 22:45 pm, 3:15 am and 9:45 am, respectively. Courtesy of G. Hakim, Univ. of Washington.

each of the jets. So we would expect the jets might be unstable, by the Rayleigh-Kuo criterion.

A snapshot from a numerical solution of the barotropic flow is shown in Fig. (29). In this simulation, the mean observed winds were used to drive the ocean, which was allowed to spin-up to a statistically steady state. The figure shows a snapshot of the sea surface height, after the model has spun up. We see that all three of the eastward jets have become unstable and are generating eddies (of both signs). The eddies drift westward, linking up with the boundary currents to their west.

Barotropic instability occurs when the lateral shear in a current is too large. The unstable waves extract energy from the mean flow, reducing the

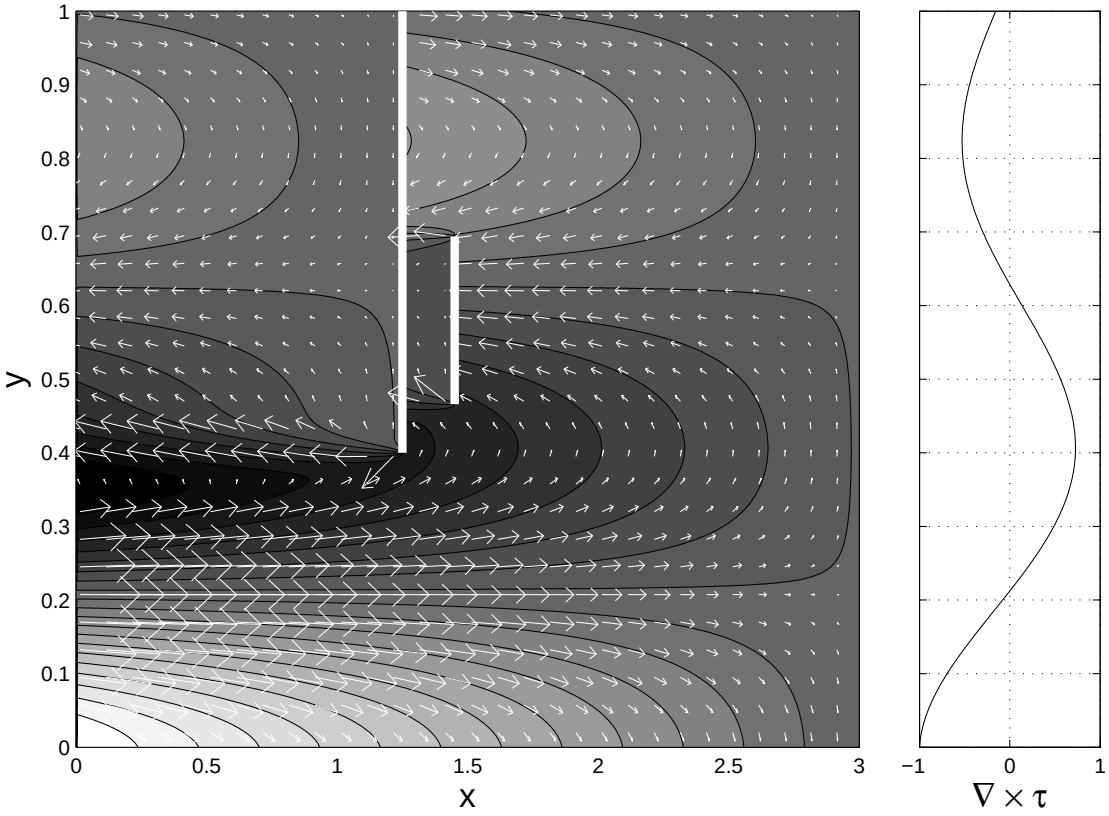

Figure 27: A Stommel-like solution for the Indian Ocean. The curl of the wind stress is indicated in the right panel. From LaCasce and Isachsen (2007).

shear by mixing momentum laterally. However, in the atmosphere *baroclinic instability* is more important, in terms of storm formation. Under baroclinic instability, the waves act to reduce the *vertical shear* of the mean flow. In order to study that, we have to take account of density changes.

2.10 ***Problems***

Problem 2.1: Topographic waves

Bottom topography, like the β-effect, can support Rossby-like waves, called *topographic waves*. To see this, use the barotropic PV equation (151), with no forcing or friction and β=0 (a constant Coriolis parameter). Assume the bottom slopes uniformly to the east:

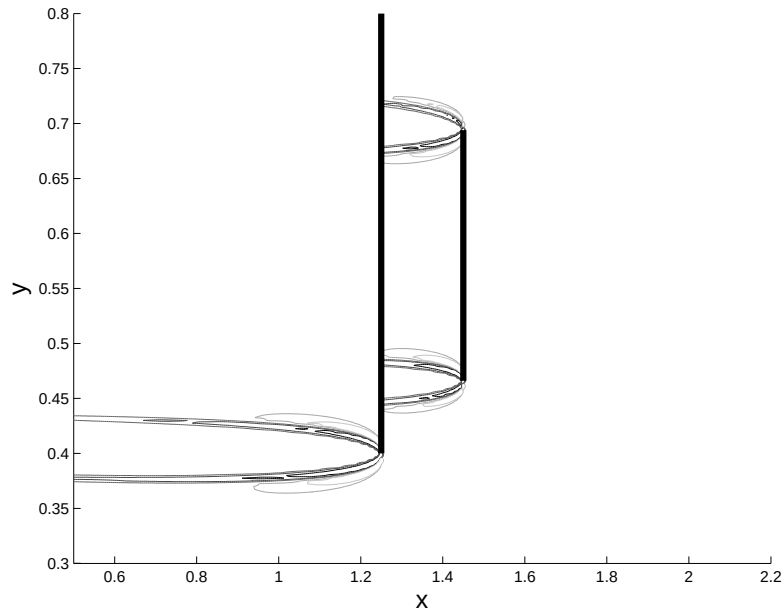

Figure 28: The PV gradient for the solution in Fig. (27). The gradient changes sign rapidly in the three jet regions. From LaCasce and Isachsen (2007).

$$H = H_0 - \alpha x \tag{227}$$

Derive the phase speed for the waves, assuming no background flow ($U = V = 0$). Which way do the waves propagate, relative to the shallower water? What if $\alpha < 0$? What about in the southern hemisphere?

Problem 2.2: Basin waves

We solved the Rossby wave problem on an infinite plane. Now consider what happens if there are solid walls. Take the one-dimensional version of the vorticity equation (set all derivatives in the y-direction to zero) with $U = 0$. Let $\psi = 0$ at $x = 0$ and $x = L$; this ensures that there is no flow into the walls. What are the solutions for ω and k?

Hint 1: Assume $\psi = A(x)cos(kx - \omega t)$, and impose the boundary

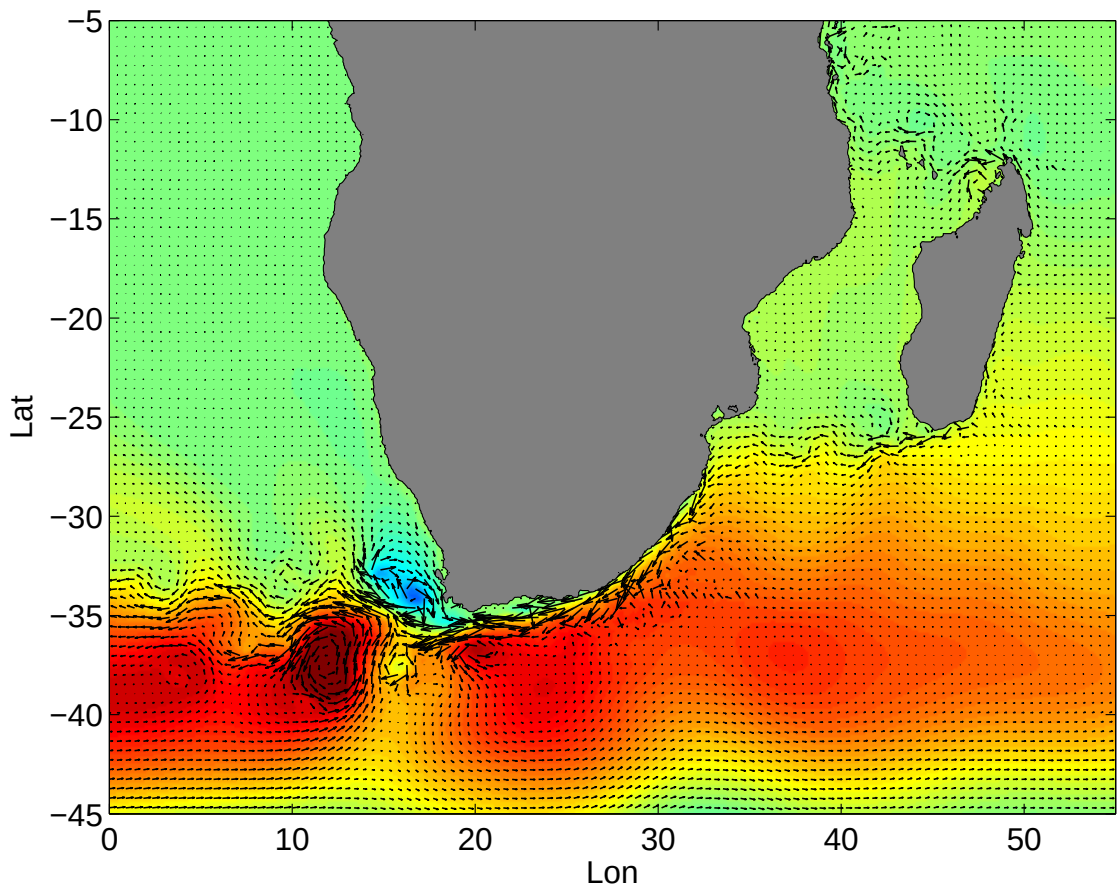

Figure 29: The sea surface height from a barotropic numerical simulation of the southern Indian and Atlantic Oceans. From LaCasce and Isachsen (2007).

conditions on A.

Hint 2: The coefficients of the sine and cosine terms should both be zero.

Hint 3: The solutions are *quantized* (have discrete values).

<u>*Problem 2.3*</u>: A large domain

What if the range of latitude is so large so that $\beta L \approx f_0$? What can you infer about the size of the vertical velocity? What does that imply about the Lagrangian derivative? You can use the Boussinesq approximation to

keep things simple.

Integrate the expression that you now have for $\frac{\partial}{\partial z}w$, assuming the fluid is barotropic. Take the bottom to be flat, and ignore the bottom Ekman layer. Say there is a surface Ekman layer. What does the equation tell you if there is a negative wind stress curl?

Problem 2.4: Free surface

Assume the layer is between z_0 and z_1. Assume the bottom is flat and there are no Ekman layers. Now let the upper surface *move*. Use the condition on w on the upper surface, then rewrite the vorticity equation. Assume the *undisturbed* surface is at $z = 0$ and the perturbed surface is at $z = \eta$. What assumption do you need to make about η to be consistent with quasi-geostrophy?

Make that assumption, and determine what the new conserved quantity is following the fluid motion. What happens when the surface is depressed (pushed down) if there is no vorticity initially?

Problem 2.5: Reflection at a northern wall

Consider Rossby waves incident on an east-west wall, located at $y = 0$. Proceed as we did in class, with one incident and one reflected wave. What can you say about the reflected wave?

Hint: there are two possibilities, depending on the sign of l_r.

Problem 2.6: Rossby waves with an isolated mountain range

We considered Rossby waves excited by a purely sinusoidal mountain range—not very realistic. A more typical case is one where the mountain

is localized. Here we will see how to deal with such a case.

Consider a mountain "range" centered at $x = 0$ with:

$$h(x, y) = h_0 \, e^{-x^2/L^2} \tag{228}$$

Because the range doesn't vary in y, we can write $\psi = \psi(x)$.

Write the wave equation, without friction. Transform the streamfunction and the mountain using the Fourier cosine transform. Then solve for the transform of ψ, and write the expression for $\psi(x)$ using the inverse transform (it's not necessary to evaluate the inverse transform).

Where do you expect the largest contribution to the integral to occur (which values of k)?

Problem 2.7: Is there really western intensification?

To convince ourselves of this, we can solve the Stommel problem in 1-D, as follows. Let the wind stress be given by:

$$\vec{\tau} = y\hat{i} \tag{229}$$

Write the vorticity equation following Stommel (linear, U=V=0, steady). Ignore variations in y, leaving a 1-D equation. Assume the domain goes from $x = 0$ to $x = L$, as before. Solve it.

Note that you should have two constants of integration. This will allow you to satisfy the boundary conditions $\psi = 0$ at $x = 0$ and $x = L$. Plot the meridional velocity $v(x)$. Assume that $(\beta\rho_0 D)^{-1} = 1$ and $L(r\rho_0 D)^{-1} = 10$. Where is the jet?

Problem 2.8: Munk's solution

Shortly after Stommel's (1948) paper came another (Munk, 1950) which also modelled the barotropic North Atlantic. The model is similar, except that Munk used lateral friction rather than bottom friction. The lateral friction was meant to represent horizontal stirring by oceanic eddies. The barotropic vorticity equation in this case is:

$$\frac{d_g}{dt}(\zeta + \beta y) = \frac{1}{\rho_0 H_0}\nabla \times \vec{\tau} + \nu \nabla^2 \zeta \tag{230}$$

Solve this problem as we did for Stommel's model. Use:

$$\tau = -\frac{L}{\pi} cos(\frac{\pi y}{L})\hat{i} \tag{231}$$

and invoke the no-slip condition ($v = 0$) at the wall. Demonstrate that the boundary layer also occurs on the western side of the basin.

Hint: The boundary layer correction has a structure like in an Ekman layer.

<u>*Problem 2.9*</u>: Barotropic instability

We have a region with $0 \leq x < 1$ and $-1 \leq y < 1$. Consider the following velocity profiles:

a) $U = 1 - y^2$
b) $U = exp(-y^2)$
c) $U = sin(\pi y)$
d) $U = \frac{1}{6}y^3 + \frac{5}{6}y$

Which profiles are unstable by the Rayleigh-Kuo criterion if $\beta = 0$? How large must β be to stabilize *all* the profiles? Note that the terms here

have been non-dimensionalized, so that β can be any number (e.g. an integer).

3 Baroclinic flows

We will now examine what happens with stratification, i.e. density variations. Now we have the possibility of vertical shear in the velocities; the winds at higher levels don't need to be parallel or of equal strength to those at lower levels. Baroclinic flows are inherently more three dimensional than barotropic ones. Nevertheless, we will see that the same type of solutions arise with baroclinic flows as with barotropic ones. We have baroclinic Rossby waves, and baroclinic instability. These phenomena involve some modifications though, as seen hereafter.

Consider the vorticity equation (75):

$$(\frac{\partial}{\partial t} - \frac{\partial \psi}{\partial y}\frac{\partial}{\partial x} + \frac{\partial \psi}{\partial y}\frac{\partial}{\partial x})(\nabla^2 \psi + f) = f_0 \frac{\partial}{\partial z} w \tag{232}$$

When we derived this, we made no demands about the vertical structure of the flows. Thus this equation works equally well with baroclinic flows as barotropic ones. The equation has two unknowns, ψ and w. For barotropic flows, we eliminate w by integrating over the depth of the fluid. Then the vertical velocity only enters at the upper and lower boundaries.

But with baroclinic flows, the vertical velocity can vary in the interior of the fluid. So it is not so simple to dispose of it. We require a second equation, which also has ψ and w in it. For this, we use the equation for the fluid density. We make the following derivation in z-coordinates. The derivation in p-coordinates is similar and given in Appendix E.

3.1 Density Equation

When we replaced the incompressibility condition for the continuity equation, we lost a prognostic equation for the density. But we still have the

original continuity equation, so if we *assume* the fluid is incompressible, this becomes:

$$\frac{\partial}{\partial t}\rho_{tot} + \vec{u} \cdot \nabla \rho_{tot} = 0 \tag{233}$$

Here:

$$\rho_{tot} = \rho_0(z) + \rho(x, y, z, t) \tag{234}$$

is the total density, of a static part, $\rho_0(z)$, and a dynamic part, $\rho(x, y, z, t)$. The constant term here also contains the constant density, used in the Boussinesq approximation. Only the dynamic part is important for horizontal motion. We assume, as always, that the dynamic part is smaller:

$$\rho_0 \gg |\rho| \tag{235}$$

Thus we have:

$$(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y})\,\rho + w\,\frac{\partial}{\partial z}\rho_0 = 0 \tag{236}$$

We neglect the term involving the vertical advection of the perturbation density, as this is smaller than the advection of background density.

Now the pressure is likewise comprised of static and dynamic parts:

$$p_{tot} = p_0(z) + p(x, y, z, t)$$

and each term is linked to the corresponding density by the hydrostatic relation. So for example, the static parts are related by:

$$\frac{\partial}{\partial z}p = -\rho g \tag{237}$$

Substituting this into (233), we get:

$$(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y})\,\frac{\partial p}{\partial z} - gw\,\frac{\partial}{\partial z}\rho_0 = 0 \tag{238}$$

We will use the quasi-geostrophic version of this equation. That obtains by approximating the horizontal velocities with their geostrophic counterparts, and by using the streamfunction defined in (72). This is:

$$(\frac{\partial}{\partial t} - \frac{\partial \psi}{\partial y}\frac{\partial}{\partial x} + \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y})\,\frac{\partial \psi}{\partial z} + \frac{N^2}{f_0}w = 0 \tag{239}$$

where N^2 is the *Brunt-Vaisala frequency*:

$$N^2 = -\frac{g}{\rho_c}\frac{d\rho_0}{dz} \tag{240}$$

The Brunt-Vaisala frequency is a measure of the stratification in z-coordinates. It reflect the frequency of oscillation of parcels in a stably stratified fluid which are displaced up or down (see problem 3.x).

Consider what the density equation means. If there is vertical motion in the presence of background stratification, the density will change. This makes sense—if a parcel moves vertically to a region with lighter density, its density will tend to decrease. There is also an interesting parallel here. The vorticity equation implies that *meridional* motion changes the parcels *vorticity*. Here we see that *vertical* motion affects its *density*. The two effects are intimately linked when you have baroclinic instability (sec. 3.8).

Equation (239) gives us a second equation involving ψ and w. Combined with the vorticity equation (75), we now have a complete system.

3.2 QG Potential vorticity

We two equations with two unknowns. But it is straightforward to combine them to produce a single equation with only one unknown. We will

eliminate w from (75) and (239). First we multiply (239) by f_0^2/N^2 and then take the derivative with respect to z:

$$\frac{\partial}{\partial z}(\frac{f_0^2}{N^2}\frac{\partial}{\partial t}\frac{\partial \psi}{\partial z}) + \frac{\partial}{\partial z}[\vec{u}_g \cdot \nabla(\frac{f_0^2}{N^2}\frac{\partial \psi}{\partial z})] = -f_0\frac{\partial}{\partial z}w \tag{241}$$

Expanding the second term:

$$\frac{\partial}{\partial z}[\vec{u}_g \cdot \nabla(\frac{f_0^2}{N^2}\frac{\partial \psi}{\partial z})] = (\frac{\partial}{\partial z}\vec{u}_g) \cdot \nabla(\frac{f_0^2}{N^2}\frac{\partial \psi}{\partial z}) + \vec{u}_g \cdot \nabla(\frac{\partial}{\partial z}(\frac{f_0^2}{N^2}\frac{\partial \psi}{\partial z})) \tag{242}$$

Consider the first term on the RHS. Writing the velocity in terms of the streamfunction, we get:

$$\frac{f_0^2}{N^2}[-\frac{\partial}{\partial z}(\frac{\partial \psi}{\partial y})\frac{\partial}{\partial x}(\frac{\partial \psi}{\partial z}) + \frac{\partial}{\partial z}(\frac{\partial \psi}{\partial x})\frac{\partial}{\partial y}(\frac{\partial \psi}{\partial z})] = 0 \tag{243}$$

The physical reason for this is that the the geostrophic velocity is parallel to the pressure; thus the dot product between $(\frac{\partial}{\partial z}\vec{u}_g)$ and the gradient of $\frac{\partial}{\partial z}\psi$ is zero. Thus (241) reduces to:

$$(\frac{\partial}{\partial t} + \vec{u}_g \cdot \nabla)\,[\frac{\partial}{\partial z}(\frac{f_0^2}{N^2}\frac{\partial \psi}{\partial z})] = -f_0\frac{\partial}{\partial z}w$$

If we combine (241) with (75), we get:

$$(\frac{\partial}{\partial t} + \vec{u}_g \cdot \nabla)\,[\nabla^2\psi + \frac{\partial}{\partial z}(\frac{f_0^2}{N^2}\frac{\partial \psi}{\partial z}) + \beta y] = 0 \tag{244}$$

This is the *quasi-geostrophic potential vorticity* (QGPV) equation. It has only one unknown, ψ. The equation implies that the potential vorticity:

$$q = \nabla^2\psi + \frac{\partial}{\partial z}(\frac{f_0^2}{N^2}\frac{\partial \psi}{\partial z}) + \beta y \tag{245}$$

is *conserved following a parcel moving with the geostrophic wind*. This is a powerful constraint. The flow evolves in such a way that q is only redistributed, not changed.

The first term in the QGPV is the QG relative vorticity and the third term is the planetary vorticity, as noted before. The second term is new; this is the *stretching vorticity*. This is related to vertical gradients in temperature.

3.3 Summary

Thus we have the baroclinic quasi-geostrophic system, for modelling motion on synoptic scales. The system is based on three fundamental assumptions, that:

- The Rossby number is small
- The N-S extent of the domain is smaller than the earth's radius
- The density (temperature) is composed of a stable background plus a weak perturbation

The QG system consists of a **vorticity equation** (75) and a **density equation** (239). These can be combined to produce a **potential vorticity equation** (244). The equations are expressed in z coordinates, but similar equations are also found in pressure coordinates (Appendix E).

Solving problems using the QGPV approach will involve *inverting* the PV relation; given the PV, we can solve for the streamfunction. Doing this requires *boundary conditions*. We consider these next.

3.4 Boundary conditions

Notice that the QGPV equation (244) doesn't contain any Ekman or topographic forcing terms. This is because the PV equation pertains to the interior. In the barotropic case, we introduced those terms by integrating between the lower and upper boundaries. But here, we must treat the boundary conditions separately.

We obtain these by evaluation the density equation (239) at the boundaries. We can rewrite the relation slightly this way:

$$\frac{f_0}{N^2}\frac{d_g}{dt}\frac{\partial\psi}{\partial z} = -w \tag{246}$$

As discussed in section (2.1), the vertical velocity at the boundary can come from either pumping from an Ekman layer or flow over topography. Thus for the lower boundary, we have:

$$\frac{f_0}{N^2}\frac{d_g}{dt}\frac{\partial\psi}{\partial z}\,|_{zb} = -u_g \cdot \nabla h - \frac{\delta}{2}\nabla^2\psi \tag{247}$$

where the velocities and streamfunction are evaluated at $z = z_b$.

The upper boundary condition is similar. For the ocean, with the ocean surface at $z = z_u$, we have:

$$\frac{f_0}{N^2}\frac{d_g}{dt}\frac{\partial\psi}{\partial z}\,|_{zu} = \frac{1}{\rho_c f_0}\nabla \times \vec{\tau}_w \tag{248}$$

The upper boundary condition for the atmosphere depends on the application. If we are considering the entire atmosphere, we could demand that the amplitude of the motion decay as $z \to \infty$, or that the energy flux must be directed upwards. However, we will primarily be interested in motion in the troposphere. Then we can treat the tropopause as a surface, either rigid or freely moving. If it is a rigid surface, we would have simply:

$$\frac{1}{N^2}\frac{d_g}{dt}\frac{\partial\psi}{\partial z^*}\,|_{zu} = 0 \tag{249}$$

3.5 Baroclinic Rossby waves

We now look at some specific solutions. We begin with seeing how stratification alters the Rossby wave solutions. We linearize the PV equation (244) assuming a constant background flow:

$$(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x})\,[\nabla^2\psi + \frac{\partial}{\partial z}(\frac{f_0^2}{N^2}\frac{\partial\psi}{\partial z})] + \beta\frac{\partial}{\partial x}\psi = 0 \tag{250}$$

We assume moreover that the domain lies between two rigid, flat surfaces, and we neglect Ekman layers on those surfaces. So the linearized boundary condition on each surface is:

$$(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x})\frac{\partial\psi}{\partial z} = 0 \tag{251}$$

This implies that the density (or temperature) doesn't change on parcels advected by the mean flow. We will assume the boundary density is just zero:

$$\frac{\partial\psi}{\partial z} = 0 \tag{252}$$

We take the surfaces to be at $z = 0$ and $z = D$.

We can assume that the dependence in x, y and t is sinusoidal, but we can't assume the same in the vertical because N^2 may be a function of z. Indeed, in the ocean, N^2 varies strongly with z. So we can write:

$$\psi = Re\{\phi(z)e^{i(kx+ly-\omega t)}\} \tag{253}$$

Substituting this into the PV equation, we get:

$$(-i\omega + ikU)[-(k^2+l^2)\phi + \frac{\partial}{\partial z}(\frac{f_0^2}{N^2}\frac{\partial \phi}{\partial z})] + i\beta k\phi = 0 \tag{254}$$

or:

$$\frac{\partial}{\partial z}(\frac{f_0^2}{N^2}\frac{\partial \phi}{\partial z}) + \lambda^2\phi = 0 \tag{255}$$

where:

$$\lambda^2 \equiv -k^2 - l^2 + \frac{\beta k}{Uk - \omega} \tag{256}$$

Equation (255) determines the vertical structure, $\phi(z)$, of the Rossby waves. With the boundary conditions (252), this constitutes an *eigenvalue* or "Sturm-Liouville" problem. Only specific values of λ will be permitted. In order to find the dispersion relation for the waves, we must first solve for the vertical structure.

3.5.1 Baroclinic modes with constant stratification

To illustrate, consider the case with $N^2 = const$. Then we have:

$$\frac{\partial^2}{\partial z^2}\phi + \frac{N^2\lambda^2}{f_0^2}\phi = 0 \tag{257}$$

This has a general solution:

$$\phi = Acos(\frac{N\lambda z}{f_0}) + Bsin(\frac{N\lambda z}{f_0}) \tag{258}$$

In order to satisfy $\frac{\partial}{\partial z}\phi = 0$ at $z = 0$, we require that $B = 0$. That leaves the condition at $z = -D$. This requires that:

$$sin(\frac{N\lambda D}{f_0}) = 0 \tag{259}$$

This can only happen if:

$$\frac{N\lambda D}{f_0} = n\pi \tag{260}$$

where $n = 0, 1, 2...$ is an integer. The vertical wavenumber is thus quantized. So we have:

$$\lambda^2 = \frac{n^2\pi^2 f_0^2}{N^2 D^2} = \frac{n^2}{L_D^2} \tag{261}$$

where

$$L_D = \frac{ND}{\pi f_0}$$

is known as the *deformation radius*. This reflects the horizontal scale over which changes in stratification are important. Combining this with the definition of λ^2, we get:

$$\frac{n^2}{L_D^2} \equiv -k^2 - l^2 + \frac{\beta k}{Uk - \omega}) \tag{262}$$

Solving for ω, we obtain:

$$\omega \equiv \omega_n = Uk - \frac{\beta k}{k^2 + l^2 + n^2/L_D^2} \tag{263}$$

This is the *dispersion relation for baroclinic Rossby waves*. The corresponding waves have the following form:

$$\psi = Acos(kx + ly - \omega_n t)cos(\frac{n\pi z}{D}) \tag{264}$$

These are the baroclinic Rossby waves.

Consider first the case with $n = 0$. Then the dispersion relation is:

$$\omega_0 = Uk - \frac{\beta k}{k^2 + l^2} \tag{265}$$

This is the dispersion relation for the barotropic Rossby wave (sec. 118). Accordingly the wave solution with $n = 0$ is

$$\psi_0 = Acos(kx + ly - \omega_n t) \tag{266}$$

The streamfunction doesn't vary in the vertical, exactly like the barotropic case we considered before. Thus the *barotropic mode* exists, even though there is stratification. All the properties that we derived before apply to this wave as well.

With $n = 1$, the streamfunction is:

$$\psi_1 = Acos(kx + ly - \omega_n t)cos(\frac{\pi z}{D}) \tag{267}$$

This is the *first baroclinic mode*. The streamfunction, and hence the velocities, change sign in the vertical. Thus if the velocity is eastward near the upper boundary, it is westward near the bottom. There is also a "zero-crossing" at $D/2$, where the velocities vanish. The waves have an associated density perturbation as well:

$$\rho_1 \propto \frac{\partial}{\partial z}\psi_1 = -\frac{n\pi}{D}Acos(kx + ly - \omega_n t)sin(\frac{\pi z}{D}) \tag{268}$$

So the density perturbation is largest at the mid-depth, where the horizontal velocities vanish. In the ocean, baroclinic Rossby waves cause large deviations in the *thermocline*, which is the subsurface maximum in the density gradient. We have assumed the surface and bottom are flat, and our solution has no density perturbations on those surfaces. But in reality, the baroclinic wave also has a surface deviation, which means one can observe it with satellite (see below).

The dispersion relation for this mode is:

$$\omega_1 = Uk - \frac{\beta k}{k^2 + l^2 + 1/L_D^2} \tag{269}$$

The corresponding zonal phase speed is:

$$c_1 = \frac{\omega_1}{k} = U - \frac{\beta}{k^2 + l^2 + 1/L_D^2} \tag{270}$$

So the first baroclinic mode also propagates westward relative to the mean flow. But the phase speed is *slower* than that of the barotropic Rossby wave. However, if the wavelength is much smaller than the deformation radius (so that $k^2 + l^2 \gg 1/L_d^2$), then we have:

$$c_1 \approx U - \frac{\beta}{k^2 + l^2} \tag{271}$$

So small scale baroclinic waves travel have a phase speed which approaches the speed for the barotropic wave of the same size. If, on the other hand, the wave is much larger than the deformation radius, then:

$$c_1 \approx U - \beta L_D^2 = U - \frac{\beta N^2 D^2}{\pi^2 f_0^2} \tag{272}$$

Thus the large waves are *non-dispersive*, because the phase speed is independent of the wavenumber. This phase speed, known as the "long wave speed", is a strong function of latitude because it varies as the inverse square Coriolis parameter. Where f_0 is small—at low latitudes—these long baroclinic waves move faster.

The phase speeds from the first four modes are plotted as a function of wavenumber in Fig. (30). We plot the function:

$$c_n = \frac{1}{2k^2 + n^2} \tag{273}$$

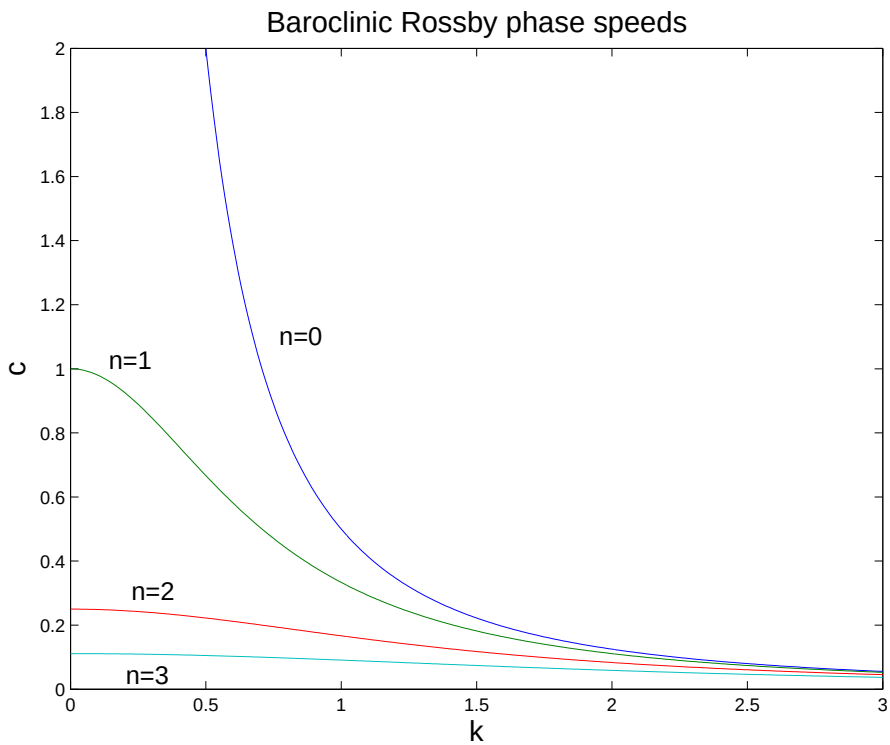

Figure 30: Rossby phase speeds as a function of wavenumber for the first four modes.

(note that the actual c is the negative of this). We have set $\beta = L_D = 1$ and $k = l$ and assume the mean flow is zero. We plot the phase speed against k. The barotropic mode ($n = 0$) has a phase speed which increases without bound as the wavenumber goes to zero. This is actually a consequence of having a rigid lid at the surface; if we had a free (moving) surface, the wave would have a finite phase speed at $k = 0$. The first baroclinic mode ($n = 1$) has a constant phase speed at low k, equal to $c = 1$. This is the long wave speed with $L_D = 1$. The second and third baroclinic modes ($n = 2, 3$) also have long wave speeds, but these are four and nine times smaller than the first baroclinic long wave speed.

3.5.2 Baroclinic modes with exponential stratification

In the preceding section, we assumed a constant Brunt-Vaisala frequency, N. This implies the density has linear profile in the vertical. In reality, the oceanic density exhibits a nonlinear variation with z. In many locations, the Brunt-Vaisala frequency exhibits a nearly exponential dependence on depth, with larger values near the surface and smaller ones at depth.

An exponential profile can also be solved analytically. Assume:

$$N^2 = N_0^2 e^{\alpha z} \tag{274}$$

Substituting (274) into (255) yields:

$$\frac{d^2\phi}{dz^2} - \alpha\frac{d\phi}{dz} + \frac{N_0^2\lambda^2}{f_0^2}e^{\alpha z}\phi = 0 \tag{275}$$

Making the substitution $\zeta = e^{\alpha z/2}$, we obtain:

$$\zeta^2\frac{d^2\phi}{d\zeta^2} - \zeta\frac{d\phi}{d\zeta} + \frac{4N_0^2\lambda^2}{\alpha^2 f_0^2}\zeta^2\phi = 0 \tag{276}$$

This is a Bessel-type equation. The solution which satisfies the upper boundary condition is:

$$\phi = Ae^{\alpha z/2}[Y_0(2\gamma)J_1(2\gamma e^{\alpha z/2}) - J_0(2\gamma)Y_1(2\gamma e^{\alpha z/2})] \tag{277}$$

where $\gamma = N_0\lambda/(\alpha f_0)$. If we then imposing the bottom boundary condition, we get:

$$J_0(2\gamma)Y_0(2\gamma e^{-\alpha H/2}) - Y_0(2\gamma)J_0(2\gamma e^{-\alpha H/2}) = 0 \tag{278}$$

Equation (278) is a *transcendental equation*. It only admits certain discrete values, γ_n. So γ_n is quantized, just as it was with constant strati-

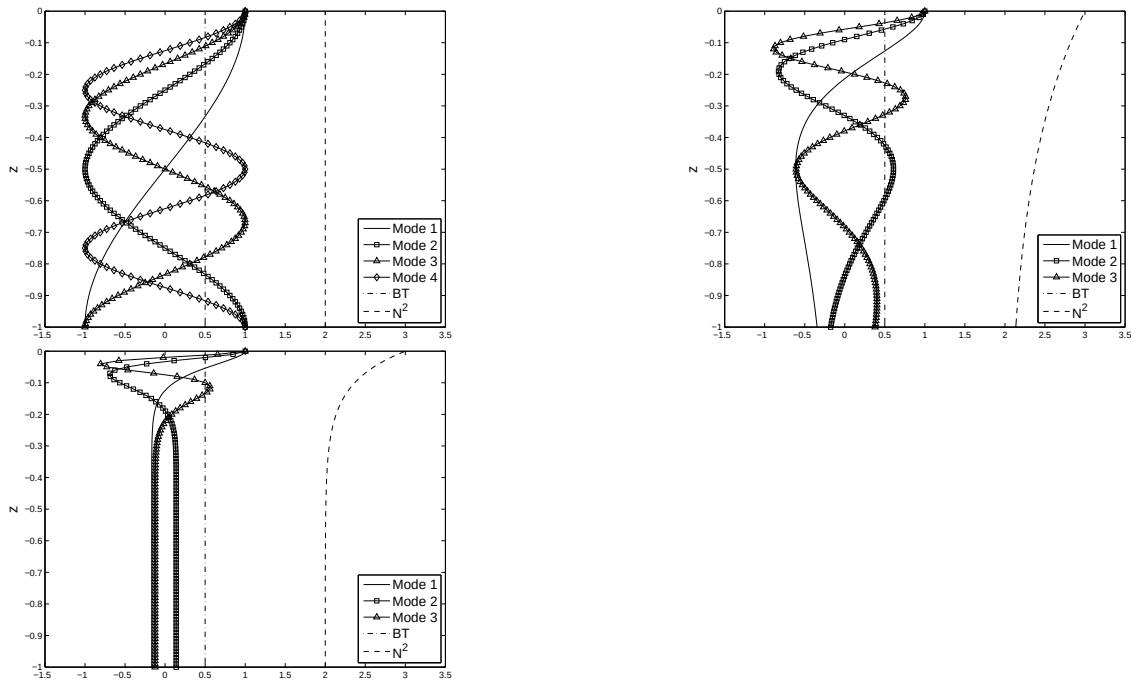

Figure 31: The baroclinic modes with N=const. (upper left panel) and with exponential N. In the upper right panel, $\alpha^{-1} = H/2$, and in the lower left, $\alpha^{-1} = H/10$. In all cases, $H = 1$. From LaCasce (2011).

fication, as in eq. (260). Once γ_n is found, the wave frequencies can be determined, from the dispersion relation, as before. Equation (278) is more difficult to solve than with constant stratification, but it's possible to do this numerically. But notice that $\gamma = 0$ is also a solution of (278)—thus there is also a barotropic mode in this case as well.

Some examples of the wave vertical structure, $\phi(z)$, are shown in Fig. (31). In the upper left panel are the cosine modes, with constant N^2. In the upper right panel are the modes with exponential stratification, for the case where α^{-1}, the e-folding depth for the stratification, is equal to half the total depth. In the lower right panel are the modes with the e-folding depth equal to 1/10th the water depth. In all cases, there is a depth-independent barotropic mode plus an infinite set of baroclinic modes. And in all cases, the first baroclinic mode has one zero crossing, the second mode has two,

and so forth. But unlike the cosine modes, the exponential modes have their largest amplitudes near the surface. So the Rossby wave velocities and density perturbations are likewise surface-intensified.

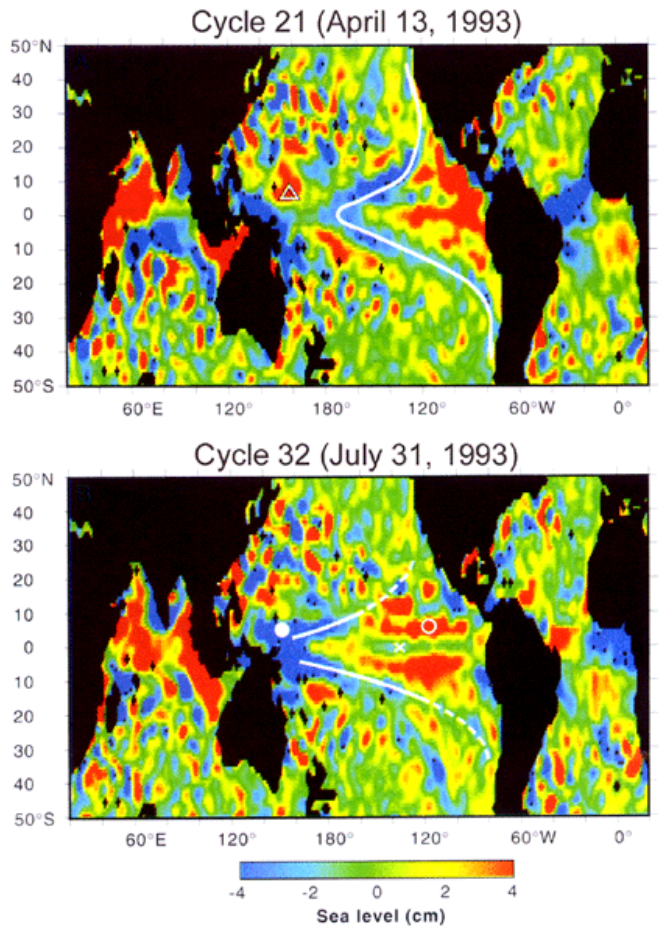

Figure 32: Sea surface height anomalies at two successive times. Westward phase propagation is clear at low latitudes, with the largest speeds occurring near the equator. From Chelton and Schlax (1996).

3.5.3 Observations of Baroclinic Rossby waves

As noted, first baroclinic Rossby waves can be seen by satellite. Shown in Fig. (32) are two sea surface height fields from satellite measurements in 1993. The fields suggest that there are large scale anomalies in the elevation of the sea surface, and that these migrate westward in time. The

speed of propagation moreover increases towards the equator.

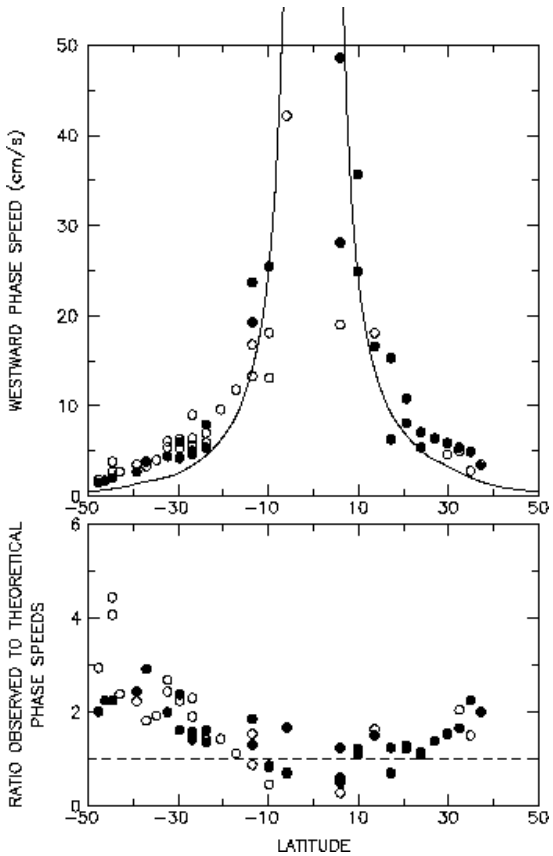

Figure 33: Westward phase speeds deduced from the motion of sea surface height anomalies, compared with the value predicted by the long wave phase speed given in (272). The lower panel shows the ratio of observed to predicted phase speed. Note the observed speeds are roughly twice as fast at high latitudes. From Chelton and Schlax (1996).

It is possible to use satellite date like this to deduce the phase speed as a function of latitude. This was done in a paper by Chelton and Schlax (1996), and the result is shown in Fig. (33). The observations are plotted over a curve showing the long wave speed, as in (272). There is reasonable agreement at most latitudes. The agreement is very good in particular below about 20 degrees of latitude; at higher latitudes there is a systematic discrepancy, with the observed waves moving perhaps twice as fast as predicted. There are a number of theories which have tried to explain

this.[5] But for our purposes, we see that the simple theory does fairly well at predicting the observed sea surface height propagation.

There are, in addition, the higher baroclinic modes (with $n > 1$). These waves are even slower than the first baroclinic mode. In addition, they have more structure in the vertical. The second baroclinic mode thus has two zero-crossings, where the horizontal velocities vanish.

3.6 Mountain waves

In sec. (2.5), we saw how a mean wind blowing over mountains could excite standing Rossby waves. Now we will consider what happens in the baroclinic case.

We consider the potential vorticity equation (244), without forcing:

$$\frac{d_g}{dt}[\nabla^2\psi + \frac{\partial}{\partial z}(\frac{f_0^2}{N^2}\frac{\partial\psi}{\partial z}) + \beta y] = 0 \tag{279}$$

As before, we consider the flow driven by a mean zonal wind:

$$U\frac{\partial}{\partial x}[\nabla^2\psi + \frac{\partial}{\partial z}(\frac{f_0^2}{N^2}\frac{\partial\psi}{\partial z})] + \beta\frac{\partial}{\partial x}\psi = 0 \tag{280}$$

The mean flow is constant, i.e. there is no vertical or lateral shear (we take up a vertically sheared flow later on). As before, we ignore the time dependence; we are looking for stationary, wave-like solutions. Again we will assume that the stratification parameter, N^2, is constant, for simplicity.

With a constant N^2, all the coefficients in the vorticity equation are constant. That means we can use a solution which is wave-like in all directions:

$$\psi = \hat{\psi}e^{ikx+ily+imz} \tag{281}$$

[5] See for example LaCasce and Pedlosky (2004) and Isachsen et al. (2007).

Substituting this into (280) yields:

$$ikU[-(k^2+l^2) - m^2\frac{f_0^2}{N^2}] + ik\beta\hat{\psi} = 0 \tag{282}$$

Rearranging, we get:

$$m = \pm\frac{N}{f_0}(\frac{\beta}{U} - k^2 - l^2)^{1/2} \tag{283}$$

The character of the solution depends on the term in the square root in (283). If this is *positive*, then m is real and we have wave-like solutions. But if the argument is *negative*, then m will be imaginary and the vertical dependence will be *exponential*. If we rule out those solutions which grow with height—recall that the source for the waves is the mountains, at the ground—then the exponential solutions are decaying upward.

But if the argument is positive, then m is real and the solution is wave-like in z. This means the waves can effectively propagate upward to infinity, leaving the troposphere and entering the stratosphere and beyond. Then the waves generated at the surface can alter the circulation higher up in the atmosphere.

In order for the argument to be positive, we require:

$$\frac{\beta}{U} > k^2 + l^2 \tag{284}$$

This implies that the mean flow, U, must be *positive*, or eastward. Rewriting the relation, we have:

$$0 < U < \frac{\beta}{k^2+l^2} \equiv U_s \tag{285}$$

So while U must be positive, neither can it be too strong. It must, in particular, be less than U_s, the speed at which the barotropic Rossby wave

is stationary (sec. 2.3.3).

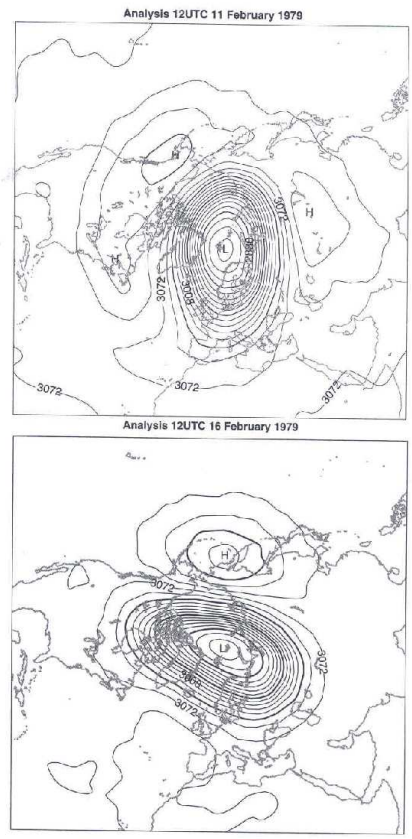

Figure 34: The geopotential height at 10 hPa on February 11 and 16, 1979. The polar vortex is being perturbed by a disturbance over the Pacific. From Holton, *An Introduction to Dynamic Meteorology*.

Why is the mean flow limited by speed of the barotropic wave? As we saw in the previous section, the barotropic mode is the *fastest* of all the Rossby modes. So upward propagating waves are possible only when the mean speed is slow enough so that one of the baroclinic Rossby modes is *stationary*.

Notice that we have not said anything about the lower boundary, where the waves are forced. In fact, the form of the mountains determines the structure of the stationary waves. But the general condition above applies

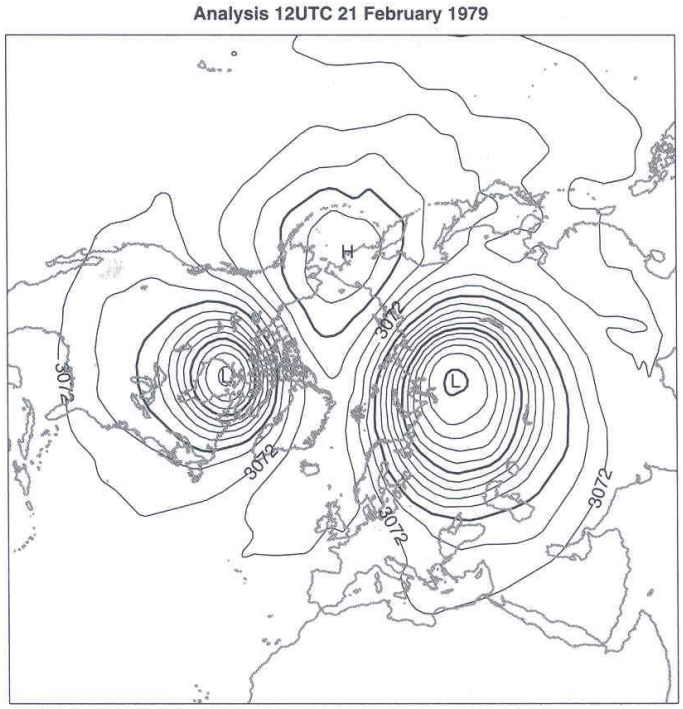

Fig. 12.10 10 hPa geopotential height analyses for February 11, 16, and 21, 1979 at 12UTC showing breakdown of the polar vortex associated with a wave number 2 sudden stratospheric warming. Contour interval: 16 dam. Analysis from ERA-40 reanalysis courtesy of the European Centre for Medium-Range Weather Forecasts (ECMWF).

Figure 35: The geopotential height at 10 hPa on February 21, 1979 (following Fig. 34). The polar vortex has split in two, appearing now as a mode 2 Rossby wave. From Holton, *An Introduction to Dynamic Meteorology*.

to all types of mountain. If the mean flow is westerly and not too strong, the waves generated over the mountains can extend upward indefinitely.

Upward propagating Rossby waves are important in the stratosphere, and can greatly disturb the flow there. They can even change the usual equator-to-pole temperature difference, a *stratospheric warming* event.

Consider Figs. (34) and (35). In the first panel of Fig. (34), we see the *polar vortex* over the Arctic. This is a region of persistent low pressure (with a correspondingly low tropopause height). In the second panel, a high pressure is developing over the North Pacific. This high intensifies, eventually causing the polar vortex has split in two, making a mode

2 planetary wave (Fig. 35). The wave has a corresponding temperature perturbation, and in regions the air actually warms moving from south to north.

Stratospheric warming events occur only in the wintertime. Charney and Drazin (1961) used the above theory to explain which this happens. In the wintertime, the winds are westerly ($U > 0$), so that upward propagation is possible. But in the summertime, the stratospheric winds are *easterly* ($U < 0$), preventing upward propagation. So Rossby waves only alter the stratospheric circulation in the wintertime.

3.7 Topographic waves

In an earlier problem, we found that a sloping bottom can support Rossby waves, just like the β-effect. The waves propagate with shallow water to their right (or "west", when facing "north" up the slope). Topographic waves exist with stratification too, and it is useful to examine their structure.

We'll use the potential vorticity equation, linearized with zero mean flow ($U = 0$) and on the f-plane ($\beta = 0$). We'll also assume that the Brunt-Vaisala frequency, N, is constant. Then we have:

$$\frac{\partial}{\partial t}(\nabla^2\psi + \frac{f_0^2}{N^2}\frac{\partial^2}{\partial z^2}\psi) = 0 \tag{286}$$

Thus the potential vorticity in the interior of the fluid *does not change in time*; it is simply constant. We can take this constant to be zero.

For the bottom boundary condition, we will assume a linear topographic slope. This can be in any direction, but we will say the depth is decreasing toward the north:

$$D = D_0 - \alpha y \tag{287}$$

so that $h = \alpha y$. In fact, this is a general choice because with f=const., the system is rotationally invariant (why?). With this topography, the bottom boundary condition (247) becomes:

$$\frac{f_0}{N^2}\frac{d_g}{dt}\frac{\partial \psi}{\partial z^*}|_{zb} = -u_g \cdot \nabla h \quad \rightarrow \quad \frac{d_g}{dt}\frac{\partial}{\partial z}\psi + \frac{N^2}{f_0}\alpha v = 0 \tag{288}$$

Let's assume further that the bottom is at $z = 0$. We won't worry about the upper boundary, as the waves will be trapped near the lower one.

To see that, assume a solution which is wave-like in x and y:

$$\psi = Re\{\hat{\psi}(z)e^{ikx+ily-i\omega t}\} \tag{289}$$

Under the condition that the PV is zero, we have:

$$(-k^2 - l^2)\hat{\psi} + \frac{f_0^2}{N^2}\frac{\partial^2}{\partial z^2}\hat{\psi} = 0 \tag{290}$$

or

$$\frac{\partial^2}{\partial z^2}\hat{\psi} - \frac{N^2\kappa^2}{f_0^2}\hat{\psi} = 0 \tag{291}$$

where $\kappa = (k^2 + l^2)^{1/2}$ is again the total wavenumber. This equation only has exponential solutions. The one that decays going up from the bottom boundary has:

$$\hat{\psi}(z) = Ae^{-N\kappa z/|f_0|} \tag{292}$$

This is the vertical structure of the topographic waves. It implies the waves have a vertical e-folding scale of:

$$H \propto \frac{|f_0|}{N\kappa} = \frac{|f_0|\lambda}{2\pi N}$$

if λ is the wavelength of the wave. Thus the vertical scale of the wave *depends on its horizontal scale*. Larger waves extend further into the interior. Note too that we have a *continuum* of waves, not a discrete set like we did with the baroclinic modes (sec. 3.5).

Notice that we would have obtained the same result with the mountain waves in the previous section. If we take (283) and set $\beta = 0$, we get:

$$m = \pm\frac{N}{f_0}(-k^2 - l^2)^{1/2} = \pm\frac{iN\kappa}{f_0} \tag{293}$$

So with $\beta = 0$, we obtain *only* exponential solutions in the vertical. The wave-like solutions require an interior PV gradient.

Now we can apply the bottom boundary condition. We linearize (288) with zero mean flow and write v in terms of the streamfunction:

$$\frac{\partial}{\partial t}\frac{\partial}{\partial z}\psi + \frac{N^2\alpha}{f_0}\frac{\partial\psi}{\partial x} = 0 \tag{294}$$

Substituting in the wave expression for ψ, we get:

$$-\frac{\omega N\kappa}{|f_0|}A - \frac{N^2\alpha k}{f_0}A = 0 \tag{295}$$

so that:

$$\omega = -\frac{N\alpha k}{\kappa}sgn(f_0) \tag{296}$$

where $sgn(f_0)$ is +1 if $f > 0$ and -1 if $f < 0$.

This is the dispersion relation for stratified topographic waves. The phase speed in the x-direction (along the isobaths, the lines of constant depth) is:

$$c_x = -\frac{N\alpha}{\kappa} sgn(f_0) \tag{297}$$

This then is "westward" in the Northern Hemisphere, i.e. with the shallow water on the right. As with planetary waves, the fastest waves are the largest ones (with small κ). These are also the waves the penetrate the highest into the water column. Thus the waves which are closest to barotropic are the fastest.

Topographic waves are often observed in the ocean, particularly over the continental slope. Observations suggest that disturbances originating at the equator propagate north (with shallow water on the right) past California towards Canada. At the same time, waves also propagate south (with the shallow water on the left) past Peru.

3.8 Baroclinic instability

Now we return to instability. As discussed before, solar heating of the earth's surface causes a temperature gradient, with a warmer equator and colder poles. This north-south temperature gradient is accompanied by a vertically sheared flow in the east-west direction. The flow is weak near the surface and increases moving upward in the troposphere.

3.8.1 Basic mechanism

The isotherms look (crudely) as sketched in Fig. (36). The temperature decreases to the north, and also increases going up. Thus the parcel A is colder (and heavier) than parcel C, which is directly above it. The air is stably stratified, because exchanging A and C would *increase* the potential energy.

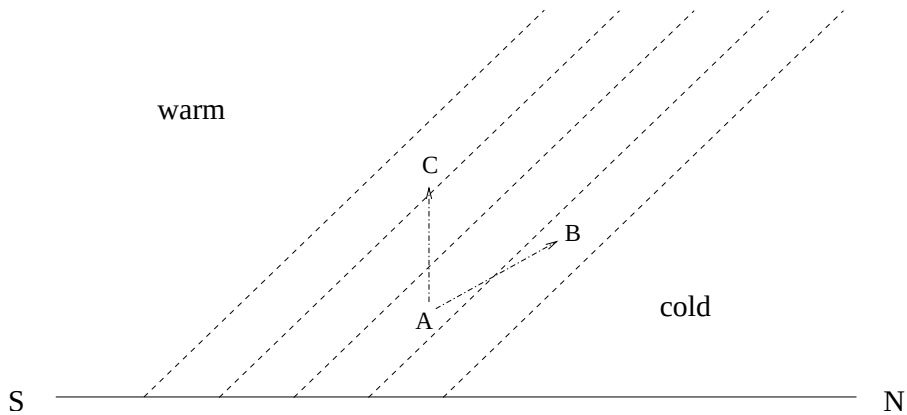

Figure 36: Slantwise convection. The slanted isotherms are accompanied by a thermal wind shear. The parcel A is colder, and thus heavier, than parcel C, implying static stability. But A is lighter than B. So A and B can be interchanged, releasing potential energy.

However, because the isotherms tilt, there is a parcel B which is above A and heavier. So A and B can be exchanged, *releasing* potential energy. This is often referred to as "slantwise" convection, and it is the basis for baroclinic instability. Baroclinic instability simultaneously *reduces the vertical shear* while *decreasing the north-south temperature gradient*. In effect, it causes the temperature contours to slump back to a more horizontal configuration, which reduces the thermal wind shear while decreasing the meridional temperature difference.

Baroclinic instability is extremely important. For one, it allows us to live at high latitudes—without it, the poles would be much colder than the equator.

3.8.2 Charney-Stern criterion

We can derive conditions for baroclinic instability, just as we did to obtain the Rayleigh-Kuo criterion for barotropic instability. We begin, as always, with the PV equation (244):

$$\frac{d_g}{dt}[\nabla^2\psi + \frac{\partial}{\partial z}(\frac{f_0^2}{N^2}\frac{\partial\psi}{\partial z}) + \beta y] = 0 \tag{298}$$

We linearize this about a mean flow, U, which varies in *both* the y and z-directions. Doing this is the same thing if we had writen the streamfunction as:

$$\psi = \Psi(y,z) + \psi'(x,y,z,t) \tag{299}$$

where the primed streamfunction is much smaller than the mean streamfunction. The mean streamfunction has an associated zonal flow:

$$U(y,z) = -\frac{\partial}{\partial y}\Psi \tag{300}$$

Note it has no meridional flow (V) because Ψ is independent of x. Using this, we see the mean PV is:

$$\frac{\partial^2}{\partial y^2}\Psi + \frac{\partial}{\partial z}(\frac{f_0^2}{N^2}\frac{\partial\Psi}{\partial z}) + \beta y \tag{301}$$

So the full linearized PV equation is:

$$(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x})[\nabla^2\psi + \frac{\partial}{\partial z}(\frac{f_0^2}{N^2}\frac{\partial\psi}{\partial z})] + (\frac{\partial}{\partial y}q_s)\frac{\partial}{\partial x}\psi = 0 \tag{302}$$

where:

$$\frac{\partial}{\partial y}q_s = \beta - \frac{\partial^2}{\partial y^2}U - \frac{\partial}{\partial z}(\frac{f_0^2}{N^2}\frac{\partial U}{\partial z}) \tag{303}$$

We saw the first two terms before, in the barotropic case. The third term however is new. It comes about because the mean velocity (and hence the mean streamfunction) varies in z.

In addition, we need the boundary conditions. We will assume flat boundaries and no Ekman layers, to make this simple. Thus we use (249), linearized about the mean flow:

$$\frac{d_g}{dt}\frac{\partial\psi}{\partial z} = (\frac{\partial}{\partial t} + U\frac{\partial}{\partial x})\frac{\partial\psi}{\partial z} + v\frac{\partial}{\partial y}\frac{\partial\Psi}{\partial z}$$

$$= (\frac{\partial}{\partial t} + U\frac{\partial}{\partial x})\frac{\partial\psi}{\partial z} - v\frac{\partial U}{\partial z} = 0 \tag{304}$$

We'll assume that we have boundaries at the ground, at $z = 0$, and an upper level, $z = D$. The latter could be the tropopause. Alternatively, we could have no upper boundary at all, as with the mountain waves. But we will use an upper boundary in the Eady model in the next section, so it's useful to include that now.

Because U is potentially a function of both y and z, we can only assume a wave structure in (x, t). So we use a Fourier solution with the following form:

$$\psi = \hat{\psi}(y, z)e^{ik(x-ct)} \tag{305}$$

Substituting into the PV equation (302), we get:

$$(U - c)[-k^2\hat{\psi} + \frac{\partial^2}{\partial y^2}\hat{\psi} + \frac{\partial}{\partial z}(\frac{f_0^2}{N^2}\frac{\partial\hat{\psi}}{\partial z})] + (\frac{\partial}{\partial y}q_s)\hat{\psi} = 0 \tag{306}$$

after canceling the factor of k. Similarly, the boundary conditions are:

$$(U - c)\frac{\partial}{\partial z}\hat{\psi} - (\frac{\partial}{\partial z}U)\hat{\psi} = 0 \tag{307}$$

We now do as we did in sec. (2.9.1): we divide (306) by $U - c$ and then multiply by the complex conjugate of $\hat{\psi}$:

$$\hat{\psi}^*[\frac{\partial^2}{\partial y^2}\hat{\psi} + \frac{\partial}{\partial z}(\frac{f_0^2}{N^2}\frac{\partial\hat{\psi}}{\partial z})] - k^2|\hat{\psi}|^2 + \frac{1}{U-c}(\frac{\partial}{\partial y}q_s)|\hat{\psi}|^2 = 0 \tag{308}$$

We then separate real and imaginary parts. The imaginary part of the equation is:

$$\hat{\psi}_r\frac{\partial^2}{\partial y^2}\hat{\psi}_i - \hat{\psi}_i\frac{\partial^2}{\partial y^2}\hat{\psi}_r + \hat{\psi}_r\frac{\partial}{\partial z}(\frac{f_0^2}{N^2}\frac{\partial\hat{\psi}_i}{\partial z}) - \hat{\psi}_i\frac{\partial}{\partial z}(\frac{f_0^2}{N^2}\frac{\partial\hat{\psi}_r}{\partial z})$$
$$+\frac{c_i}{|U-c|^2}(\frac{\partial}{\partial y}q_s)|\hat{\psi}|^2 = 0 \tag{309}$$

We have again used:

$$\frac{1}{U-c} = \frac{1}{U-c_r-ic_i} = \frac{U-c_r+ic_i}{|U-c|^2}$$

As we did previously, we use a channel domain and demand that $\hat{\psi} = 0$ at the north and south walls, at $y = 0$ and $y = L$. We integrate the PV equation in y and then invoke integration by parts. Doing this yields, for the first two terms on the LHS:

$$\int_0^L(\hat{\psi}_i\frac{\partial^2}{\partial y^2}\hat{\psi}_r - \hat{\psi}_r\frac{\partial^2}{\partial y^2}\hat{\psi}_i)\,dy = \hat{\psi}_i\frac{\partial}{\partial y}\hat{\psi}_r|_0^L - \int_0^L\frac{\partial}{\partial y}\hat{\psi}_i\frac{\partial}{\partial y}\hat{\psi}_r\,dy$$
$$-\hat{\psi}_r\frac{\partial}{\partial y}\hat{\psi}_i|_0^L + \int_0^L\frac{\partial}{\partial y}\hat{\psi}_r\frac{\partial}{\partial y}\hat{\psi}_i\,dy = 0 \tag{310}$$

We can similarly integrate the PV equation in the vertical, from $z = 0$ to $z = D$, and again integrate by parts. This leaves:

$$\hat{\psi}_r\frac{f_0^2}{N^2}\frac{\partial\hat{\psi}_i}{\partial z}|_0^D - \hat{\psi}_i\frac{f_0^2}{N^2}\frac{\partial\hat{\psi}_r}{\partial z}|_0^D \tag{311}$$

(because the leftover integrals are the same and cancel each other). We then evaluate these two terms using the boundary condition. We rewrite that as:

$$\frac{\partial}{\partial z}\hat{\psi} = (\frac{\partial}{\partial z}U)\frac{\hat{\psi}}{U-c} \tag{312}$$

The real part of this is:

$$\frac{\partial}{\partial z}\hat{\psi}_r = (\frac{\partial}{\partial z}U)[\frac{(U-c_r)\hat{\psi}_r}{|U-c|^2} - \frac{c_i\hat{\psi}_i}{|U-c|^2}] \tag{313}$$

and the imaginary part is:

$$\frac{\partial}{\partial z}\hat{\psi}_i = (\frac{\partial}{\partial z}U)[\frac{(U-c_r)\hat{\psi}_i}{|U-c|^2} + \frac{c_i\hat{\psi}_r}{|U-c|^2}] \tag{314}$$

If we substitute these into (311), we get:

$$\frac{f_0^2}{N^2}(\frac{\partial}{\partial z}U)\frac{c_i\hat{\psi}_i^2}{(U-c_r)^2+c_i^2}|_0^D + \frac{f_0^2}{N^2}(\frac{\partial}{\partial z}U)\frac{c_i\hat{\psi}_r^2}{(U-c_r)^2+c_i^2}|_0^D =$$

$$\frac{f_0^2}{N^2}(\frac{\partial}{\partial z}U)\frac{c_i\hat{|}\psi|^2}{(U-c_r)^2+c_i^2}|_0^D \tag{315}$$

So the doubly-integrated (311) reduces to:

$$c_i[\int_0^L\int_0^D \frac{|\hat{\psi}|^2}{|U-c|^2}(\frac{\partial}{\partial y}q_s)\,dydz + \int_0^L \frac{f_0^2}{N^2}\frac{\hat{|}\psi|^2}{|U-c|^2}(\frac{\partial}{\partial z}U)\;|_0^D\,dy] = 0 \tag{316}$$

This is the *Charney-Stern criterion* for instability. In order to have instability, $c_i > 0$ and that requires that the term in brackets vanish.

Note that the first term is identical to the one we got for the Rayleigh-Kuo criterion (221). In that case we had:

$$\frac{\partial}{\partial y}q_s = \beta - \frac{\partial^2}{\partial y^2}U \tag{317}$$

For instability, we required that $\frac{\partial}{\partial y}q_s$ had to be zero somewhere in the domain.

The baroclinic condition is similar, except that now the background PV is given by (303), so:

$$\frac{\partial}{\partial y}q_s = \beta - \frac{\partial^2}{\partial y^2}U - \frac{\partial}{\partial z}(\frac{f_0^2}{N^2}\frac{\partial U}{\partial z}) = 0$$

So now the vertical shear can also cause the PV gradient to vanish.

In addition, the boundary contributions also come into play. In fact we have *four* possibilities:

- $\frac{\partial}{\partial y}q_s$ vanishes in the interior, with $\frac{\partial}{\partial z}U = 0$ on the boundaries
- $\frac{\partial}{\partial z}U$ at the upper boundary has the opposite sign as $\frac{\partial}{\partial y}q_s$
- $\frac{\partial}{\partial z}U$ at the lower boundary has the same sign as $\frac{\partial}{\partial y}q_s$
- $\frac{\partial}{\partial z}U$ has the same sign on the boundaries, with $\frac{\partial}{\partial y}q_s = 0$ in the interior

The first condition is the Rayleigh-Kuo criterion. This is the only condition in the baroclinic case too if the vertical shear vanishes at the boundaries. Note that from the thermal wind balance:

$$\frac{\partial}{\partial z}U \propto \frac{\partial}{\partial y}T$$

So having zero vertical shear at the boundaries implies the temperature is *constant* on them. So the boundaries are important if there is a temperature gradient on them.

The fourth condition applies when the PV (and hence the gradient) is zero in the interior. Then the two boundaries can interact to produce instability. This is Eady's (1949) model of baroclinic instability, which we consider in the next section.

In the atmosphere, the mean relative vorticity is generally smaller than the β-effect. So the interior gradient is positive (and approximately equal to β). Then the main effect is for the lower boundary to cancel the interior term. This is what happens in Charney's (1947) model of baroclinic instability.

It is also possible to construct a model with zero shear at the boundaries and where the gradient of the interior PV vanishes because of the vertical gradient. This is what happens in Phillip's (1954) model of instability. His model has two fluid layers, with the flow in each layer being barotropic. Thus the shear at the upper and lower boundaries is zero. But because there are two layers, the PV in each layer can be different. If the PV in the layers is of opposite sign, then they can potentially sum to zero. Then Philip's model is unstable.

As with the Rayleigh-Kuo criterion, the Charney-Stern criteria represent a necessary condition for instability but not a sufficient one. So satisfying one of the conditions above indicates instability *may* occur. Note that only one needs to be satisfied. But if none of the conditions are satisfied, the flow is stable.

3.9 The Eady model

The simplest model of baroclinic instability with continuous stratification is that of Eady (1949). This came out two years after Charney's (1947) model, which also has continuous stratification *and* the β-effect—something not included in the Eady model. But the Eady model is comparatively simple, and illustrates the major aspects.

The configuration for the Eady model is shown in Fig. (37). We will make the following assumptions:

The Eady Model

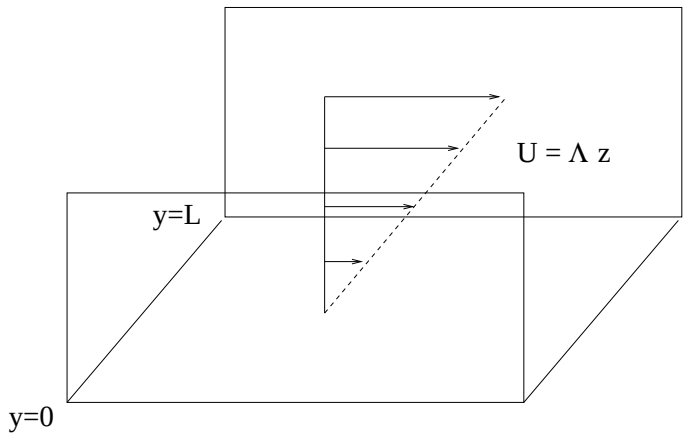

Figure 37: The configuration for the Eady model.

- A constant Coriolis parameter ($\beta = 0$)
- Uniform stratification ($N^2 = const.$)
- The mean velocity has a constant shear, so $U = \Lambda z$
- The motion occurs between two rigid plates, at $z = 0$ and $z = D$
- The motion occurs in a channel, with $v = 0$ on the walls at $y = 0, L$

The uniform stratification assumption is reasonable for the troposphere but less so for the ocean (where the stratification is greater near the surface, as we have seen). The rigid plate assumption is also unrealistic, but simplifies the boundary conditions.

From the Charney-Stern criteria, we see that the model can be unstable because the vertical shear is the same on the two boundaries. The interior PV on the other hand is zero, so this cannot contribute to the instability. We will see that the interior in the Eady model is basically passive. It is the interaction between temperature anomalies on the boundaries which are important.

We will use a wave solution with the following form:

$$\psi = \hat{\psi}(z) sin(\frac{n\pi y}{L}) e^{ik(x-ct)}$$

The sin term satisfies the boundary conditions on the channel walls because:

$$v = \frac{\partial}{\partial x}\psi = 0 \quad \rightarrow \quad ik\hat{\psi} = 0 \tag{318}$$

which implies that $\hat{\psi} = 0$. The sin term vanishes at $y = 0$ and $y = L$.

The linearized PV equation for the Eady model is:

$$(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x})(\nabla^2\psi + \frac{f_0^2}{N^2}\frac{\partial^2}{\partial z^2}\psi) = 0 \tag{319}$$

Because there is no β term, the PV is constant on air parcels advected by the mean flow. Inserting the wave solution in yields:

$$(U - c)[(-(k^2 + \frac{n^2\pi^2}{L^2})\hat{\psi} + \frac{f_0^2}{N^2}\frac{\partial^2}{\partial z^2}\hat{\psi}] = 0 \tag{320}$$

So either the phase speed equals the mean velocity or the PV itself is zero. The former case defines what is known as a *critical layer*; we won't be concerned with that at the moment. So we assume instead the PV is zero. This implies:

$$\frac{\partial^2}{\partial z^2}\hat{\psi} = \alpha^2\hat{\psi} \tag{321}$$

where

$$\alpha \equiv \frac{N\kappa}{f_0}$$

and where $\kappa = (k^2 + (n\pi/L)^2)^{1/2}$ is the total horizontal wavenumber. This is exactly the same as in the topographic wave problem in (3.7). Equation (321) determines the vertical structure of the waves.

First, let's consider what happens when the vertical scale factor, α, is large. This is the case when the waves are short, because κ is then large. In this case the solutions to (321) are exponentials which decay away from the boundaries:

$$\hat{\psi} = Ae^{-\alpha z}, \quad \hat{\psi} = Be^{\alpha(z-D)} \tag{322}$$

near $z = 0$ and $z = D$, respectively. The waves are thus trapped on each boundary and have a vertical structure like topographic waves.

To see how the waves behave, we use the boundary condition. This is:

$$(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x})\frac{\partial \psi}{\partial z} - \frac{\partial \psi}{\partial x}\frac{dU}{dz} = 0 \tag{323}$$

Inserting the wave solution and the mean shear, this is simply:

$$(\Lambda z - c)\frac{\partial \psi}{\partial z} - \Lambda\hat{\psi} = 0 \tag{324}$$

after cancelling the factor of ik. At $z = 0$, this is:

$$(\alpha c - \Lambda)A = 0 \tag{325}$$

after inserting the vertical dependence at the lower boundary. At $z = D$, we have:

$$[\alpha(\Lambda D - c) - \Lambda]B = 0 \tag{326}$$

To have non-trivial solutions, A and B are non-zero. So we require:

$$c = \frac{\Lambda}{\alpha}, \quad c = \Lambda D - \frac{\Lambda}{\alpha} \tag{327}$$

at $z = 0, D$ respectively.

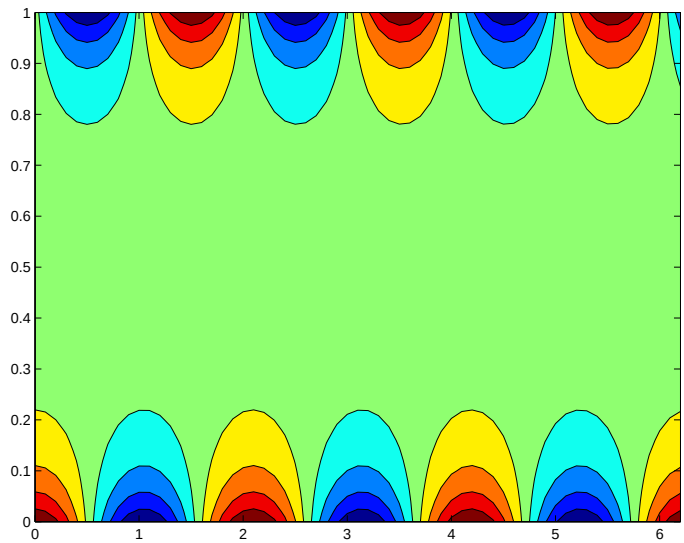

Figure 38: The Eady streamfunction in the limit of large α.

First we notice that the phase speeds are *real*—so there is no instability. The waves are simply propagating on each boundary. In the limit that α is large (the decay from the boundaries is rapid), these are:

$$c \approx 0, \quad c \approx \Lambda D \tag{328}$$

So the phase speeds are equal to the mean velocities on the boundaries. Thus the waves are just swept along by the background flow.

If α is not so large, the boundary waves propagate at speeds different than the mean flow.

The solution is shown in Fig. (38). We have two waves, each advected by the mean flow at its respective boundary and each decaying exponentially away from the boundary. These waves are *independent* because they decay so rapidly with height; they do not interact with each other.

Now let's look at the case where α is not so large, so that the waves extend further into the interior. Then we would write for the wave solution:

$$\hat{\psi} = Ae^{\alpha z} + Be^{-\alpha z} \tag{329}$$

This applies over the whole interior, including both boundaries. Plugging into the boundary equation (324) we get, at $z = 0$:

$$(-c\alpha - \Lambda)A + (\alpha c - \Lambda)B = 0 \tag{330}$$

while at the upper boundary, at $z = D$, we get:

$$(\alpha(\Lambda D - c) - \Lambda)e^{\alpha D}A + (-\alpha(\Lambda D - c) - \Lambda)e^{-\alpha D}B = 0 \tag{331}$$

We can rewrite these equations in matrix form as follows:

$$\begin{pmatrix} c\alpha + \Lambda & -c\alpha + \Lambda \\ (-\alpha c + \Lambda(\alpha D - 1))e^{\alpha D} & (\alpha c - \Lambda(\alpha D + 1))e^{-\alpha D} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{332}$$

Note we multiplied the first equation through by -1. Because this system is homogeneous, solutions exist *only* if the determinant of the coefficients vanishes. Multiplying this out, we get:

$$c^2\alpha^2(-e^{\alpha D} + e^{-\alpha D}) + c\alpha(\Lambda - \Lambda\alpha D - \Lambda)e^{-\alpha D} + c\alpha(\Lambda\alpha D - \Lambda + \Lambda)e^{\alpha D} -$$

$$\Lambda^2(\alpha D + 1)e^{-\alpha D} - \Lambda^2(\alpha D - 1)e^{\alpha D} = 0 \tag{333}$$

or:

$$-2c^2\alpha^2 sinh(\alpha D) + 2c\alpha^2\Lambda D sinh(\alpha D) - 2\Lambda^2\alpha D cosh(\alpha D)$$

$$+2\Lambda^2 sinh(\alpha D) = 0 \tag{334}$$

Dividing through by $-2\alpha^2 sinh(\alpha D)$:

$$c^2 - \Lambda D c + \frac{\Lambda^2 D}{\alpha} coth(\alpha D) - \frac{\Lambda^2}{\alpha^2} = 0 \tag{335}$$

This quadratic equation has the solutions:

$$c = \frac{\Lambda D}{2} \pm \frac{\Lambda D}{2}[1 - \frac{4}{\alpha D} coth(\alpha D) + \frac{4}{\alpha^2 D^2}]^{1/2} \tag{336}$$

We can rewrite the part in the square root using the identity:

$$cothx = \frac{1}{2}[tanh\frac{x}{2} + coth\frac{x}{2}]$$

Then, pulling in a factor of $\alpha D/2$, the solution is:

$$\begin{aligned} c &= \frac{\Lambda D}{2} \pm \frac{\Lambda}{\alpha}[\frac{\alpha^2 D^2}{4} - \frac{\alpha D}{2} coth(\frac{\alpha D}{2}) - \frac{\alpha D}{2} tanh(\frac{\alpha D}{2}) + 1]^{1/2} \\ &= \frac{\Lambda D}{2} \pm \frac{\Lambda}{\alpha}[(\frac{\alpha D}{2} - coth[\frac{\alpha D}{2}])(\frac{\alpha D}{2} - tanh[\frac{\alpha D}{2}])]^{1/2} \end{aligned} \tag{337}$$

Now for all x, $x > tanh(x)$; so the second factor in the root is always positive. Thus if:

$$\frac{\alpha D}{2} > coth[\frac{\alpha D}{2}] \tag{338}$$

the term inside the root is positive. Then we have two phase speeds, both of which are real. This occurs when α is large. In particular, if $\alpha \gg (2/D)coth(\alpha D/2)$, these phase speeds are:

$$c = 0, \quad \Lambda D \tag{339}$$

So we recover the trapped-wave solutions that we derived first.

If, on the other hand:

$$\frac{\alpha D}{2} < coth[\frac{\alpha D}{2}] \tag{340}$$

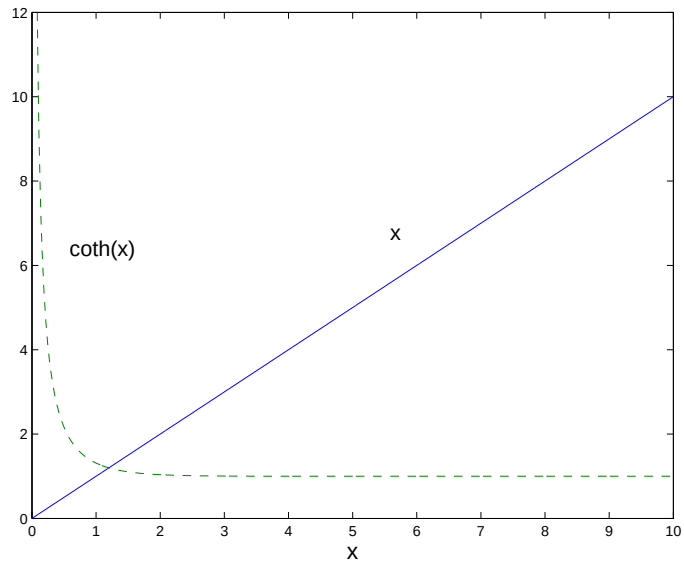

Figure 39: x and $coth(x)$.

the term inside the root of (337) is negative. In Fig. (39), we plot x and $coth(x)$. You can see that x is less for small values of x. Thus the condition for instability is met when α is small. Since we have:

$$\alpha = \frac{N}{f_0}(k^2 + \frac{n^2\pi^2}{L^2})^{1/2}$$

this occurs when the wavenumbers, k and n, are small. Thus large waves are more unstable.

When this condition is met, we can write the phase speed as:

$$c = \frac{\Lambda D}{2} \pm ic_i \tag{341}$$

where

$$c_i = \frac{\Lambda}{\alpha}[(coth[\frac{\alpha D}{2}] - \frac{\alpha D}{2})(\frac{\alpha D}{2} - tanh[\frac{\alpha D}{2}])]^{1/2}$$

Putting this into the wave expression, we have that:

$$\psi \propto e^{ik(x-ct)} = e^{ik(x-\Lambda Dt/2)\mp kc_i t} \tag{342}$$

Thus at each wavenumber there is a growing wave and a decaying wave. The growth rate is equal to kc_i.

The real part of the phase speed is:

$$c_r = \frac{\Lambda D}{2} \tag{343}$$

This is how fast the wave is propagating. We see that the speed is equal to the mean flow speed at the midpoint in the vertical. So it is moving slower than the mean flow speed at the upper boundary and faster than that at the lower boundary. We call the midpoint, where the speeds are equal, the *steering level*.

The growth rate is just kc_i. This is plotted in Fig. (40) for the $n = 1$ mode in the y-direction. We use the following parameters:

$$N = 0.01\, sec^{-1}, \quad f_0 = 10^{-4}\, sec^{-1}, \quad \Lambda = 0.005\, sec^{-1},$$

$$D = 10^4\, m, \quad L = 2 \times 10^6\, m$$

This shear parameter yields a velocity of 50 m/sec at the tropopause height (10 km), similar to the peak velocity in the Jet Stream. For these values, the Eady model yields complex phase speeds, indicating the troposphere is baroclinically unstable.

The growth rate increases from zero as k increases, reaches a maximum value and then goes to zero. For k larger than a critical value, the waves are stable. Thus there is a *short wave cut-off* for the instability. The shorter the waves are, the more trapped they are at the boundaries and thus less able to interact with each other.

The growth rate is a maximum at $k = 1.25 \times 10^{-6}\, m$, corresponding to a wavelength of $2\pi/k = 5027$ km. The wave with this size will grow

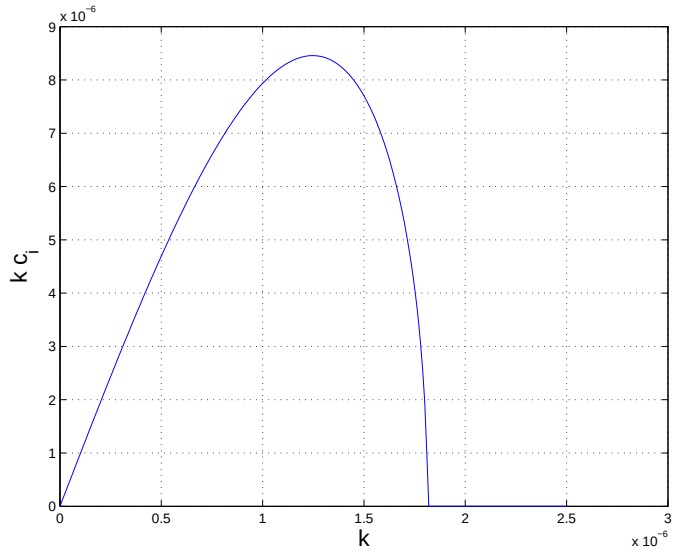

Figure 40: The Eady growth rate as a function of the wavenumber, k.

faster than any other. If we begin with a random collection of waves, this one will dominate the field after a period of time.

The distance from a trough to a crest is one-fourth of a wavelength, or roughly 1250 km for this wave. So this is the scale we'd expect for storms. The maximum value of kc_i is 8.46 $\times 10^{-6}$ sec^{-1}, or equivalently $1/1.4\ day^{-1}$. Thus the growth time for the instability is on the order of a day. So both the length and time scales in the Eady model are consistent with observations of storm development in the troposphere.

Using values typical of oceanic conditions:

$$N = 0.0005\, sec^{-1}, \quad f_0 = 10^{-4}\, sec^{-1}, \quad \Lambda = 0.0001\, sec^{-1},$$

$$D = 5 \times 10^3\, m, \quad L = 2 \times 10^6\, m$$

we get a maximum wavelength of about 100 km, or a quarter wavelength of $25\, km$. Because the deformation radius is so much less in the ocean, the "storms" are correspondingly smaller. The growth times are also roughly ten times longer than in the troposphere. But these values should be taken

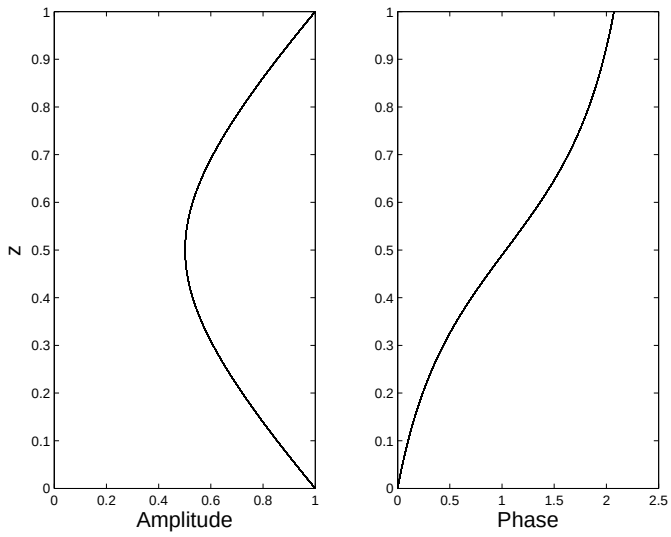

Figure 41: The amplitude (left) and phase (right) of the Eady streamfunction vs. height.

as very approximate, because N in the ocean varies greatly between the surface and bottom.

Let's see what the unstable waves look like. To plot them, we rewrite the solution slightly. From the condition at the lower boundary, we have:

$$(c\alpha + \Lambda)A + (-c\alpha + \Lambda)B = 0$$

So the wave solution can be written:

$$\psi = A[e^{\alpha z} + \frac{c\alpha + \Lambda}{c\alpha - \Lambda} e^{-\alpha z}] sin(\frac{n\pi y}{L}) e^{ik(x-ct)}$$

Rearranging slightly, we get:

$$\psi = A[cosh(\alpha z) - \frac{\Lambda}{c\alpha} sinh(\alpha z)] sin(\frac{n\pi y}{L}) e^{ik(x-ct)} \tag{344}$$

We have absorbed the αc into the unknown A. Because c is complex, the second term in the brackets will affect the phase of the wave. To take this into account, we rewrite the streamfunction thus:

$$\psi = A\Phi(z)sin(\frac{n\pi y}{L})cos[k(x - c_r t) + \gamma(z)]e^{kc_i t} \tag{345}$$

where

$$\Phi(z) = [(cosh(\alpha z) - \frac{c_r \Lambda}{|c|^2 \alpha} sinh(\alpha z))^2 + (\frac{c_i \Lambda}{|c|^2 \alpha} sinh(\alpha z))^2]^{1/2}$$

is the magnitude of the amplitude and

$$\gamma = tan^{-1}[\frac{c_i \Lambda sinh(\alpha z)}{|c|^2 \alpha cosh(\alpha z) - c_r \Lambda sinh(\alpha z)}]$$

is its phase. These are plotted in Fig. (41). The amplitude is greatest near the boundaries. But it is not negligible in the interior, falling to only about 0.5 at the mid-level. Rather than two separate waves, we have one which spans the depth of the fluid. Also, the phase changes with height. So the streamlines *tilt* in the vertical.

We see this in Fig. (42), which shows the streamfunction, temperature, meridional and vertical velocity for the most unstable wave. The streamfunction extends between the upper and lower boundaries, and the streamlines tilt to the west going upward. This means the wave is tilted *against* the mean shear. You get the impression the wave is working against the mean flow, trying to reduce its shear (which it is). The meridional velocity (third panel) is similar, albeit shifted by 90 degrees. The temperature on the other hand tilts toward the east with height, and so is offset from the meridional velocity.

We can also derive the vertical velocity for the Eady wave. Inverting the linearized temperature equation, we have:

$$w = -\frac{f_0}{N^2}(\frac{\partial}{\partial t} + \Lambda z \frac{\partial}{\partial x})\frac{\partial \psi}{\partial z} + \frac{f_0}{N^2}\Lambda \frac{\partial \psi}{\partial x} \tag{346}$$

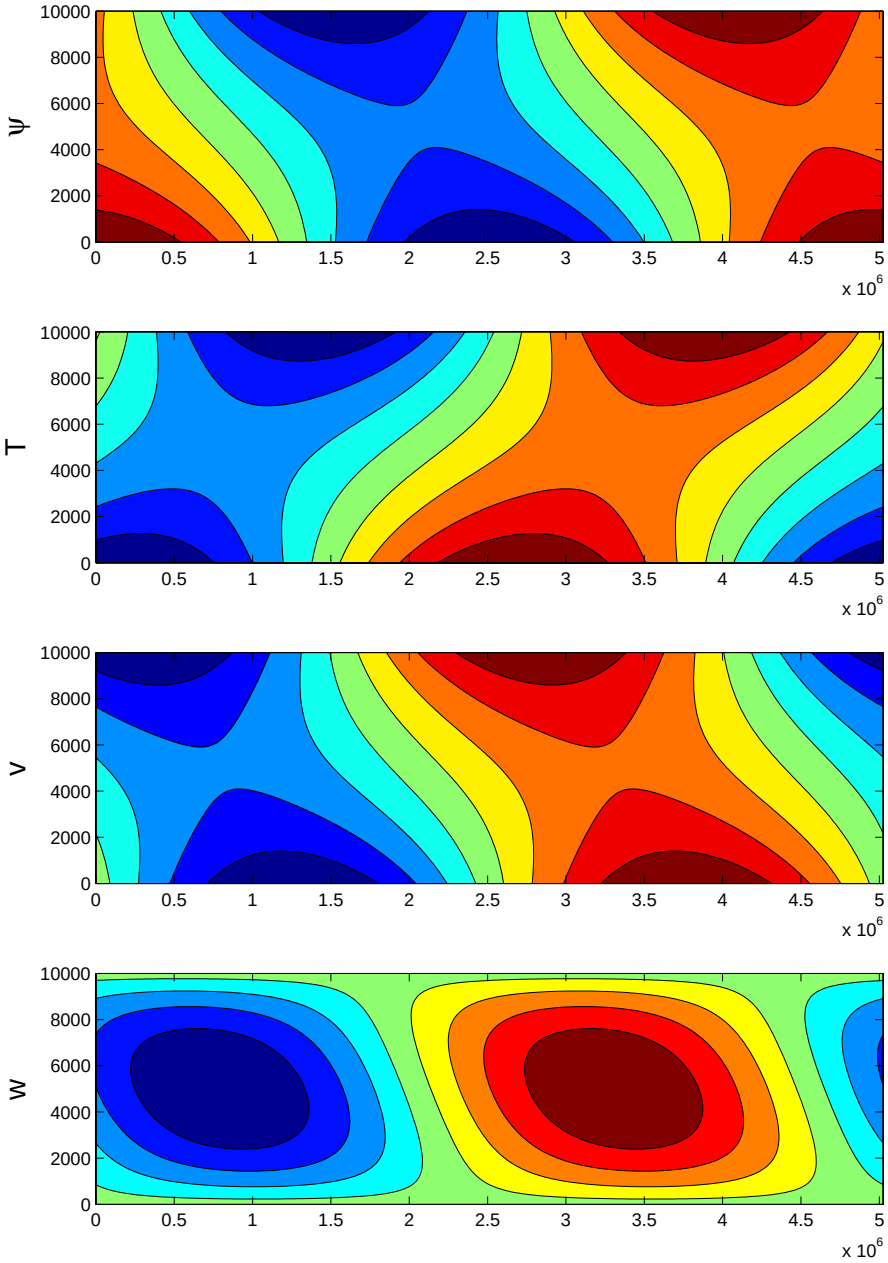

Figure 42: The streamfunction (upper), temperature (second), meridional velocity (third) and vertical velocity for the most unstable wave in the Eady problem.

This is shown in the bottom panel for the most unstable wave. There is generally downward motion when the flow is toward the south and upward motion when toward the north. This fits exactly with our expectations for slantwise convection, illustrated in Fig. (36). Fluid parcels which are higher up and to the north are being exchanged with parcels lower down to the south. So the Eady model captures most of the important elements of baroclinic instability.

However, the Eady model lacks an interior PV gradient (it has no β-effect). Though this greatly simplifies the derivation, the atmosphere possesses such gradients, and it is reasonable to ask how they alter the instability. Interior gradients are considered in both the the Charney (1947) and Phillips (1954) models. Details are given by Pedlosky (1987) and by Vallis (2006).

3.10 ***Problems***

Problem 3.1: Normal modes

In sec. (3.5.1), we solved for the baroclinic modes assuming the the upper and lower boundaries were flat surfaces, with $w = 0$. As a result, the waves have non-zero flow at the bottom. But if the lower boundary is *rough*, a better condition is to assume that the horizontal velocity vanishes, i.e. $u = v = 0$.

Find the modes with this boundary condition. Compare the solutions to those in sec. (3.5.1). What happens to the barotropic mode?

Problem 3.2: Baroclinic Rossby waves

a) What is the phase velocity for a long first baroclinic Rossby wave in

the ocean at 10N? Assume that $N/f = 31.4$ and that the ocean depth is 5 km.

b) What about at 30N?

c) What is the group velocity for long first baroclinic Rossby waves?

d) What would happen to a long wave if it encountered a western wall?

Problem 3.3: Mountain waves

Suppose that a stationary linear Rossby wave is forced by flow over sinusoidal topography with height $h(x) = h_0\, cos(kx)$. Show that the lower boundary condition on the streamfunction can be expressed as:

$$\frac{\partial}{\partial z}\psi = -\frac{hN^2}{f_0} \tag{347}$$

Using this, and an appropriate upper boundary condition, solve for $\psi(x, z)$. What is the position of the crests relative to the mountain tops?

Problem 3.4: Topographic waves

Say we are in a region where there is a steep topographic slope rising to the east, as off the west coast of Norway. The bottom decreases by 1 km over a distance of about 20 km. Say there is a southward flow of 10 cm/sec over the slope (which is constant with depth). Several fishermen have seen topographic waves which span the entire slope. But they disagree about which way they are propagating—north or south. Solve the problem for them, given that $N \approx 10 f_0$ and that we are at 60N.

Problem 3.5: Instability and the Charney-Stern relation

Consider a region with $-1 \leq y < 1$ and $0 \leq z \leq D$. We have the

following velocity profiles:

a) $U = Acos(\frac{\pi z}{D})$
b) $U = Az + B$
c) $U = z(1 - y^2)$

Which profiles are stable or unstable if $\beta = 0$? What if $\beta \neq 0$?

(Note the terms have been non-dimensionalized, so β can be any number, e.g. 1, 3.423, .5, etc.).

Problem 3.6: Eady waves

a) Consider a mean flow $U = -Bz$ over a flat surface at $z = 0$ with no Ekman layer and no upper surface. Assume that $\beta = 0$ and that $N = const.$. Find the phase speed of a perturbation wave on the lower surface.

b) Consider a mean flow with $U = -Bz^2$. What is the phase speed of the wave at $z = 0$ now? What is the mean temperature gradient on the surface?

c) Now imagine a sloping bottom with zero mean flow. How is the slope oriented and how steep is it so that the topographic waves are propagating at the same speed as the waves in (a) and (b)?

Problem 3.7: Eady heat fluxes

Eady waves can flux heat. To see this, we calculate the correlation between the northward velocity and the temperature:

$$\overline{vT} \propto \overline{\frac{\partial \psi}{\partial x}\frac{\partial \psi}{\partial z}} \equiv \frac{1}{L}\int_0^L \frac{\partial \psi}{\partial x}\frac{\partial \psi}{\partial z} dx$$

where L is the wavelength of the wave. Calculate this for the Eady wave and show that it is positive; this implies that the Eady waves transport

warm air northward. You will also find that the heat flux is *independent of height*.

• Hint: use the form of the streamfunction given in (345).

• Hint:

$$\int_0^L sin(k(x-ct))\, cos(k(x-ct))\; dx = 0$$

• Hint:

$$\frac{d}{dz} tan^{-1}\frac{y}{x} = \frac{x^2}{x^2+y^2}\,(\frac{xdy/dz - ydx/dz}{x^2}) = \frac{xdy/dz - ydx/dz}{x^2+y^2}$$

• Hint: The final result will be proportional to c_i. Note that c_i is *positive* for a growing wave.

<u>*Problem 3.8*</u>: Eady momentum fluxes

Unstable waves can flux momentum. The zonal *momentum flux* is defined as:

$$\overline{uv} \propto -\overline{\frac{\partial\psi}{\partial y}\frac{\partial\psi}{\partial x}} \equiv -\frac{1}{L}\int_0^L \frac{\partial\psi}{\partial y}\frac{\partial\psi}{\partial x} dx$$

Calculate this for the Eady model. Why do you think you get the answer you do?

4 Appendices

4.1 Appendix A: Kelvin's theorem

The vorticity equation can be derived in an elegant way. This is based on the *circulation*, which is the integral of the vorticity over a closed area:

$$\Gamma \equiv \iint \vec{\zeta} \cdot \hat{n}\, dA \tag{348}$$

where $\hat{n}$ is the normal vector to the area. From Stoke's theorem, the circulation is equivalent to the integral of the velocity around the circumference:

$$\Gamma = \iint (\nabla \times \vec{u}) \cdot \hat{n}\, dA = \oint \vec{u} \cdot\, \vec{dl} \tag{349}$$

Thus we can derive an equation for the circulation if we integrate the momentum equations around a closed circuit. For this, we will use the momentum equations in vector form. The derivation is somewhat easier if we work with the fixed frame velocity:

$$\frac{d}{dt}\vec{u}_F = -\frac{1}{\rho}\nabla p + \vec{g} + \vec{F} \tag{350}$$

If we integrate around a closed area, we get:

$$\frac{d}{dt}\Gamma_F = -\oint \frac{\nabla p}{\rho} \cdot \vec{dl} + \oint \vec{g} \cdot \vec{dl} + \oint \vec{F} \cdot \vec{dl} \tag{351}$$

The gravity term vanishes because it can be written in terms of a potential (the geopotential):

$$\vec{g} = -g\hat{k} = \frac{\partial}{\partial z}(-gz) \equiv \nabla\Phi \tag{352}$$

and because the closed integral of a potential vanishes:

$$\oint \nabla\Phi \cdot \vec{dl} = \oint d\Phi = 0 \tag{353}$$

So:

$$\frac{d}{dt}\Gamma_F = -\oint \frac{dp}{\rho} + \oint \vec{F} \cdot \vec{dl} \tag{354}$$

Now the circulation, Γ_F, has two components:

$$\Gamma_F = \oint \vec{u}_F \cdot \vec{dl} = \iint \nabla \times \vec{u}_F \cdot \hat{n}\, dA = \iint (\vec{\zeta} + 2\vec{\Omega}) \cdot \hat{n}\, dA \tag{355}$$

As noted above, the most important components of the vorticity are in the vertical. So a natural choice is to take an area which is in the horizontal, with $\hat{n} = \hat{k}$. Then:

$$\Gamma_F = \iint (\zeta + f)\, dA \tag{356}$$

Putting this back in the circulation equation, we get:

$$\frac{d}{dt}\iint (\zeta + f)\, dA = -\oint \frac{dp}{\rho} + \oint \vec{F} \cdot \vec{dl} \tag{357}$$

Now, the first term on the RHS of (357) is zero under the Boussinesq approximation because:

$$\oint \frac{dp}{\rho} = \frac{1}{\rho_c}\oint dp = 0$$

It is also zero if we use pressure coordinates because:

$$\oint \frac{dp}{\rho}|_z \rightarrow \oint d\Phi|_p = 0$$

Thus, in both cases, we have:

$$\frac{d}{dt}\Gamma_a = \oint \vec{F} \cdot \vec{dl} \tag{358}$$

So the absolute circulation can only change under the action of friction. If $\vec{F} = 0$, the absolute circulation is conserved on the parcel. This is Kelvin's theorem.

4.2 Appendix B: Solution in the Ekman layer

The solution for velocities in the Ekman layer is as follows. Substituting the parametrized stresses (86) into the boundary layer equations (78-79) yields:

$$-f_0 v_a = -A_z \frac{\partial^2}{\partial z^2} u_a \tag{359}$$

$$f_0 u_a = -A_z \frac{\partial^2}{\partial z^2} v_a \tag{360}$$

Note that the geostrophic velocity was assumed to be independent of height, so it doesn't contribute to the RHS. If we define a variable χ thus:

$$\chi \equiv u_a + i v_a \tag{361}$$

we can combine the two equations into one:

$$\frac{\partial^2}{\partial z^2}\chi = i\frac{f_0}{A_z}\chi \tag{362}$$

The general solution to this is:

$$\chi = A\, exp(\frac{z}{\delta_E})\, exp(i\frac{z}{\delta_E}) + B\, exp(-\frac{z}{\delta_E})\, exp(-i\frac{z}{\delta_E}) \tag{363}$$

where:

$$\delta_E = \sqrt{\frac{2A_z}{f_0}} \tag{364}$$

This is the "Ekman depth". So the depth of the Ekman layer is determined by the mixing coefficient and by the Coriolis parameter.

To proceed, we need boundary conditions. The solutions should decay moving upward, into the interior of the fluid, as the boundary layer solutions should be confined to the boundary layer. Thus we can set:

$$A = 0$$

From the definition of χ, we have:

$$u_a = Re\{\chi\} = Re\{B\}\, exp(-\frac{z}{\delta_E})\, cos(\frac{z}{\delta_E})$$
$$+Im\{B\}\, exp(-\frac{z}{\delta_E})\, sin(\frac{z}{\delta_E}) \tag{365}$$

and

$$v_a = Im\{\chi\} = -Re\{B\}\, exp(-\frac{z}{\delta_E})\, sin(\frac{z}{\delta_E})$$
$$+Im\{B\}\, exp(-\frac{z}{\delta_E})\, cos(\frac{z}{\delta_E}) \tag{366}$$

Thus there are two unknowns. To determine these, we evaluate the velocities at $z = 0$. To satisfy the no-slip condition, we require:

$$u_a = -u_g, \quad v_a = -v_g \quad at \;\; z = 0$$

Then the total velocity will vanish. So we must have:

$$Re\{B\} = -u_g$$

and

$$Im\{B\} = -v_g$$

Now we must integrate the velocities to obtain the transports. Strictly speaking, the integrals are over the depth of the layer. But as the ageostrophic velocities decay with height, we can just as well integrate them to infinity. So, we have:

$$U_a = -u_g \int_0^\infty exp(-\frac{z}{\delta_E}) \, cos(\frac{z}{\delta_E}) \, dz - v_g \int_0^\infty exp(-\frac{z}{\delta_E}) \, sin(\frac{z}{\delta_E}) \, dz$$
$$= -\frac{\delta}{2}(u_g + v_g) \tag{367}$$

(using a standard table of integrals). Likewise:

$$V_a = u_g \int_0^\infty exp(-\frac{z}{\delta_E}) \, sin(\frac{z}{\delta_E}) \, dz - v_g \int_0^\infty exp(-\frac{z}{\delta_E}) \, cos(\frac{z}{\delta_E}) \, dz$$
$$= \frac{\delta}{2}(u_g - v_g) \tag{368}$$

4.3 Appendix C: Rossby wave energetics

Another way to derive the group velocity is via the energy equation for the waves. For this, we first need the energy equation for the wave. As the wave is barotropic, it has only kinetic energy. This is:

$$E = \frac{1}{2}(u^2 + v^2) = \frac{1}{2}[(-\frac{\partial \psi}{\partial y})^2 + (\frac{\partial \psi}{\partial x})^2] = \frac{1}{2}|\nabla \psi|^2$$

To derive an energy equation, we multiply the wave equation (110) by ψ. The result, after some rearranging, is:

$$\frac{\partial}{\partial t}(\frac{1}{2}|\nabla \psi|^2) + \nabla \cdot [-\psi \nabla \frac{\partial}{\partial t}\psi - \hat{i}\beta \frac{1}{2}\psi^2] = 0 \tag{369}$$

We can also write this as:

$$\frac{\partial}{\partial t}E + \nabla \cdot \vec{S} = 0 \tag{370}$$

So the kinetic energy changes in response to the divergence of an energy *flux*, given by:

$$\vec{S} \equiv -\psi \nabla \frac{\partial}{\partial t}\psi - \hat{i}\beta \frac{1}{2}\psi^2$$

The energy equation is thus like the continuity equation, as the density also changes in response to a divergence in the velocity. Here the kinetic energy changes if there is a divergence in $\vec{S}$.

Let's apply this to the wave. We have

$$E = \frac{k^2 + l^2}{2} A^2 sin^2(kx + ly - \omega t) \tag{371}$$

So the energy varies sinusoidally in time. Let's average this over one wave period:

$$< E > \equiv \int_0^{2\pi/\omega} E\, dt = \frac{1}{4}\left(k^2 + l^2\right) A^2 \tag{372}$$

The flux, $\vec{S}$, on the other hand is:

$$\vec{S} = -(k\hat{i} + l\hat{j})\, \omega\, A^2 cos^2(kx + ly - \omega t) - \hat{i}\beta\, \frac{A^2}{2} cos^2(kx + ly - \omega t) \tag{373}$$

which has a time average:

$$< S > = \frac{A^2}{2}[-\omega(k\hat{i} + l\hat{j}) - \frac{\beta}{2}\hat{i}] = \frac{A^2}{4}[\beta \frac{k^2 - l^2}{k^2 + l^2}\, \hat{i} + \frac{2\beta kl}{k^2 + l^2}\, \hat{j}] \tag{374}$$

Rewriting this slightly:

$$< S >= [\beta \frac{k^2 - l^2}{(k^2 + l^2)^2} \hat{i} + \frac{2\beta kl}{(k^2 + l^2)^2} \hat{j}] \; E \equiv \vec{c}_g < E > \tag{375}$$

So the mean flux is the product of the mean energy and the group velocity, $\vec{c}_g$. It is straightforward to show that the latter is the same as:

$$c_g = \frac{\partial \omega}{\partial k} \hat{i} + \frac{\partial \omega}{\partial l} \hat{j} \tag{376}$$

Since c_g only depends on the wavenumbers, we can write:

$$\frac{\partial}{\partial t} < E > + \vec{c}_g \cdot \nabla < E >= 0 \tag{377}$$

We could write this in Lagrangian form then:

$$\frac{d_c}{dt} < E >= 0 \tag{378}$$

where:

$$\frac{d_c}{dt} = \frac{\partial}{\partial t} + \vec{c}_g \cdot \nabla \tag{379}$$

In words, this means that the energy is conserved when moving at the group velocity. The group velocity then is the relevant velocity to consider when talking about the energy of the wave.

4.4 Appendix D: Fjørtoft's criterion

This is an alternate condition for barotropic instability, derived by Fjørtoft (1950). This follows from taking the real part of (216):

$$(\hat{\psi}_r \frac{\partial^2}{\partial y^2} \hat{\psi}_r + \hat{\psi}_i \frac{\partial^2}{\partial y^2} \hat{\psi}_i) - k^2 |\hat{\psi}|^2 + (U - c_r) \frac{|\hat{\psi}|^2}{|U - c|^2} \frac{\partial}{\partial y} q_s = 0 \tag{380}$$

If we again integrate in y and rearrange, we get:

$$\int_0^L (U - c_r)\frac{|\hat{\psi}|^2}{|U-c|^2}\frac{\partial}{\partial y}q_s =$$

$$-\int_0^L (\hat{\psi}_r\frac{\partial^2}{\partial y^2}\hat{\psi}_r + \hat{\psi}_i\frac{\partial^2}{\partial y^2}\hat{\psi}_i)dy + \int_0^L k^2|\hat{\psi}|^2 dy \tag{381}$$

We can use integration by parts again, on the first term on the RHS. For instance,

$$\int_0^L \hat{\psi}_r\frac{\partial^2}{\partial y^2}\hat{\psi}_r\, dy = \hat{\psi}_r\frac{\partial}{\partial y}\hat{\psi}_r|_0^L - \int_0^L (\frac{\partial}{\partial y}\hat{\psi}_r)^2\, dy \tag{382}$$

The first term on the RHS vanishes because of the boundary condition. So (380) can be written:

$$\int_0^L (U-c_r)\frac{|\hat{\psi}|^2}{|U-c|^2}\frac{\partial}{\partial y}q_s\, dy = \int_0^L (\frac{\partial}{\partial y}\hat{\psi}_r)^2 + (\frac{\partial}{\partial y}\hat{\psi}_i)^2 + k^2|\hat{\psi}|^2\, dy \tag{383}$$

The RHS is always *positive*. Now from Rayleigh's criterion, we know that:

$$\int_0^L \frac{|\hat{\psi}|^2}{|U-c|^2}\frac{\partial}{\partial y}q_s\, dy = 0 \tag{384}$$

So we conclude that:

$$\int_0^L (U - c_r)\frac{|\hat{\psi}|^2}{|U-c|^2}\frac{\partial}{\partial y}q_s > 0 \tag{385}$$

We don't know what c_r is, but the condition states essentially that this integral must be positive for *any* real constant, c_r.

To test this, we can just pick a value for c_r. The usual procedure is to pick some value of the velocity, U; call that U_s. A frequent choice is to use the value of U at the point where $\frac{\partial}{\partial y}q_s$ vanishes; Then we must have that:

$$(U - U_s)\frac{\partial}{\partial y}q_s > 0 \tag{386}$$

somewhere in the domain. If this fails, the flow is stable.

Fjørtoft's criterion is also a necessary condition for instability. It represents an additional constraint to Rayleigh's criterion. Sometimes a flow will satisfy the Rayleigh criterion but not Fjørtoft's—then the flow is stable. Interestingly, it's possible to show that Fjørtoft's criterion requires the flow have a relative vorticity maximum somewhere in the domain interior, not just on the boundaries.

4.5 Appendix E: QGPV in pressure coordinates

The PV equation in pressure coordinates is very similar to that in z-coordinates. First off, the vorticity equation is given by:

$$\frac{d_H}{dt}(\zeta + f) = -(\zeta + f)(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) \tag{387}$$

Using the incompressibility condition (42), we rewrite this as:

$$\frac{d_H}{dt}(\zeta + f) = (\zeta + f)\frac{\partial \omega}{\partial p} \tag{388}$$

The quasi-geostrophic version of this is:

$$\frac{d_g}{dt}(\zeta + f) = f_0\frac{\partial \omega}{\partial p} \tag{389}$$

where $\zeta = \nabla^2\Phi/f_0$.

To eliminate ω, we use the potential temperature equation (14). For simplicity we assume no heating, so the equation is simply:

$$\frac{d\theta}{dt} = 0 \tag{390}$$

We assume:

$$\theta_{tot}(x,y,p,t) = \theta_0(p) + \theta(x,y,p,t)\ , \quad |\theta| \ll |\theta_0|$$

where θ_{tot} is the full temperature, θ_0 is the "static" temperature and θ is the "dynamic" temperature. Substituting these in, we get:

$$\frac{\partial\theta}{\partial t} + u\frac{\partial\theta}{\partial x} + v\frac{\partial\theta}{\partial y} + w\frac{\partial}{\partial p}\theta_0 = 0 \tag{391}$$

We neglect the term $\omega\partial\theta/\partial p$ because it is much less than the term with θ_0.

The geopotential is also dominated by a static component:

$$\Phi_{tot} = \Phi_0(p) + \Phi(x,y,p,t)\ , \quad |\Phi| \ll |\Phi_0| \tag{392}$$

Then the hydrostatic relation (44) yields:

$$\frac{d\Phi_{tot}}{dp} = \frac{d\Phi_0}{dp} + \frac{d\Phi}{dp} = -\frac{RT_0}{p} - \frac{RT'}{p} \tag{393}$$

and where:

$$T_{tot} = T_0(p) + T(x,y,p,t)\ , \quad |T| \ll |T_0| \tag{394}$$

Equating the static and dynamic parts, we find:

$$\frac{d\Phi}{dp} = -\frac{RT'}{p} \tag{395}$$

Now we need to rewrite the hydrostatic relation in terms of the potential temperature. From the definition of potential temperature, we have:

$$\theta = T\,(\frac{p_s}{p})^{R/c_p}, \quad \theta_0 = T_0\,(\frac{p_s}{p})^{R/c_p}$$

where again we have equated the dynamic and static parts. Thus:

$$\frac{\theta}{\theta_0} = \frac{T}{T_0} \tag{396}$$

So:

$$\frac{1}{T_0}\frac{d\Phi}{dp} = -\frac{RT}{pT_0} = -\frac{R\theta}{p\theta_0} \tag{397}$$

So, dividing equation (391) by θ_0, we get:

$$(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y})\,\frac{\theta'}{\theta_0} + \omega\,\frac{\partial}{\partial p}ln\theta_0 = 0 \tag{398}$$

Finally, using (397) and approximating the horizontal velocities by their geostrophic values, we obtain the QG temperature equation:

$$(\frac{\partial}{\partial t} + u_g\frac{\partial}{\partial x} + v_g\frac{\partial}{\partial y})\,\frac{\partial\Phi}{\partial p} + \sigma\omega = 0 \tag{399}$$

The parameter:

$$\sigma(p) = -\frac{RT_0}{p}\frac{\partial}{\partial p}ln(\theta_0)$$

reflects the static stratification and is proportional to the Brunt-Vaisala frequency (sec. 3.1). We can write this entirely in terms of Φ and ω:

$$(\frac{\partial}{\partial t} - \frac{1}{f_0}\frac{\partial}{\partial y}\Phi\frac{\partial}{\partial x} + \frac{1}{f_0}\frac{\partial}{\partial x}\Phi\frac{\partial}{\partial y})\,\frac{\partial\Phi}{\partial p} + \omega\sigma = 0 \tag{400}$$

As in sec. (3.2), we can combine the vorticity equation (389) and the temperature equation (400) to yield a PV equation. In pressure coordinates, this is:

$$(\frac{\partial}{\partial t} - \frac{1}{f_0}\frac{\partial}{\partial y}\Phi\frac{\partial}{\partial x} + \frac{1}{f_0}\frac{\partial}{\partial x}\Phi\frac{\partial}{\partial y})\,[\frac{1}{f_0}\nabla^2\Phi + \frac{\partial}{\partial p}(\frac{f_0^2}{\sigma}\frac{\partial\psi}{\partial p}) + \beta y] = 0 \tag{401}$$