IN2080

5b. Pumping: Bits and Ends

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Pumping Lemma revisited

Recall example from last week:

$$L = \{a^n b^n \mid n \geq 0\}$$

We used the pumping lemma to show that this language was not regular.

How did we do that?

Pumping Lemma revisited

What about the following language, for $\Sigma = \{a, b, c\}$:

$$L = \{ab^n c^n \mid n \geq 0\} \cup \{a^k w \mid k \neq 1, \text{ and } w \in \Sigma^* \text{ doesn't start with } a\}$$

- Union of two languages:
- First language: all words of the form $ab^n c^n$
- Second language: all Σ^* words that start with either 0 or 2 or more a's.

$\rightarrow$ L is a disjoint union

Pumping Lemma revisited

Lemma (Pumping Lemma)

If A is a regular language, then there is a number p, called the pumping length, where if w is a word in A of length $\geq p$ then w can be divided into three parts, $w = xyz$, such that

1. $xy^i z \in A$ *for every* $i \geq 0$,
2. $|y| > 0$,
3. $|xy| \leq p$.

$$L = \{ab^n c^n \mid n \geq 0\} \cup \{a^k w \mid k \neq 1, \text{ and } w \in \Sigma^* \text{ doesn't start with } a\}$$

Does L satisfy the pumping lemma?

Pumping Lemma revisited

Lemma (Pumping Lemma, shortened)

If A is a regular, $|w| \geq p$ can be divided into three parts, $w = xyz$, such that

1. $xy^i z \in A$ *for every* $i \geq 0$,
2. $|y| > 0$, $|xy| \leq p$.

$$L = \{ab^n c^n \mid n \geq 0\} \cup \{a^k w \mid k \neq 1, \text{ and } w \in \Sigma^* \text{ doesn't start with } a\}$$

- Let p be the pumping length.
- Each $w \in L$ is either of the form $ab^n c^n$ or $a^k w$.

Pumping Lemma revisited

Lemma (Pumping Lemma, shortened)

If A is a regular, $|w| \geq p$ can be divided into three parts, $w = xyz$, such that

1. $xy^i z \in A$ *for every* $i \geq 0$,
2. $|y| > 0$, $|xy| \leq p$.

$$L = \{ab^n c^n \mid n \geq 0\} \cup \{a^k w \mid k \neq 1, \text{ and } w \in \Sigma^* \text{ doesn't start with } a\}$$

- **Case 1:** Assume $s = ab^n c^n$, where n is such that $|s| \geq p$.
- Choose $x = \varepsilon, y = a, z = b^n c^n$. Then $|y| > 0$ and $|xy| \leq p$.
- The string $xz = b^n c^n = a^0 b^n c^n$ is of the form $a^k w$ for $k \neq 1$ and $w \in \Sigma^*$ not starting with a. $\Rightarrow xz \in L$.

Pumping Lemma revisited

Lemma (Pumping Lemma, shortened)

If A is a regular, $|w| \geq p$ can be divided into three parts, $w = xyz$, such that

1. $xy^i z \in A$ *for every* $i \geq 0$,
2. $|y| > 0$, $|xy| \leq p$.

$$L = \{ab^n c^n \mid n \geq 0\} \cup \{a^k w \mid k \neq 1, \text{ and } w \in \Sigma^* \text{ doesn't start with } a\}$$

- **Case 1:** Assume $s = ab^n c^n$, where n is such that $|s| \geq p$.
- Choose $x = \varepsilon, y = a, z = b^n c^n$. Then $|y| > 0$ and $|xy| \leq p$.
- The string $xy^i z = b^n c^n = a^i b^n c^n$ for $i \geq 2$ is of the form $a^k w$ for $k \neq 1$ and $w \in \Sigma^*$ not starting with a. $\Rightarrow xy^i z \in L$.

Pumping Lemma revisited

$$L = \{ab^n c^n \mid n \geq 0\} \cup \{a^k w \mid k \neq 1, \text{ and } w \in \Sigma^* \text{ doesn't start with } a\}$$

- **Case 2:** Assume $s = a^k w_1 w_2 \cdots w_n$, for $k \neq 1$ and $w \in \Sigma^*$ not starting with a, where n, k are such that $|s| \geq p$.
- If $k = 0$, choose $x = \varepsilon, y = w_1, z = w_2 \cdots w_n$. Then $|y| > 0$ and $|xy| \leq p$.
- The strings xz and $xy^i z$ for $i > 2$ are in Σ^* and do not start with $a \Rightarrow xz, xy^i z \in L$.

Pumping Lemma revisited

$$L = \{ab^n c^n \mid n \geq 0\} \cup \{a^k w \mid k \neq 1, \text{ and } w \in \Sigma^* \text{ doesn't start with } a\}$$

- **Case 2:** Assume $s = a^k w_1 w_2 \cdots w_n$, for $k \neq 1$ and $w \in \Sigma^*$ not starting with a, where n, k are such that $|s| \geq p$.
- If $k = 2$, choose $x = \varepsilon, y = aa, z = w_1 w_2 \cdots w_n$. Then $|y| > 0$ and $|xy| \leq p$.
- The string xz is in Σ^* and doesn't start with a. $xz \in L$
- The string $xy^i z$ for $i \geq 1$ starts with 2 or more a's, followed by a word $w \in \Sigma^*$ that does not start with an a. $\Rightarrow xy^i z \in L$.

Pumping Lemma revisited

$$L = \{ab^n c^n \mid n \geq 0\} \cup \{a^k w \mid k \neq 1, \text{ and } w \in \Sigma^* \text{ doesn't start with } a\}$$

- Assume $s = a^k w_1 w_2 \cdots w_n$, for $k \neq 1$ and $w \in \Sigma^*$ not starting with a, where n, k are such that $|s| \geq p$.
- If $k \geq 3$, choose $x = \varepsilon, y = a, z = a^{k-1} w_2 \cdots w_n$. Then $|y| > 0$ and $|xy| \leq p$.
- The string xz is of the form $a^{k-1}w$, where $w \in \Sigma^*$ and doesn't start with a. $xz \in L$
- The string $xy^i z$ for $i \geq 1$ is of the form $a^{k+i-1}w$ where $w \in \Sigma^*$ does not start with an a. $\Rightarrow xy^i z \in L$.

Pumping Lemma revisited

$$L = \{ab^n c^n \mid n \geq 0\} \cup \{a^k w \mid k \neq 1, \text{ and } w \in \Sigma^* \text{ doesn't start with } a\}$$

can be pumped!! Does that mean L is regular?

- If L is regular, then so is $L \cap ab\Sigma^*$ (recall: regular languages are closed under intersection).
- $L \cap ab\Sigma^* = \{ab^n c^n \mid n \geq 1\}$
- Exercise: show that this language is nonregular! (analogous to proof for $a^n b^n$)
- So L is nonregular...is this a counter-example to the pumping lemma?
  No, pumping lemma is not an if and only if statement!