Computer problems

connected to lecture notes on:

Fundamentals of Atmospheres and Oceans on Computers

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Fall 2008

PREFACE

Presented are computer problems connected to the lecture notes on "Fundamental of Atmospheres and Oceans on Computers". A certain number of these are compulsary and must be submitted for evaluation. The students may, however, decide which problems to turn in themselves. The mandatory problems replace the mid term exam. The compulsory problems must be approved before the student is allowed to take the (oral) exam.

Solving atmospheric and oceanographic problems, sometimes referred to as metocean problems, using numerical methods consist of three stages. The first is to develop a numerical analogue of the continuous mathematical problem formulated based on the physical problem at hand. The second is to construct a computer code (or program) on a given computer. This stage includes debugging and verification. Debugging means to check that the code has no formal errors, and that the code is a true replica of the numerical analogue. Verification is a bit harder. It implies checking the results against what is known about the true solution, e.e.g, analytic solutions. The final and third stage is to be able to visualize the results, and finally to discuss them in light of what physics the solution represent. All three stages are equally important. I therefore strongly recommend all students to solve as many of the problems as possible in order to get the neccessary hands-on experience and insight into stages two and three.

The problems will be continuously amended to adjust to the lecture notes. The author would like to thank the many colleagues who has contributed to develop these exercises over the years, and to the many students for pointing out misprints and other mistakes.
Good luck!

Blindern, November 21, 2008
Lars Petter Røed (sign.)

Contents

List of Figures

Computer problem 1:
Truncation error in a recursion formula with two terms

a.

Let

$$\pi = 4\arctan(1),\ Z_1 = \pi, \text{ and } S_1 = \pi. \tag{1}$$

Compute

$$Z_{i+1} = 3.1Z_i - 2.1Z_1 \text{ and } S_{i+1} = \left(\frac{9.}{5.}\right) S_i - \left(\frac{4.}{5.}\right) S_1 \tag{2}$$

for $i = 1(1)100$. Compute also the *relative* error (e.g., $\epsilon_i = Z_{i+1} - Z_i$) for each i. Write π, Z_i, S_i and the relative error in percent. The output should be readable and self explanatory, e.g., should have headings for each column. Enclose the program code and the printout when you turn in your paper. Do the problem on different platforms (from handholds to portables, PCs and supercomputers) available to you. Experiment by using different constants in the recursion formulas. Does it make a difference in the answer?

The purpose of the exercise is twofold: 1) It is simple enough to enable you to refresh your knowledge of FORTRAN or FORTRAN skills without having to write lengthy codes, and 2) it demonstrates the dramatic consequences of the presence of insignificant truncation errors always present in numerical computations.

b.

Show analytically why the recursion formulas for Z_i and S_i do not compute π correctly.

Computer problem 2:
Diffusive processes in the ocean and atmosphere

In the ocean and atmosphere the vertical (and horizontal) heat exchange is dominantly a turbulent process. Commonly the vertical turbulent heat exchange is parametrized as a diffusion process, that is, governed by the equation

$$\partial_t \theta + \partial_z F = 0, \tag{3}$$

where $\theta = \theta(z,t)$ is the potential temperature, z is the vertical coordinate, t is time and F is the vertical component of the diffusive flux vector.

In its simplest form the heat exchange is parametrized as a down the gradient diffusion process. Thus

$$F = -\kappa \partial_z \theta, \tag{4}$$

where κ is the diffusion coefficient, and (3) becomes

$$\partial_t \theta = \partial_z \left(\kappa \partial_z \theta \right). \tag{5}$$

Note that since the heat exchange is due to turbulent mixing the diffusion or mixing coefficient κ is normally a function of space and time. We underscore that θ can be any active tracer like temperature, humidity and salinity, or a passive tracer like CO_2.

In this exercise we assume that the diffusion coefficient is constant. Under these circumstances (5) reduces to

$$\partial_t \theta = \kappa \partial_z^2 \theta. \tag{6}$$

You are are asked to solve (6) numerically for two applications. The first is associated with mixing in the atmospheric planetary boundary layer, while the second is associated with mixing in the oceanic mixed layer. The mixing or diffusion coefficient in the two spheres are dramatically different. While the mixing coefficient for the atmospheric boundary layer is $\kappa = 30$ m^2s^{-1}, the similar mixing coefficient for the oceanic mixed layer is a factor 10^{-4} less, namely $\kappa = 3 \cdot 10^{-3}$ m^2s^{-1} (*Gill*, 1982).

In this exercise we assume that the fluid is contained between two fixed z-levels, where the boundary conditions replace (6). Regarding the atmospheric application we assume that the bottom level is located at $z = 0$, while the top level $z = D$ is at the top of the planetary mixed layer. For the ocean application the bottom level is the mixed layer at depth $z = -D$ while the top level is the surface at $z = 0$. Thus $z \in < 0, D >$ for the atmospheric application and $z \in < -D, 0 >$ for the oceanic application. For both cases you are asked to find the numerical solution for $t \in [0, N\Delta t]$ where $N = 401$, that is, $t = n\Delta t, n = 0(1)N$, where n is the time step counter, Δt is the time step and N is the total number of time levels. We assume that the mixed layers are $D = 100$ m deep in both cases. Furthermore we let $D = (J_{max} - 1)\Delta z$ where Δz is the space increment[1] and $J_{max} = 27$ is the total number of grid points including the two boundary points at the bottom and top levels.

[1]Also commonly referred to as the mesh or grid size.

a.

Develop a numerical scheme (or finite difference approximation) that is forward in time and centered in space for (6). Show that the derived scheme is stable under the condition $K \leq \frac{1}{2}$, where

$$K = \frac{\kappa \Delta t}{\Delta z^2}. \tag{7}$$

b.

Next develop a finite difference approximation for (6) that is centered in both time and space, and show that this algorithm is unconditionally unstable in a numerical sense.

c.

Let $K = 0.45$. Compute the time step you will have to use for the atmospheric and oceanic applications, respectively. Discuss why there is such a huge difference and its possible consequences.

d.

Consider first an atmospheric application. We assume that initially the temperature distribution is a sinusoidal function of height as shown in Figure 1. Thus we assume

$$\theta(z, 0) = \theta_0 \sin(\frac{\pi z}{D}), \quad z \in [0, D], \tag{8}$$

where $\theta_0 = 10^o$C. Furthermore we let the temperature at the bottom level (or surface) $z = 0$ and the top level $z = D$ be fixed at the freezing point for all times, that is,

$$\theta(0, t) = \theta(D, t) = 0^o\text{C} \quad \forall t. \tag{9}$$

Use the stable forward in time, centered in space scheme developed under a. to find the numerical solution to (6) for two cases; one with $K = 0.45$ and a second with $K = 0.55$. Plot the results for $n = 0$, $n = 50$, $n = 100$ and $n = 200$ in which the height and temperature are made dimensionless by dividing through by D and θ_0, respectively.

Assess and discuss the solutions. Explain in particular why the solution for $K = 0.55$ develops a “saw tooth” pattern.

e.

Consider next an oceanographic application. In this case we assume that the initial condition is

$$\theta(z, 0) = 0^o\text{C}, \quad z \in [0, D], \tag{10}$$

throughout the water column. Thus the initial condition is the trivial solution to (6). In contrast to the atmospheric application the diffusion process is generated by letting the ocean surface be

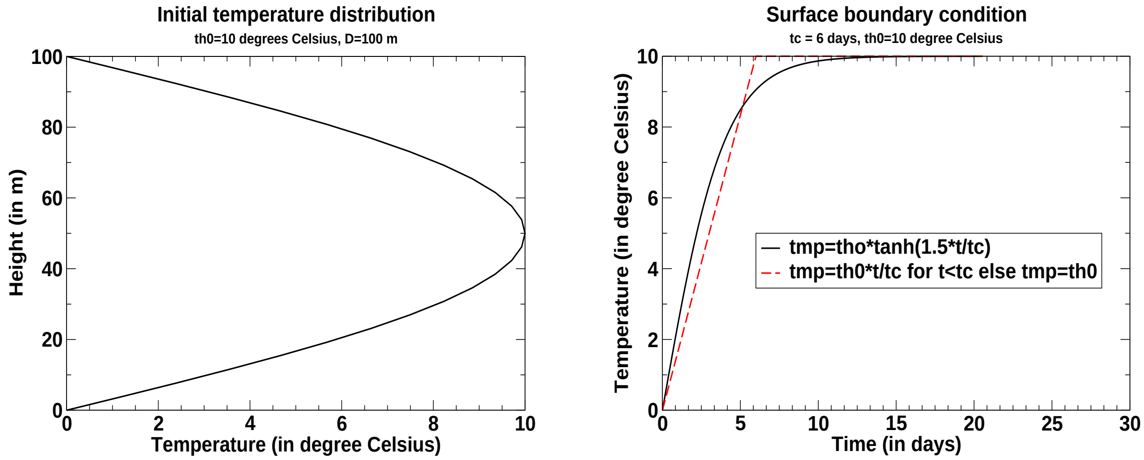

Figure 1: Left-hand panel displays the initial temperature distribution according to (8), while the right-hand panel shows the time evolution of the surface temperature according to (11) and (12), respectively.

heated from above. Specifically, we let the boundary condition at $z = 0$ increase from zero to a fixed temperature $\theta_0 = 10^o$C after som finite time. This can be achieved either by letting the boundary condition be specified according to

$$\theta(0,t) = \theta_0 \left\{ \begin{array}{lll} \frac{t}{t_c} & ; & 0 < t < t_c \\ 1 & ; & t \geq t_c \end{array} \right. , \tag{11}$$

where $t_c = 6$ days determines how fast the surface temperature reaches its final temperature θ_0 (cf. the left-hand panel of Figure 1), or by using a hyperbolic tangent (a good function), that is,

$$\theta(0,t) = \theta_0 \tanh(\gamma \frac{t - t_c}{t_c}) \tag{12}$$

where $\gamma = 1.5$ together with t_c determines how fast the temperature approaches its final temperature (cf. the right-hand panel of Figure 1). At the bottom of the ocean mixed layer $z = -D$ the temperature is fixed at the freezing point. Thus

$$\theta(-D,t) = 0^o\mathrm{C} \tag{13}$$

Again choose the time step so that $K = 0.45$. Plot the results for $n = 0$, $n = 100$, $n = 200$ and $n = 400$. Use either (11) or (12) as your surface boundary condition.

Assess and discuss the solution. In particular you are asked to compare the solution with the steady state solution to (6), that is, the solution as $t \rightarrow \infty$ given the above initial and boundary conditions.

Computer problem 3:
Advection in atmosphere and oceans

Since tracers such as temperature, salinity and humidity has a decisive impact on the dynamics of the atmosphere and ocean through its influence on the pressure distribution through density, advection (transport) of these tracers is of zero order importance. Moreover, transport of contaminants in the ocean and atmosphere is one crucial element when discussing environmental issues. For instance emissions of radionuclide in one location are transported via atmospheric and oceanic circulation patterns to quite other locations. Other examples are trans-boundary advection of chemical substances such as sulfur (mostly atmosphere) and nutrients (mostly ocean). In the ocean advection processes are also of crucial importance regarding search and rescue, oil drift, and drifting objects (e.g, fish larvae, rafts, human beings, ship wrecks, etc.)

Commonly all transport and spreading of the above are governed by an advection equation, say

$$\partial_t \theta + \nabla \cdot (\mathbf{v}\theta) = \nabla \cdot (\kappa \nabla \theta) \tag{14}$$

where θ is the concentration of the tracer, $\mathbf{v}$ is the three-dimensional wind or current vector and κ is the mixing or diffusion coefficient. We note that commonly the transport is associated with the advection part of (14), while the spreading is associated with the mixing part of (14). While the mixing was exemplified in Computer Problem #2 we focus on the advection part in this Computer Problem #3. Thus we will neglect the mixing part in the remainder of this problem except in the very last question.

To make the problem as simple as possible, but no simpler, we reduce the advection problem to one dimension in space. Furthermore we let the advection speed be constant, say $\mathbf{v} = u_0\mathbf{i}$. Thus we will consider numerical solutions to the equation

$$\partial_t \theta + u_0 \partial_x \theta = 0 \quad \text{for} \quad x \in < 0, L_x > \tag{15}$$

with appropriate boundary and initial conditions. To this end we will make use of three schemes, namely the *leapfrog scheme*,

$$\frac{\theta_j^{n+1} - \theta_j^{n-1}}{2\Delta t} + u_0 \frac{\theta_{j+1}^n - \theta_{j-1}^n}{2\Delta x} = 0, \tag{16}$$

the *upwind scheme* (or *upstream scheme*)

$$\frac{\theta_j^{n+1} - \theta_j^n}{\Delta t} + \frac{F_j^n - F_{j-1}^n}{\Delta x} = 0, \tag{17}$$

where

$$F_j^n = \frac{1}{2}(u_0 + |u_0|)\theta_j^n + \frac{1}{2}(u_0 - |u_0|)\theta_{j+1}^n \tag{18}$$

and the *Lax-Wendroff scheme* consisting of the two steps

$$\frac{\theta_j^{n+1} - \frac{1}{2}\left(\theta_{j+1}^n + \theta_{j-1}^n\right)}{\Delta t} = -u_0 \frac{\theta_{j+1}^n - \theta_{j-1}^n}{2\Delta x}. \tag{19}$$

$$\frac{\theta_j^{n+2} - \theta_j^n}{2\Delta t} = -u_0 \frac{\theta_{j+1}^{n+1} - \theta_{j-1}^{n+1}}{2\Delta x}. \tag{20}$$

for $j = 1(1)J$ where $x = 0$ is associated with $j = 1$ and $x = L$ with $j = J$.

We note that in contrast to the leapfrog scheme, which is centered in time and space, the upwind scheme is a simple forward in time and one-sided in space scheme. The Lax-Wendroff scheme is special. It consists of a diffusive step that alternates with a leapfrog step. Note that the diffusive step looks like a forward in time, centered in space scheme, except that θ_j^n is replaced by $\frac{1}{2}(\theta_{j+1}^n + \theta_{j-1}^n)$.

Part 1:

a.

Show that all schemes are numerically stable under the condition

$$|C| \leq 1 \quad \text{where} \quad C = \frac{u_0 \Delta t}{\Delta x} \tag{21}$$

is the Courant number.

b.

Show that a forward in time, centered in space (FTCS) finite difference approximation applied to (15) results in an unconditionally unstable scheme.

c.

Show that the upwind scheme (17) inherently includes a numerical diffusion with a diffusion coefficients given by

$$\frac{1}{2}|u_0|\Delta x(1 - C) \tag{22}$$

where C is the Courant number given in (21).

d.

Show that the $\mathcal{O}(\Delta x^2)$ and $\mathcal{O}(\Delta t^2)$ terms neglected in the Lax-Wendroff scheme are[2]

$$-\frac{2}{3}u_0 \Delta x^2 (1 - C^2) \left(\partial_x^3 \theta\right)_j^n , \tag{23}$$

and hence that the Lax-Wendroff scheme avoids the low order diffusion inherent in the upstream scheme diffusion.

[2]Hint: Combine (19) and (20) by substituting (19) into (20) and then plug in the respective Taylor series to the order needed.

Part 2:

To solve (15) we need conditions at the two boundaries $x = 0$ and $x = L_x$, as well an initial condition at time $t = 0$. At $x = 0$ and $x = L_x$ we will make use of periodic boundary conditions. Hence we require that $\theta(x,t) = \theta(x + L_x, t)$. The numerical analogue is $\theta_1^n = \theta_J^n$, $\theta_2^n = \theta_{J+1}^n$, and so forth, where $j = 1$ is associated with $x = 0$ and $j = J$ is associated with $x = L_x$. We let the initial condition be a Gaussian bell, that is, initially (at time $t = 0$) the tracer concentration has the distribution

$$\theta(x,0) = \theta_0 e^{-\left(\frac{x-x_0}{\sigma}\right)^2} \quad ; \quad \forall x \in < 0, L_x >, \tag{24}$$

where θ_0 is the maximum tracer concentration, x_0 is the position of the initial maximum tracer concentration, σ is a measure of the width of the bell (the larger σ is, the wider the bell is) and L_x is the width of the computational domain.

Below we will make use of dimensionless variables. Thus we introduce θ', x' and t' as our dimensionless variables. Furthermore, since the advection speed is constant the dimensionless speed becomes $u_0' = 1$. Scaling the tracer concentration by its maximum initial concentration θ_0 and x by the width of the computational domain L_x we get

$$\theta' = \frac{\theta}{\theta_0}, \quad x' = \frac{x}{L_x}, \quad t' = \frac{t}{T} \tag{25}$$

where T is some as yet unknown time scale. If we substitute these dimensionless variables into (15) we get

$$R\partial_{t'}\theta' + \partial_{x'}\theta' = 0 \quad \text{for} \quad x \in < 0, 1 > \tag{26}$$

where the dimensionless number R is

$$R = \frac{L_x}{u_0 T} = 1 \tag{27}$$

and hence that $T = L_x/u_0$.

In summary, dropping primes, the system you are required to solve is the dimensionless advection equation

$$\partial_t\theta + \partial_x\theta = 0 \quad \text{for} \quad x \in < 0, 1 > \tag{28}$$

subject to the initial condition

$$\theta(x,0) = e^{-\left(\frac{x-x_0}{\sigma}\right)^2} \quad ; \quad \forall x \in < 0, 1 >, \tag{29}$$

at time $t = 0$, and the the periodic boundary condition at $x = 0, 1$.

You are asked to perform two experiments one in which the width of the Gaussian bell specified in (29) is wide ($\sigma = L_x/10 = 0.1$), and one in which it is narrow ($\sigma = 0.001$). The latter experiment is constructed to display the peculiarities of the various schemes in the presence of fronts (large gradients).

e.

Solve (28) using each of the three numerical schemes above subject to the specified initial condition (29) and the periodic boundary condition. Let $x_j = (j-1)\Delta x$ and $J = 101$, and let $t^{(n)=n\Delta t}$. Stop the computations after 10 cycles. Do one experiment with the Courant number $C = 0.5$ and another with $C = 1$. Please also feel free to experiment with other Courant numbers $\frac{1}{2} < C < 1$. Plot the solution after 1/2, 1, 2, 5 and 10 cycles together with the initial tracer distribution for each of the two Courant number values. Plot the graphs for each scheme together in one graph (six graphs for each scheme) for the two Courant numbers, that is, a total of six plots (3 schemes x 2 Courant numbers).

f.

Discuss the solutions based on the plots. What characterizes the solution as it evolves in time? Which of the solutions are diffusive and which are dispersive? What are the characteristics of these processes?

g.

Finally we consider the simplified advection-diffusion equation

$$\partial_t \theta + u_0 \partial_x \theta = \kappa \partial_x^2 \theta. \tag{30}$$

where the advection speed u_0 as well as the mixing coefficient κ is constant. Construct a scheme that is stable and consistent for (30) and state the stability condition. Explain your choices.

Computer problem 4:
Yoshida's equatorial jet current

We consider an "infinite" equatorial ocean consisting of two immiscible layers with a density difference $\Delta\rho$ (Figure 2). The density of the lower layer equals the reference density ρ_0. The lower layer is thick with respect to the upper layer. At time $t = 0$ the ocean is at rest, at which time the thickness of the upper layer equals its equilibrium depth H. At this particular time the ocean is forced into motion by turning on a westerly wind (wind from the west).

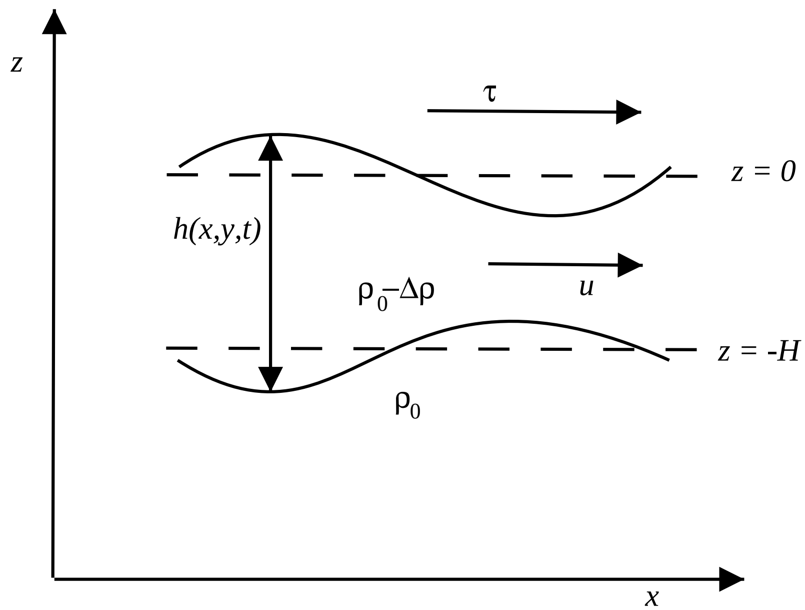

Figure 2: Sketch of a reduced gravity ocean model consisting of two layers with a density difference given by $\Delta\rho$.

The governing equations of such a "reduced gravity" model of the ocean, is

$$\partial_t u - \beta y v = \frac{\tau^x}{\rho_0 H} \tag{31}$$

$$\partial_t v + \beta y u = -g' \partial_y h \tag{32}$$

$$\partial_t h + H \partial_y v = 0 \tag{33}$$

Here $u = u(y,t)$ and $v = v(y,t)$ are the respectively the east-west and north-south components of the velocity in a Cartesian coordinate system (x, y, z) with x directed eastward along the equator, y directed northwards with $y = 0$ at the equator, and z directed along the negative gravitational direction as displayed in Figure 2. The impact of the Earth's rotation is given by the Coriolis parameter $f = 2\Omega \sin\phi$ where Ω is the Earth's rotation rate and ϕ is the latitude. The westerly wind is given by the wind stress component τ^x which is fixed in time. Furthermore, we define the reduced gravity by $g' \equiv g(\Delta\rho_0/\rho)$ where g is the gravitational acceleration. The instantanuous thickness of the upper layer is given by $h = h(y,t)$.

Note that at the equator $f = 0$ and that it increases with increasing latitude. A simplified parameterization of this effect is through the so called β-plane approximation,

$$f = \beta y, \quad \text{hvor} \quad \beta = \partial_y f|_{y=0}. \tag{34}$$

We note the β is just a measure of the first term in a Taylor series of f at the equator. Thus it represents the first order effect effect of the impact of the change in the Earth's rotation rate with latitude.

Part 1:

a.

Show that the inertial oscillation[3] is eliminated by neglecting $\partial_t v$ in (32).

b.

Show that the system of equations (31) - (33) reduces to the ordinary differential equation

$$L^4 \partial_y^2 v - y^2 v = aLy \tag{35}$$

where

$$L = \sqrt{\frac{c}{\beta}}, \quad a = \frac{\tau^x}{\rho_0 \beta L H}, \quad c = \sqrt{g' H} \tag{36}$$

under the condition that the inertial oscillation is eliminated.

c.

Explain why we are allowed to specify two boundary conditions. In the following we will assume that they are $v|_{y=0} = 0$ and $v|_{y\to\infty} = 0$.

d.

We make (35) dimensionless by letting $y = L\hat{y}$, $(u,v) = a(\hat{u},\hat{v})$, and $t = (\beta L)^{-1}\hat{t}$. Use the a direct elliptic solver, e.g., Gauss elimination, to solve the dimensionless expression of (35). Let $\Delta y = 0.1$ and plot $\hat{v}$ and $\hat{u}$ at time $\hat{t}$ = 1 as a function of $\hat{y}$ from $\hat{y} = 0$ to $\hat{y} = 8$. We note that $v|_{y\to\infty} = 0$ and hence that $\hat{v}$ is different from zero at $\hat{y} = 8$. Explain how make use of the condition that $v|_{y\to\infty} = 0$.

[3] An oscillation in which the frequency equals the inertial frequency f.

e.

Discuss the numerical solution. Let $\tau^x = 0.1Pa$, $\beta = 2. \cdot 10^{-11}(ms)^{-1}$, $L = 275$km, $\rho = 10^3 kgm^{-3}$ and $H = 200m$. What is the maximum current in the equatorial jet for $\hat{t} = 1$?

f.

Solve (35) analytically. Hint: Make a series using Hermitian polynomials (se for instance *Abramowitz and Stegun*, 1965).

Computer problem 5: Geostrophic adjustment - Rossby and gravity waves

One of the most important and strongest balances in the atmosphere and ocean, confirmed over and over again by observations, is geostrophy. When the fluid motion is in geostrophic balance we have a balance between the Coriolis acceleration and the pressure forcing, that is,

$$f\mathbf{k} \times \mathbf{u}_g = -\frac{1}{\rho_0}\nabla_H p, \quad \text{or} \quad v_g = \frac{1}{\rho_0 f}\partial_x p, \quad u_g = -\frac{1}{\rho_0 f}\partial_y p, \tag{37}$$

where $f = 2\Omega \sin\phi$ is the Coriolis parameter, $\mathbf{k}$ is the unit vector along the vertical z-axis, $\mathbf{u}_g$ is the (horizontal) geostrophic velocity with components u_g, v_g along the x-axis and y-axis, respectively, ρ_0 is the density, $\nabla_H = \mathbf{i}\partial_x + \mathbf{j}\partial_y$ is the horizontal component of the three-dimensional del-operator, and p is pressure. Note that (37) contains three unknowns, namely p, u_g, and v_g, but only two equations. Hence the system is undetermined. Only by specifying one of them, say the pressure p, can we find the two other variables.

A fundamental question is therefore how the atmosphere and ocean actually adjust from an unbalanced state to one in geostrophic balance under gravity. This problem, coined geostrophic adjustment (under gravity), was first raised by Carl Gustav Rossby[4] back in the 1930s (*Rossby*, 1937, 1938), and is the background for this computer problem. As usual we make the problem as simple as possible, but no simpler. Thus, we consider the one-dimensional (1-D) shallow water equations for this purpose. It also conveniently serves the purpose of illustrating solution modes, the role of initial conditions and the use of an open boundary condition (FRS).

We recall that the shallow water equations assumes a hydrostatic balance and hence that $p = \rho_0 g h$, where h is the geopotential height. Thus the governing equations, inherently non-linear, are

$$\partial_t h = -\nabla_H \cdot (h\mathbf{u}), \tag{38}$$
$$\partial_t \mathbf{u} = -f\mathbf{k} \times \mathbf{u} - \mathbf{u} \cdot \nabla_H \mathbf{u} - g\nabla_H h, \tag{39}$$

where the Coriolis parameter is $f = 1.26 \cdot 10^{-4}\text{s}^{-1}$ (corresponding to its value at 60^{o}N). As is common we may regard h as the geopotential height of a pressure surface in the atmosphere and as the depth of a water column in the ocean. The equilibrium height of h in the atmosphere is associated with a pressure surface of ≈ 900hPa, while the equilibrium depth in the ocean is commonly ≈ 1km.

[4]Carl-Gustaf Arvid Rossby (1898 - 1957) was a Swedish-U.S. meteorologist who pioneered explaining the large-scale motions of the atmosphere in terms of fluid mechanics. Rossby came into meteorology and oceanography while studying under Vilhelm Bjerknes in Bergen in 1919, where Bjerknes' group was developing the concept of a polar front (the Bergen School of Meteorology). His name is associated with various quantities and phenomena in meteorology and oceanography, e.g., the Rossby number, Rossby's radius of deformation, and Rossby waves.

Part 1:

a.

Show that by introducing $\mathbf{U} = h\mathbf{u}$ and $h = h$ as new variables (38) and (39) become

$$\begin{aligned} \partial_t h &= -\nabla_H \cdot \mathbf{U}, && (40) \\ \partial_t \mathbf{U} &= -f\mathbf{k} \times \mathbf{U} - \nabla_H \cdot \left(\frac{\mathbf{U}\mathbf{U}}{h} \right) - \frac{1}{2} g \nabla_H h^2. && (41) \end{aligned}$$

b.

Show that by letting $\mathbf{u}'(x,t)$ and $h'(x,t)$ denote the deviations away from a background state in geostrophic balance, that is,

$$\mathbf{u} = u_g \mathbf{i} + \mathbf{u}'(x,t), \quad h = H(y) + h'(x,t), \quad \text{where} \quad u_g = -\frac{g}{f}\partial_y H \tag{42}$$

then (38) and (39) reduces to

$$\begin{aligned} \partial_t h &= -u\partial_x h - h\partial_x u, && (43) \\ \partial_t u &= fv - u\partial_x u - g\partial_x h, && (44) \\ \partial_t v &= -f(u - u_g) - u\partial_x v, && (45) \end{aligned}$$

where primes on u, v and h are dropped for clarity.

c.

If you were to solve the system (43) - (45), how many boundary and initial conditions do you have at your disposal? Explain how you derived the number of conditions.

Part 2:

We will solve the system (43) - (45) using numerical methods for a limited domain $x \in \langle 0, D \rangle$. To this end we need boundary conditions at $x = 0, D$ and initial conditions at time $t = 0$. We assume that the motion is started from one at rest in which the geopotential height (or ocean surface) is perturbed. Thus the initial conditions are

$$u = v = 0, \quad \text{and} \quad h(x,t) = H_0 + Ae^{-\left(\frac{x-x_m}{\sigma}\right)^2} \tag{46}$$

where $H_0 = 1000$ m, $A = 15$ m, $u_g = 0$ ms^{-1}, $x_m = D/2$ is the middle point of the domain, and σ is a measure of the width of the Gaussian bell.

d.

To solve (43) - (45) we will adopt the leapfrog scheme. Construct the scheme so that

$$\frac{h_j^{n+1} - h_j^{n-1}}{2\Delta t} = -DIVH, \tag{47}$$

$$\frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} = CORU + ADVU + PRES, \tag{48}$$

$$\frac{v_j^{n+1} - v_j^{n-1}}{2\Delta t} = CORV + ADVV \tag{49}$$

where Δt is the time step, $DIVH$ is the divergence term in (43), $CORU$, $CORV$ are the respective Coriolis terms and $ADVU$, $ADVV$ the respective advection terms in (44) and (45), and $PRES$ is the pressure term in (44).

Describe in some detail how you derive the finite difference approximation to the various terms. Explain why you make the choices you make.

e.

Is the scheme stable and consistent? If so, why and under what condition(s) is the scheme stable? Describe in some detail how you analyzed the stability and consistency of the scheme[5]. How long time step Δt can be used? Explain your choice.

f.

Solve the above equations using the scheme (47) - (49) you have constructed for the domain $x \in \langle 0, D \rangle$. Assume that the variables u, v, and h retain their initial values at the boundaries $x = 0$ and $x = D$. Further, let the grid length be $\Delta x = 100$km, $D = 62\Delta x$ and $\sigma = 5\Delta x$.

Plot h after 1.5, 3.0, 4.5, 6 and 10 hours into the future. Discuss the solution. Try to make a movie spanning $t \in [0, 10]$hrs. What kind of waves do you observe?

g.

Repeat the above computation using the the FRS method to relax the inner solution towards the externally specified values $(\hat{u}, \hat{v}, \hat{h}) = (0, 0, H_0)$ in a buffer zone seven points wide where the relaxation parameter λ_j is given in Table 1 on page 19.

Compare the solution to the one you obtained performing the computation in item **f**, for instance by plotting the difference between them. Explain and discuss any differences you observe.

[5]Hint: Neglect the non-linear terms when performing the stability analysis.

j	λ_j	j
1	1.0	j_{max}
2	0.69	$j_{max} - 1$
3	0.44	$j_{max} - 2$
4	0.25	$j_{max} - 3$
5	0.11	$j_{max} - 4$
6	0.03	$j_{max} - 5$
7	0.0	$j_{max} - 6$

Table 1: Values of the relaxation parameter used in Part 2, item g.

h.

Compute the geostrophic component of the velocity

$$v_g = \frac{g}{f}\partial_x h \tag{50}$$

using the solution for h at $t = 6$ hours. Compare v_g and v at $t = 6$ hours and describe and discuss what you observe. What do you think have happened?

i.

Finally, replace the initial condition for v in (46) by

$$v = \frac{g}{f}\partial_x h. \tag{51}$$

and repeat item **g.**. Discuss the solution by comparing it to the solution obtained through item **g.** above.

Computer Problem 6:
Solving the advection equation using flux correction

We continue to consider numerical solutions to the advection equation (15) in which the advection speed is not necessarily a constant. Writing the advection equation in flux form we get

$$\partial_t \theta + \partial_x (u\theta) = 0 \tag{52}$$

where θ is the tracer concentration.

Part 1:

a.

Show that

$$\theta_j^{n+1} = \theta_j^n - \left(F_j^n - F_{j-1}^n\right), \tag{53}$$

is a first order finite difference approximation (in a non-staggered grid) to (52) where

$$F_j^n = \frac{1}{2}\left[\left(u_j^n + |u_j^n|\right)\theta_j^n + \left(u_{j+1}^n - |u_{j+1}^n|\right)\theta_{j+1}^n\right]\frac{\Delta t}{\Delta x}. \tag{54}$$

b.

Show that the scheme in (53) has a truncation error of order Δt og Δx.

c.

Show that the scheme in (53) is a second approximation to the advection-diffusion equation

$$\partial_t \theta + \partial_x (u\theta) = \kappa \partial_x^2 \theta \quad \text{hvor} \quad \kappa = \frac{1}{2}|u|(\Delta x - |u|\Delta t), \tag{55}$$

assuming that the velocity is a slowly varying function in time and space.

d.

Equation (55) tells us that (53) has an inherent diffusion with a diffusion coefficient given by κ. Visualize this by solving (53) numerically for $x \in \langle 0, L\rangle$ where L =2000km. Let the space increment be Δx =20km, and the velocity be constant, say $u = u_{\max}$ =1m/s, and $t_n = n\Delta t$ where n is the time counter and Δt is the time step. As in Computer Problem #3 on page 9 we make use of cyclic boundary conditions at $x = 0$ and $x = L$. The initial condition is

$$\theta|_{t=0} = \theta_0 e^{-\left(\frac{2x-L}{4\Delta x}\right)^2} \tag{56}$$

where the tracer amplitude is $\theta_0 = 1$.

Plot the results after 1,3, 5 and 10 days together with the initial tracer concentration. To this end you need to specify the time step. Explain and discuss your choice.

Describe and discuss what you observe by comparing the evolution of the tracer concentration with the initial tracer distribution. Explain what have happened.

Part 2:

According to *Smolarkiewicz* (1983) it is possible to counteract the inherent numerical diffusion in the upwind scheme by adding a correction term, or an advective flux u^*C, to (52). The velocity u^* is the so called *antidiffusive velocity*, and is defined by

$$u^* = \kappa \begin{cases} \partial_x \theta / th & , \quad \theta > 0 \\ 0 & , \quad \theta \leq 0 \end{cases} . \tag{57}$$

that is solving the equation

$$\partial_t \theta + \partial_x [(u + u^*)\theta] = 0. \tag{58}$$

rather than (52).

e.

Solve (58) using the MPDATA method, that is, the predictor-corrector method. Use first the iterative method with at least two steps, then the simple method of scaling the antidiffusive velocity. Let the parameters and initial condition be as in Part 1, item **d.**. When scaling use a scaling factor of $S_c = 1.3$. When computing the antidiffusive velocity use a centered in space finite difference approximation, and ensure that you add, as suggested by *Smolarkiewicz* (1983), a small number $\epsilon = 10^{-15}$ in the denominator.

f.

Why do we have to add the small number ϵ to the denominator?

g.

Make experiments varying the scaling factor S_c. Try out other finite difference approximations to the antidiffusive velocity. Discuss the results.

Computer Problem 7:
The storm surge problem

We consider below the so called storm surge problem. The purpose is to give you experience in constructing numerical solutions to geophysical problems that include more than one dependent variable.

In contrast to the atmosphere the astronomical forcing gives rise to an important periodic water level response called tides. In addition to this phenomenon the water level in the ocean also changes due to atmospheric wind and sea level pressure. The latter is called the storm surge response and the water level change caused by it the storm surge. From time to time the joint occurence of high tides and high storm surges can lead to devastating high water levels even along the Norwegian coast. One such example is from mid October 1987 where the water level in Oslo Harbour reached 1.96 meters above normal sea level. In fact since the early 1980s the Norwegian Meteorological Institute has forecasted sea level changes due to storm surges using numerical models.

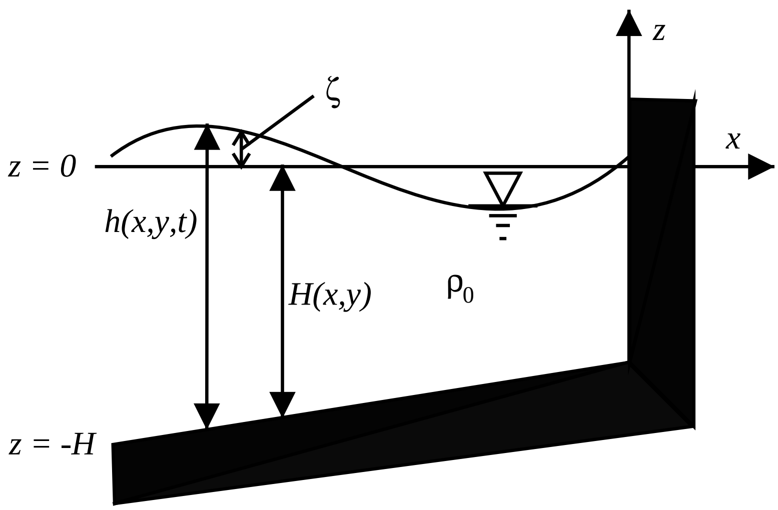

Figure 3: Sketch of a storm surge model along a straight coast conveniently showing some of the notation used.

Many of the earlier studies of storm surges, (e.g., *Røed*, 1979; *Gjevik and Røed*, 1976; *Martinsen et al.*, 1979, to mention a few of the Norwegian ones), have shown that the storm surge is mainly a barotropic response. Storm surge models therefore commonly assume that the density is constant in time and space. The equations therefore reduce to the well known shallow water eqautions. Let

$$\mathbf{U} = \int_{-H}^{\zeta} \mathbf{u} dz, \tag{59}$$

with components (U, V) along the x, y-axes, respectively, be the transport of water in a water column of depth $h = H + \zeta$ where ζ is the sea level deviation away from the equilibrium depth H (see Figure 3). Then the shallow water equations may be written

$$\begin{aligned} \partial_t \mathbf{U} + \nabla_H \cdot (h^{-1}\mathbf{U}\mathbf{U}) + f\mathbf{k} \times \mathbf{U} = -gH\nabla_H(h - H) + \rho_0^{-1}(\boldsymbol{\tau}_s - \boldsymbol{\tau}_b), \\ \partial_t h + \nabla_H \cdot \mathbf{U} = 0. \end{aligned} \tag{60}$$

where $\boldsymbol{\tau}_s$ and $\boldsymbol{\tau}_b$ are respectively the wind and bottom stresses with components (τ_s^x, τ_s^y) og (τ_b^x, τ_b^y), g is the gravitational acceleration and ρ_0 is the (uniform in time and space) density. Linearizing (60) and neglecting variations in the y direction then gives

$$\begin{aligned} \partial_t U - fV = -gH\partial_x h + \rho_0^{-1}(\tau_s^x - \tau_b^x), \\ \partial_t V + fU = \rho_0^{-1}(\tau_s^y - \tau_b^y), \\ \partial_t h + \partial_x U = 0. \end{aligned} \tag{61}$$

Part 1:

In the following we will assume that changes in the equilibrium depth are so small that H to a good approximation can be considered as being constant.

a.

Show that (61) follows by linearizing (60) under the assumption that changes in the equilibrium depth H are insignificant and that $|\mathbf{U}|^2 \ll |\mathbf{U}|$.

b.

What changes are introduced to (61) if the changes in the equilibrium depth H are significant?

c.

We will solve (61) using numerical methods. To this end we will use a centered in space and forward-backward in time scheme[6]. Hence, one such scheme, called the Sielecki scheme (*Sielecki*, 1968), is

$$\begin{aligned} \frac{U_j^{n+1} - U_j^n}{\Delta t} &= fV_j^n - gH\frac{h_{j+1}^n - h_{j-1}^n}{2\Delta x} + \frac{(\tau_s^x)_j^n - (\tau_b^x)_j^n}{\rho_0}, \\ \frac{V_j^{n+1} - V_j^n}{\Delta t} &= -fU_j^{n+1} + \frac{(\tau_s^y)_j^n - (\tau_b^y)_j^n}{\rho_0}, \\ \frac{h_j^{n+1} - h_j^n}{\Delta t} &= -\frac{U_{j+1}^{n+1} - U_{j-1}^{n+1}}{2\Delta x}. \end{aligned} \tag{62}$$

[6]Forward-backward in time means that as soon as one dependent variable is updated (in time) we use these values when updating the other dependent variables.

Here we have assumed that all variables are evaluated at the same point in time and space, that is, are evaluated in a non-staggered grid (Arakawa A-grid). Show that the scheme (62) is numerically stable under the condition

$$\Delta t \leq \frac{2\Delta x}{\sqrt{gH}}\sqrt{1-\left(\frac{f\Delta t}{2}\right)^2} \tag{63}$$

using von Neumanns method[7].

Part 2:

In the following we will first solve the storm surge problem analytically. Although the analytic solution constitutes an approximation to the problem, it can nevertheless be used to verify that the numerical solution is well behaved. To make things as simple as possible, but no simpler, we investigate storm surges along a straight coast ($\partial_y = 0$). Furthermore we will consider the linear problem only. Thus we will investigate analytic and numerical solution to the linearized version of (61) along a straight coast.

When you solve the problem the parameters appearing in the goverening equations are set to:

$$\begin{aligned} \boldsymbol{\tau}_s &= 0.1\text{Pa}\mathbf{j} \quad , \qquad \rho_0 = 10^3\text{kg/m}^3, \\ \boldsymbol{\tau}_b &= \rho_0 R\frac{\mathbf{U}}{H} \quad , \qquad R = 2.4\cdot 10^{-3}\text{m/s}. \end{aligned} \tag{64}$$

if not explicitly deviated.

The initial condition is an ocean at rest and in equilibrium, that is,

$$\mathbf{U}(x,0) = 0 \quad \text{and} \quad \zeta = 0. \tag{65}$$

Since the ocean is limited by the straight coast at $x = 0$ the natural boundary condition here is no flow through the coast. Thus $\mathbf{i}\cdot\mathbf{U} = 0$ at $x = 0$. In principle the ohter boundary is at $x \to -\infty$ which is open. Since this boundary is far from the coast in the sense that it is several Rossby radii away, the natural open boundary condition is that the solution should approach the Ekman solution at a distance sufficiently far away from the coast. Thus,

$$\lim_{x\to-\infty} h = H \quad \text{or} \quad \lim_{x\to-\infty} \partial_x h = 0. \tag{66}$$

d.

Show that the inertial oscillations are avoided if we neglect the term $\partial_t U$ in (61). Explain why it is natural to use the Ekman solution (66) as the open boundary condtion when $x \to -\infty$.

[7]Hint: When analysing the instability neglect all forcing (stress) terms. Also let the discrete Fourier representation of the dependent variables be $U = U_n e^{i\alpha(j-1)\Delta x}$, $V = V_n e^{i\alpha(j-1)\Delta x}$, and $h = H_n e^{i\alpha(j-1)\Delta x}$, respectively. To arrive at (63) eliminate first U_n, U_{n+1} and V_n, V_{n+1} to arrive at one equation involving only H_n and H_{n+1}.

e.

In the following we assume that the motion along the coast (in the y-direction) is in geostrophic balance. Thus we assume that the first equation appearing in (61) reduces to

$$fV = gH\partial_x h. \tag{67}$$

Show that the analytic solution to (61), under the assumption that $R = 0$ (no bottom stress), becomes

$$\begin{aligned} U &= U_E\left(1 - e^{x/\lambda}\right) && (68)\\ V &= ftU_E e^{x/\lambda} && (69)\\ h &= H\left(1 + \frac{tU_E}{\lambda H}e^{x/\lambda}\right) && (70) \end{aligned}$$

where $\lambda = \sqrt{gH}/f$ is Rossby's radius of deformation and

$$U_E = \frac{\tau_s^y}{\rho_0 h}, \tag{71}$$

is the Ekman transport, that is, the transport you get when solving the steady state version ($\partial_t = 0$) of (61) with $\partial_x = 0$ og $R = 0$.

f.

Solve (61) analytically under the assumption that $\tau_b^x = 0, \tau_b^y = \rho_0 Rv$, and that the term $\partial_t U$ in (61) can be neglected[8].

g.

Plot the analytical solutions of h, U and V derived under **e.** and **f.** in a $x - t$ diagram, sometimes referred to as a Hovmöller diagram.

h.

Solve the storm surge problem using numerical methods using the full equations and the parameters listed in (64) including the bottom stress. Use a staggered grid so that the h points are located half way between the U, V points. Let the distance between adjacent h points be Δx. Choose Δx so that the (Rossby's) deformation radius, λ, is well resolved, that is, $\Delta x \leq \lambda/10$. We furthermore assume that the Ekman solution is valid sufficiently far away from the coast, that is, for $x \geq 10\lambda$ and use this as your open boundary condition for $x >> \lambda$. The boundary condition at $x = 0$ is, as alluded to above, the slip condition, that is, $U = 0, V \neq 0$.

[8] Advice: Use Laplace transforms

i.

Plot the numerical solution of the dependent variables h, U og V in a Hovmöller diagram. Compare the numerical solution with thos derived analytically under **f.**. Discuss the differences and the similarities.

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