IN2080

4. Pumping Lemma for Regular Languages

Einar Broch Johnsen

University of Oslo

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Department of Informatics



University of Oslo

Last lecture

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  $\Rightarrow$ Pumping Lemma!

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- Long words in regular languages must have repeating parts. Why is that?

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- What happens if $i > |Q|$?

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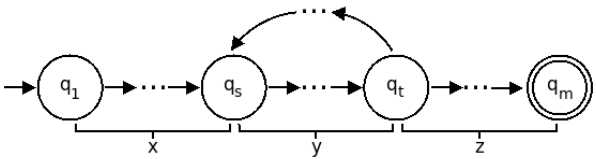
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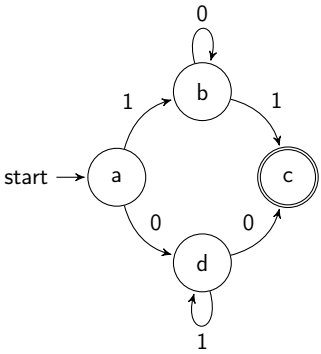
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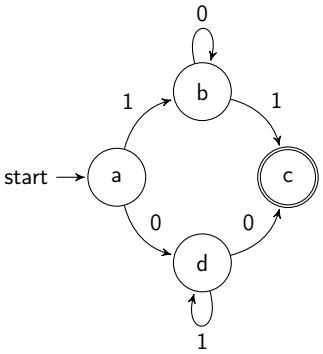
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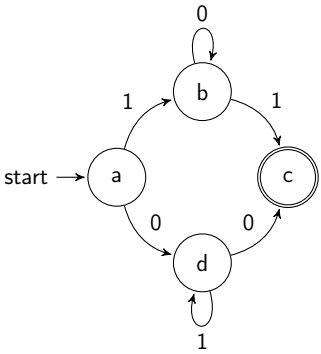


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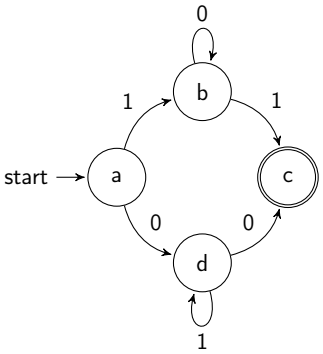
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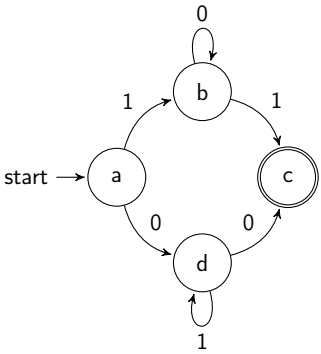
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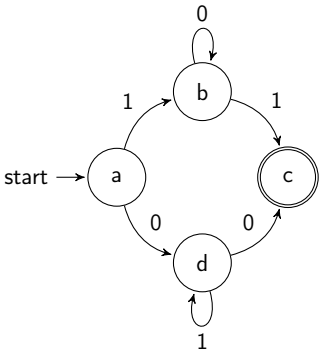
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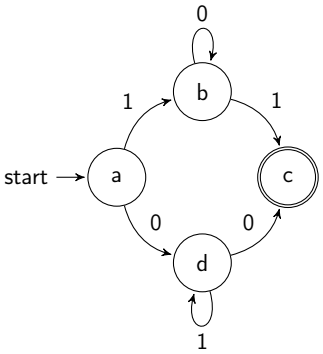
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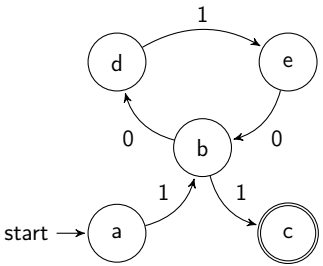
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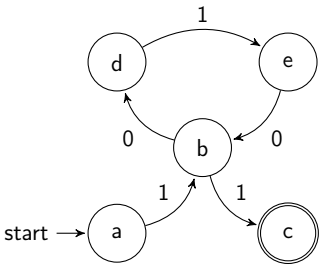
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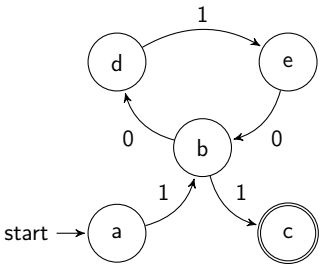
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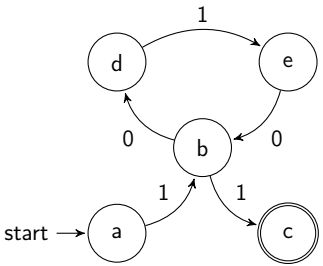


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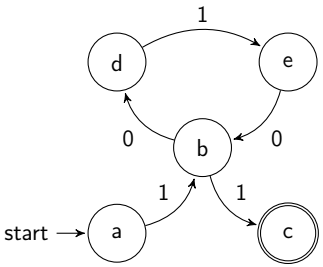


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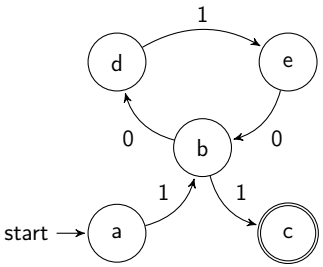
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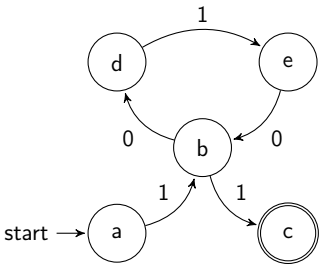
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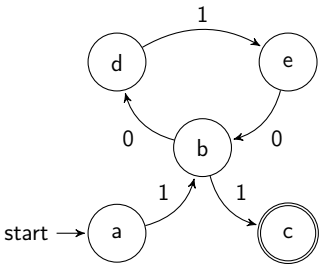
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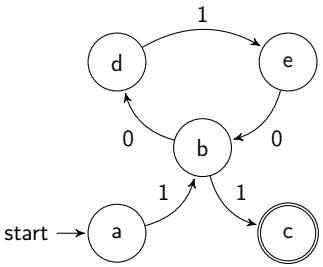
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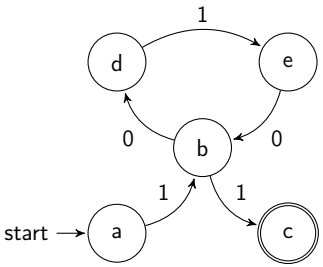
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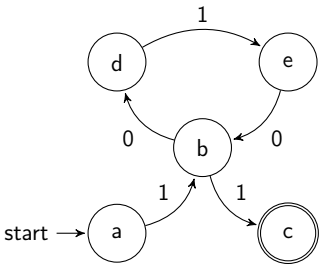
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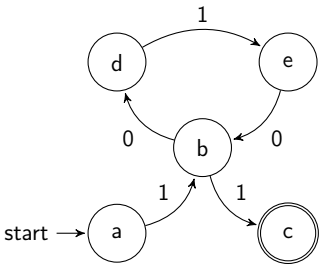
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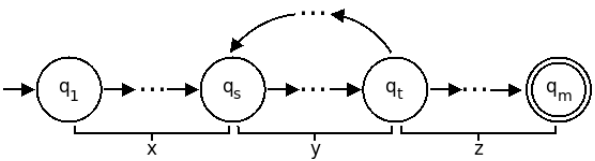
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- Since $n + 1 \geq p + 1$, one state must occur twice within the first $p + 1$ elements of the accepting sequence (pigeonhole principle).
- Let these occurrences be s_j and s_l. Since these occur in the first $p + 1$ elements of the accepting sequence, we have $j < l \leq p + 1$.

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- Then $|y| > 0$ and $|xy| \leq p$.

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- Thus the word $xy^i z$ takes M from the start state s_0 to s_j, follows the path from s_j to s_l i times (recall that $s_j = s_l$), then takes M from s_l to s_n.

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- Thus the word xy^iz takes M from the start state s_0 to s_j, follows the path from s_j to s_l i times (recall that $s_j = s_l$), then takes M from s_l to s_n.
- Thus M accepts any word xy^iz for $i \geq 0$.

Pumping Lemma

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If A is a regular language, then there is a number p, called the pumping length, where if w is a word in A of length $\geq p$ then w can be divided into three parts, $w = xyz$, such that

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  $\rightarrow$ We need to find the counterexample!

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- *If A is regular*, then the pumping lemma gives us a pumping length p.
- Let $w = 0^p 1^p$.

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  $\Rightarrow$ Then xy^iz for $i \geq 2$ has more 0s than 1s. Contradiction! $\Rightarrow$ A cannot be regular!

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- We have to be careful!

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- Did we just prove that B is regular?

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$\Rightarrow$ Contradiction, so B cannot be regular!

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- Another way of showing B is nonregular is to reduce it to the nonregularity of A:

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- Argument about non-regularity by contradiction!