Selected solutions week 13 - IN2080

This week we'll take a look at some previous exam questions regarding NP-completeness. The exams can be found **here** and **here**.

2016: 8

Let G be a graph with n nodes. It contains a Hamiltonian path from s to t if there is a path from s to t which visits all nodes in the graph once. The idea behind the reduction is to have one white domino for each node in the graph. Each white domino has the form $(2k, 2k+1)$, where $k \in \mathbb{N}$ and $0 < k \leq n$. Each black domino represents an edge in the graph, and is on the form $(2k+1, 2j)$, where $k, j \in \mathbb{N}$ and $0 < k, j \leq n$. We also have the black brick $(2n+1, 2 \cdot 1)$, where the nth node is t and the 1st node is s. We remove any other black dominoes on the form $(2n+1, x)$. For there to be a domino circle containing all white dominoes, there must be a sequence of dominoes that switches between white and black bricks. This sequence models a Hamiltonian path from s to t in the graph.

Now we show that $\langle G, s, t\rangle \in HAMPATH \iff \langle B\rangle \in HVIT\text{-}DS$, where B is the collection of dominoes described above. For the forward direction, we have that there is a Hamiltonian path from s to t. Given our construction, we have the sequence of dominoes $(2, 2+1), (2+1, i), (i, i+1) \ldots, (2n, 2n+1), (2n+1, 2)$, which means that $\langle B\rangle \in HVIT\text{-}DS$. For the other direction, there is a sequence of dominoes $(2i, 2i+1), (2i+1, 2j), (2j, 2j+1) \ldots, (2k, 2k+1), (2k+1, 2i)$ such that all white bricks occur in it. Since all white bricks occur, the white brick representing t, $(2n, 2n+1)$, has to occur in the sequence. As there is only one black brick that can be placed after this one, namely $(2n+1, 2)$, it must be the last one in the sequence. As such, the first brick must be $(2, 2+1)$, the white brick representing s. Because this construction models the nodes and edges of G, there must be a Hamiltonian path from s to t in G.

This construction is done in polynomial time, as we make $O(n)$ white dominoes, and $O(n^2)$ black ones.

2017: 3.3

To show that $EXPLEX$ is NP-complete, we'll give a reduction from $HAMPATH$. For a graph G, and nodes s, t in G, we make a new graph G'. G' is identical to G, except for three things:

- There is a new node t'.
- The only edge to t' is from t, and t' has no outgoing edges.
- Each node is labeled with the number 1.

This construction can be done in polynomial time, as we only add a single node and edge, and label each node.

Now we need to show that $\langle G, s, t\rangle \in HAMPATH \iff \langle G', s, k\rangle \in EXPLEX$ where k is the number of nodes in G'. For the forward direction, we know there is a path from s to t going through all the nodes in G. Since G' is near identical to G, the same path exists in G'. Since t has an edge to t', the path can be extended to reach this final node. The sum of each number in each node on the path will be k, and so $\langle G', s, k\rangle \in EXPLEX$. For the other direction, we know that there is a Hamiltonian path from s to t'. The reason is that we can't repeat any nodes in Explex, and since t' has no outgoing edges, the path must end with that node. Since G is identical to G' except for the exclusion of t', there is a Hamiltonian path from s to t in G as well.

2019: 6

Let $S \subseteq \mathbb{N}$, and $t \in \mathbb{N}$. The idea behind this reduction is to provide two numbers t_2 and t_3, such that they cannot be used to sum to t. We pick $t_2 = t+1, t_3 = t+2$ and let $f(\langle S, t\rangle) = \langle S \cup \{t_2, t_3\}, t, t_2, t_3\rangle$.

Now we show that

$$\langle S, t\rangle \in \text{SUBSET-SUM} \iff \langle S \cup \{t_2, t_3\}, t, t_2, t_3\rangle \in \text{TRIPPEL-SUBSET-SUM}$$

For the forward direction, we have that t is a sum of numbers from S. Since we add t_2 and t_3 to the set, and they are different from each other, the subsets summing up to them are $\{t_2\}$ and $\{t_3\}$ respectively. We also know that these numbers are not used in the subset summing to t, as $t < t_2 < t_3$. For the other way, we have that some subset of $S \cup \{t_2, t_3\}$ sum up to t. As neither t_2 or t_3 can be in this subset since they are greater than t, we have that there must be some subset of S which sum to t. Thus, $\langle S, t\rangle \in SUBSET\text{-}SUM$. This construction can be done in polynomial time, as we simply only do some addition.