IN3070/4070 – Logic – Autumn 2019

Lecture : Slide set for the Exam

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Disclaimer

This slide set contains a selection of

- Syntax
- Semantics
- Calculi

for many of the logics discussed in the lecture. These slides will be available in Inspera during the exam. There is *no* guarantee that all of these will be needed or useful for the exam.

Propositional Logic

Syntax — Formulae

Formulae are made up of atomic formulae and the logical connectives $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction), $\rightarrow$ (implication).

Definition 1.1 (Atomic Formulae).

Let $\mathcal{P} = \{p_1, p_2, \ldots\}$ be a countable set of symbols called atomic formulae (or atoms), denoted by lower case letters $p, q, r, \ldots$.

Definition 1.2 (Propositional Formulae).

The propositional formulae, denoted A, B, C, F, G, H, are inductively defined as follows:

1. *Every atom $A \in \mathcal{P}$ is a formula.*
2. *If A and B are formulae, then $(\neg A)$, $(A \wedge B)$, $(A \vee B)$ and $(A \rightarrow B)$ are formulae.*

Let $\mathcal{F}$ be the set of all (legal) formulae.

Semantics — Truth Value

Definition 1.3 (Interpretation).

Let A be a formula and $\mathcal{P}_A$ the set of atoms in A.
An interpretation for A is a total function $\mathcal{I}_A : \mathcal{P}_A \rightarrow \{T, F\}$ that assigns one of the truth values T or F to every atom in $\mathcal{P}_A$.

Definition 1.4 (Truth Value).

Let $\mathcal{I}_A$ be an interpretation for $A \in \mathcal{F}$. The truth value $v_{\mathcal{I}_A}(A)$ (shortly $v(A)$) of A under $\mathcal{I}_A$ is defined inductively as follows. For an atomic formula A, $v_{\mathcal{I}_A}(A) = \mathcal{I}_A(A)$. For composite formulae:

A	$v(A_1)$	$v(A_2)$	$v(A)$
$\neg A_1$	T		F
$\neg A_1$	F		T
$A_1 \vee A_2$	F	F	F
$A_1 \vee A_2$	*otherwise*		T

A	$v(A_1)$	$v(A_2)$	$v(A)$
$A_1 \wedge A_2$	T	T	T
$A_1 \wedge A_2$	*otherwise*		F
$A_1 \rightarrow A_2$	T	F	F
$A_1 \rightarrow A_2$	*otherwise*		T

LK – Axiom and Propositional Rules

- axiom

$$\dfrac{}{\Gamma, A \;\Rightarrow\; A, \Delta}\ \text{axiom}$$

- rules for $\wedge$ (conjunction)

$$\dfrac{\Gamma, A, B \;\Rightarrow\; \Delta}{\Gamma, A \wedge B \;\Rightarrow\; \Delta}\ \wedge\text{-left} \qquad \dfrac{\Gamma \;\Rightarrow\; A, \Delta \qquad \Gamma \;\Rightarrow\; B, \Delta}{\Gamma \;\Rightarrow\; A \wedge B, \Delta}\ \wedge\text{-right}$$

- rules for $\vee$ (disjunction)

$$\dfrac{\Gamma, A \;\Rightarrow\; \Delta \qquad \Gamma, B \;\Rightarrow\; \Delta}{\Gamma, A \vee B \;\Rightarrow\; \Delta}\ \vee\text{-left} \qquad \dfrac{\Gamma \;\Rightarrow\; A, B, \Delta}{\Gamma \;\Rightarrow\; A \vee B, \Delta}\ \vee\text{-right}$$

- rules for $\rightarrow$ (implication)

$$\dfrac{\Gamma \;\Rightarrow\; A, \Delta \qquad \Gamma, B \;\Rightarrow\; \Delta}{\Gamma, A \rightarrow B \;\Rightarrow\; \Delta}\ \rightarrow\text{-left} \qquad \dfrac{\Gamma, A \;\Rightarrow\; B, \Delta}{\Gamma \;\Rightarrow\; A \rightarrow B, \Delta}\ \rightarrow\text{-right}$$

- rules for $\neg$ (negation)

$$\dfrac{\Gamma \;\Rightarrow\; A, \Delta}{\Gamma, \neg A \;\Rightarrow\; \Delta}\ \neg\text{-left} \qquad \dfrac{\Gamma, A \;\Rightarrow\; \Delta}{\Gamma \;\Rightarrow\; \neg A, \Delta}\ \neg\text{-right}$$

First-order Logic

Syntax — Terms

Terms are built up of constant (symbols), variable (symbols), and function (symbols).

Definition 2.1 (Terms).

Let $\mathcal{A} = \{a, b, \ldots\}$ be a countable set of constant symbols,
$\mathcal{V} = \{x, y, z, \ldots\}$ be a countable set of variable symbols, and
$\mathcal{F} = \{f, g, h, \ldots\}$ be a countable set of function symbols.

Terms, denoted t, u, v, are inductively defined as follows:

1. *Every variable $x \in \mathcal{V}$ is a term.*
2. *Every constant $a \in \mathcal{A}$ is a term.*
3. *If $f \in \mathcal{F}$ is an n-ary function (symbol) $n>0$ and $t_1, \ldots, t_n$ are terms, then $f(t_1, \ldots, t_n)$ is a term.*

Example: a, x, $f(a, x)$, $f(g(x), b)$, and $g(f(a, g(y)))$ are terms.

Syntax — First-Order Formulae

Formulae are built up of atomic formulae and the logical connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$, and $\forall$ (universal quantifier), $\exists$ (existential quantifier).

Definition 2.2 (Atomic Formulae).

Let $\mathcal{P} = \{p, q, r, \ldots\}$ be a countable set of predicate symbols. If $p \in \mathcal{P}$ is an n-ary predicate (symbol) $n \geq 0$ and $t_1, \ldots, t_n$ are terms, then $p(t_1, \ldots, t_n)$, $\top$, and $\bot$ are atomic formulae (or atoms).

Definition 2.3 ((First-Order) Formulae).

(First-order) formulae, denoted A, B, C, F, G, H, are inductively defined as follows:

1. *Every atomic formula p is a formula.*
2. *If A and B are formulae and $x \in \mathcal{V}$, then $(\neg A)$, $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$, $\forall x\, A$, and $\exists x\, A$ are formulae.*

Semantics — Interpretation

An interpretation assigns concrete objects, functions and relations to constant symbols, function symbols, and predicate symbols.

Definition 2.4 (Interpretation/Structure).

An interpretation (or structure) $\mathcal{I} = (D, \iota)$ consists of the following elements:

1. *Domain D is a non-empty set*
2. *Interpretation of constant symbols assigns each constant $a \in \mathcal{A}$ an element $a^\iota \in D$*
3. *Interpretation of function symbols assigns each n-ary function symbol $f \in \mathcal{F}$ with $n{>}0$ a function $f^\iota : D^n \to D$*
4. *Interpretation of propositional variables assigns each 0-ary predicate symbol $p \in \mathcal{P}$ a truth value $p^\iota \in \{T, F\}$*
5. *Interpretation of predicate symbols assigns each n-ary predicate symbol $p \in \mathcal{P}$ with $n{>}0$ a relation $p^\iota \subseteq D^n$*

Semantics — Variable Assignments, Value of Terms

The interpretation doesn't tell what to do about variables.
We need something additional.

Definition 2.5 (Variable Assignment).

Given the set of variables $\mathcal{V}$, and an interpretation $\mathcal{I} = (D, \iota)$, a variable assignment α for $\mathcal{I}$ is a function $\alpha : \mathcal{V} \to D$.

Ben-Ari (7.18) writes this $\sigma_{\mathcal{I}_A}$

Definition 2.6 (Term Value).

Let $\mathcal{I} = (D, \iota)$ be an interpretation, and α an variable assignment for $\mathcal{I}$. The term value $v_{\mathcal{I}}(\alpha, t)$ of a term $t \in \mathcal{T}$ under $\mathcal{I}$ and α is inductively defined:

1. $v_{\mathcal{I}}(\alpha, x) = \alpha(x)$ *for a variable* $v \in \mathcal{V}$
2. $v_{\mathcal{I}}(\alpha, a) = a^{\iota}$ *for a constant symbol* $a \in \mathcal{A}$
3. $v_{\mathcal{I}}(\alpha, f(t_1, \ldots, t_n)) = f^{\iota}(v_{\mathcal{I}}(\alpha, t_1), \ldots, v_{\mathcal{I}}(\alpha, t_n))$ *for an n-ary* $f \in \mathcal{F}$

Semantics — Modification of an assignment

Definition 2.7 (Modification of a variable assignment).

Given an interpretation $\mathcal{I} = (D, \iota)$ and a variable assignment α for $\mathcal{I}$.
Given also a variable $y \in \mathcal{V}$ and a domain element $d \in D$.
The modified variable assignment $\alpha\{y \leftarrow d\}$ is defined as

$$\alpha\{y \leftarrow d\}(x) = \begin{cases} d & \text{if } x = y \\ \alpha(x) & \text{otherwise} \end{cases}$$

- $\mathcal{I} = (\mathbb{N}, \iota)$
- $\mathcal{V} = \{x, y\}$
- $\alpha(x) = 3 \in \mathbb{N}$ and $\alpha(y) = 5 \in \mathbb{N}$ is an assignment for $\mathcal{I}$
- $\alpha\{y \leftarrow 7\}(x) = 3$ and $\alpha\{y \leftarrow 7\}(y) = 7$

Semantics — Truth Value

Definition 2.8 (Truth Value).

Let $\mathcal{I} = (D, \iota)$ be an interpretation and α an assignment for $\mathcal{I}$. The truth value $v_{\mathcal{I}}(\alpha, A) \in \{T, F\}$ of a formula A under $\mathcal{I}$ and α is defined inductively as follows:

1. $v_{\mathcal{I}}(\alpha, p) = T$ *for* 0*-ary* $p \in \mathcal{P}$ *iff* $p^{\iota} = T$, *otherwise* $v_{\mathcal{I}}(\alpha, p) = F$
2. $v_{\mathcal{I}}(\alpha, p(t_1, \ldots, t_n)) = T$ *for* $p \in \mathcal{P}$, $n > 0$, *iff* $(v_{\mathcal{I}}(\alpha, t_1), \ldots, v_{\mathcal{I}}(\alpha, t_n)) \in p^{\iota}$, *otherwise* $v_{\mathcal{I}}(\alpha, p(t_1, \ldots, t_n)) = F$
3. $v_{\mathcal{I}}(\alpha, \neg A) = T$ *iff* $v_{\mathcal{I}}(\alpha, A) = F$, *otherwise* $v_{\mathcal{I}}(\alpha, \neg A) = F$
4. $v_{\mathcal{I}}(\alpha, A \wedge B) = T$ *iff* $v_{\mathcal{I}}(\alpha, A) = T$ *and* $v_{\mathcal{I}}(\alpha, B) = T$, *otherwise* $v_{\mathcal{I}}(\alpha, A \wedge B) = F$
5. $v_{\mathcal{I}}(\alpha, A \vee B) = T$ *iff* $v_{\mathcal{I}}(\alpha, A) = T$ *or* $v_{\mathcal{I}}(\alpha, B) = T$, *otherwise* $v_{\mathcal{I}}(\alpha, A \vee B) = F$
6. $v_{\mathcal{I}}(\alpha, A \rightarrow B) = T$ *iff* $v_{\mathcal{I}}(\alpha, A) = F$ *or* $v_{\mathcal{I}}(\alpha, B) = T$, *otherwise* $v_{\mathcal{I}}(\alpha, A \rightarrow B) = F$
7. $v_{\mathcal{I}}(\alpha, \forall x A) = T$ *iff* $v_{\mathcal{I}}(\alpha\{x \leftarrow d\}, A) = T$ *for all* $d \in D$, *otherwise* $v_{\mathcal{I}}(\alpha, \forall x A) = F$
8. $v_{\mathcal{I}}(\alpha, \exists x A) = T$ *iff* $v_{\mathcal{I}}(\alpha\{x \leftarrow d\}, A) = T$ *for some* $d \in D$, *otherwise* $v_{\mathcal{I}}(\alpha, \exists x A) = F$
9. $v_{\mathcal{I}}(\alpha, \top) = T$ *and* $v_{\mathcal{I}}(\alpha, \bot) = F$

Ground LK – Rules for Universal and Existential Quantifier

- rules for $\forall$ (universal quantifier)

$$\frac{\Gamma, A[x\backslash t], \forall x\, A \;\Rightarrow\; \Delta}{\Gamma, \forall x\, A \;\Rightarrow\; \Delta}\ \forall\text{-left} \qquad \frac{\Gamma \;\Rightarrow\; A[x\backslash a], \Delta}{\Gamma \;\Rightarrow\; \forall x\, A, \Delta}\ \forall\text{-right}$$

  - t is an arbitrary closed term
  - Eigenvariable condition for the rule $\forall$-right: a must not occur in the conclusion, i.e. in Γ, Δ, or A
  - the formula $\forall x\, A$ is preserved in the premise of the rule $\forall$-left

- rules for $\exists$ (existential quantifier)

$$\frac{\Gamma, A[x\backslash a] \;\Rightarrow\; \Delta}{\Gamma, \exists x\, A \;\Rightarrow\; \Delta}\ \exists\text{-left} \qquad \frac{\Gamma \;\Rightarrow\; \exists x\, A, A[x\backslash t], \Delta}{\Gamma \;\Rightarrow\; \exists x\, A, \Delta}\ \exists\text{-right}$$

  - t is an arbitrary closed term
  - Eigenvariable condition for the rule $\exists$-left: a must not occur in the conclusion, i.e. in Γ, Δ, or A
  - the formula $\exists x\, A$ is preserved in the premise of the rule $\exists$-right

Free-variable LK: γ-rules

Definition 2.9 (γ-rules in free-variable LK).

The γ-rules in free variable LK *are:*

$$\frac{\Gamma, \forall x\, A, A[x \backslash U] \;\Rightarrow\; \Delta}{\Gamma, \forall x\, A \;\Rightarrow\; \Delta}\ \forall\text{-}left \qquad \frac{\Gamma \;\Rightarrow\; \Delta, \exists x\, A, A[x \backslash U]}{\Gamma \;\Rightarrow\; \Delta, \exists x\, A}\ \exists\text{-}right$$

U is a new free variable

- By *new*, we mean that it does not previously occur in the derivation.

Free-variable LK: δ-rules

Definition 2.10 (δ-rules in free-variable LK).

The δ-rules in free-variable LK *are:*

$$\frac{\Gamma, A[x \backslash f(\vec{U})] \ \Rightarrow \ \Delta}{\Gamma, \exists x\, A \ \Rightarrow \ \Delta}\ \exists\text{-}left \qquad \frac{\Gamma \ \Rightarrow \ \Delta, A[x \backslash f(\vec{U})]}{\Gamma \ \Rightarrow \ \Delta, \forall x\, A}\ \forall\text{-}right$$

f is a new Skolem function
$\vec{U} = U_1, \ldots, U_n$ are the free variables occurring in $\exists x\, A$.

- By *new*, we mean that f does not previously occur in the derivation.

The First-Order Resolution Calculus

The resolution rule is generalized by performing unification as part of the rule and an additional factorization rule is added.

Definition 2.11 (First-Order Resolution Calculus).

$$\frac{}{C_1, \ldots, \{\}, \ldots, C_n}\ \textit{axiom}$$

$$\frac{C_1, \ldots, C_i \cup \{L_1\}, \ldots, C_j \cup \{L_2\}, \ldots, C_n, \sigma(C_i \cup C_j)}{C_1, \ldots, C_i \cup \{L_1\}, \ldots, C_j \cup \{L_2\}, \ldots, C_n}\ \textit{resolution}$$

with σ a m.g.u. of L_1 and $\overline{L_2}$.

$$\frac{C_1, \ldots, C_i \cup \{L_1, \ldots, L_m\}, \ldots, C_n, \sigma(C_i \cup \{L_1\})}{C_1, \ldots, C_i \cup \{L_1, \ldots, L_m\}, \ldots, C_n}\ \textit{factorization}$$

with σ a m.g.u. of $L_1 \ldots L_m$.

- a resolution proof for a set of clauses S is a derivation of S in the resolution calculus; the substitution σ is local for every rule application; variables in every clause C can be renamed

Modal Logic

Kripke Frames

Definition 3.1 (Kripke Frame).

A (Kripke) frame $F = (W, R)$ *consists of*

- *a non-empty set of worlds* W
- *a binary accessibility relation* $R \subseteq W \times W$ *on the worlds in* W

Definition 3.2 (Reminder: Propositional Interpretation).

A propositional interpretation is a function $\mathcal{I} : \mathcal{P} \to \{T, F\}$ *that assigns a truth value to every propositional variable.*

Definition 3.3 (Modal Interpretation).

A modal interpretation (Kripke model) $\mathcal{I}_M := (F, \{\mathcal{I}(w)\}_{w \in W})$ *consists of*

- *a Kripke frame* $F = (W, R)$
- *one propositional interpretation* $\mathcal{I}(w)$ *for each* $w \in W$

Modal Truth Value

Definition 3.4 (Modal Truth Value).

Let $\mathcal{I}_M = ((W, R), \{\mathcal{I}(w)\}_{w \in W})$ be a Kripke structure. The modal truth value $v_{\mathcal{I}_M}(w, A)$ of a formula A in the world w in the structure $\mathcal{I}_M$ is T (true) if "w forces A under $\mathcal{I}_M$", denoted $w \Vdash A$, and F (false), otherwise.

The forcing relation $w \Vdash A$ is defined inductively as follows:

- $w \Vdash p$ *for* $p \in \mathcal{P}$ *iff* $\mathcal{I}(w)(p) = T$
- $w \Vdash \neg A$ *iff not* $w \Vdash A$
- $w \Vdash A \wedge B$ *iff* $w \Vdash A$ *and* $w \Vdash B$
- $w \Vdash A \vee B$ *iff* $w \Vdash A$ *or* $w \Vdash B$
- $w \Vdash A \to B$ *iff not* $w \Vdash A$ *or* $w \Vdash B$
- $w \Vdash \Diamond A$ *iff* $v \Vdash A$ *for some* $v \in W$ *with* $(w, v) \in R$
- $w \Vdash \Box A$ *iff* $v \Vdash A$ *for all* $v \in W$ *with* $(w, v) \in R$

Satisfiability and Validity

In modal logic a formula F is valid, if it evaluates to *true* in all worlds of all Kripke structures.

Definition 3.5 (Satisfiable,Model,Unsatisfiable,Valid,Invalid).

Let A be a formula. and $\mathcal{I}_M$ be a Kripke structure.

- $\mathcal{I}_M$ *is a* model in modal logic *for A, denoted $\mathcal{I}_M \models A$, iff $v_{\mathcal{I}_M}(w, A) = T$ for all $w \in W$.*
- *A is satisfiable in modal logic iff $\mathcal{I}_M \models A$ for some Kripke structure $\mathcal{I}_M$.*
- *A is unsatisfiable in modal logic iff A is not satisfiable.*
- *A is valid, denoted $\models A$, iff $\mathcal{I}_M \models A$ for all modal interpretations $\mathcal{I}_M$.*
- *A is invalid/falsifiable in modal logic iff A is not valid.*

More Modal Logics

modal logic	condition on R	axioms
K	(no condition)	–
K4	transitive	$\Box A \to \Box\Box A$
D	serial	$\Box A \to \Diamond A$
D4	serial, transitive	$\Box A \to \Diamond A$, $\Box A \to \Box\Box A$
T	reflexive	$\Box A \to A$
S4	reflexive, transitive	$\Box A \to A$, $\Box A \to \Box\Box A$
S5	equivalence (reflexive, euclidean)	$\Box A \to A$, $\Diamond A \to \Box\Diamond A$

(A relation $R \subseteq W \times W$ is *serial* iff for all $w_1 \in W$ there is some $w_2 \in W$ with $(w_1, w_2) \in R$; a relation $R \subseteq W \times W$ is *euclidean* iff for all $w_1, w_2, w_3 \in W$ the following holds: if $(w_1, w_2) \in R$ and $(w_1, w_3) \in R$ then $(w_2, w_3) \in R$.)

Lemma: if a relation is reflexive and euclidean, it is also symmetric and transitive, i.e. an equivalence relation.

A Sequent Calculus for **K**

- Let $\mathcal{L}$ be a set of labels
- A labeled formula is a pair $u : A$ where $u \in \mathcal{L}$ and A a formula.
- An accessibility formula has the shape uRv for two labels $u, v \in \mathcal{L}$.
- Use labeled sequents, containing labeled formulae and accessibility formulae
- Propositional rules for labeled formulas: just copy labels, e.g.

$$\frac{\Gamma \;\Rightarrow\; u : A, \Delta \qquad \Gamma \;\Rightarrow\; u : B, \Delta}{\Gamma \;\Rightarrow\; u : A \wedge B, \Delta} \wedge\text{-right}$$

- The $\Diamond$-left rule creates a new label:

$$\frac{\Gamma, uRv, v : A \;\Rightarrow\; \Delta}{\Gamma, u : \Diamond A \;\Rightarrow\; \Delta} \Diamond\text{-left} \qquad \text{for a fresh label } v$$

- The $\Box$-left rule transfers info to other labels:

$$\frac{\Gamma, uRv, v : A, u : \Box A \;\Rightarrow\; \Delta}{\Gamma, uRv, u : \Box A \;\Rightarrow\; \Delta} \Box\text{-left}$$

- Axioms require same labels: $u : A, \Gamma \;\Rightarrow\; u : A, \Gamma$

Rules for the Succedent

- The $\Box$-right rule creates a new label:

$$\frac{\Gamma, uRv \;\Rightarrow\; v : A,\ \Delta}{\Gamma \;\Rightarrow\; u : \Box A, \Delta}\ \Box\text{-right} \qquad \text{for a fresh label } v$$

- The $\Diamond$-right rule transfers info to other labels:

$$\frac{\Gamma, uRv \;\Rightarrow\; v : A,\ u : \Diamond A,\ \Delta}{\Gamma, uRv \;\Rightarrow\; u : \Diamond A,\ \Delta}\ \Diamond\text{-right}$$

Intuitionistic Logic

Kripke Semantics

- is a formal semantics created in the late 1950s and early 1960s by Saul Kripke and André Joyal; was first used for modal logics, later adapted to intuitionistic logic and other non-classical logics

Definition 4.1 (Kripke Frame).

A (Kripke) frame $F = (W, R)$ consists of a

- *a non-empty set of worlds W*
- *a binary accessibility relation $R \subseteq W \times W$ on the worlds in W*

Definition 4.2 (Intuitionistic Frame).

An intuitionistic frame $F_J = (W, R)$ is a Kripke frame (W, R) with a reflexive and transitive accessibility relation R.

($R \subseteq W \times W$ is reflexive iff $(w_1, w_1) \in R$ for all $w_1 \in W$; R is transitive iff for all $w_1, w_2, w_3 \in W$: if $(w_1, w_2) \in R$ and $(w_2, w_3) \in R$ then $(w_1, w_3) \in R$)

Intuitionistic Interpretation

Definition 4.3 (Intuitionistic Interpretation).

An intuitionistic interpretation (J-structure) $\mathcal{I}_J{:=}(F_J, \{\mathcal{I}_C(w)\}_{w \in W})$ *consists of*

- *an intuitionistic frame* $F_J = (W, R)$
- *a set of class. interpretations* $\{\mathcal{I}_C(w)\}_{w \in W}$ *with* $\mathcal{I}_C(w){:=}(D^w, \iota^w)$ *assigning a domain* D^w *and an interpretation* ι^w *to every* $w{\in}W$

Furthermore, the following holds:

1. *cumulative domains, i.e. for all* $w, v{\in}W$ *with* $(w, v){\in}R$*:* $D^w{\subseteq}D^v$
2. *interpretations only "increase", i.e. for all* $w, v{\in}W$ *with* $(w, v){\in}R$*:*
   a. $a^{\iota^w} = a^{\iota^v}$ *for every constant* a
   b. $f^{\iota^w} \subseteq f^{\iota^v}$ *for every function* f
   c. $p^{\iota^w} = T$ *implies* $p^{\iota^v}{=}T$ *for every* $p \in \mathcal{P}^0$
   d. $p^{\iota^w} \subseteq p^{\iota^v}$ *for every predicate* $p \in \mathcal{P}^n$ *with* $n > 0$

   ($g{\subseteq}h$ *holds for* g *and* h *iff* $g(x){=}h(x)$ *for all* x *of the domain of* g*)*

Intuitionistic Truth Value

Definition 4.4 (Intuitionistic Truth Value).

Let $\mathcal{I}_J = ((W, R), \{(D^w, \iota^w)\}_{w \in W})$ be a J-structure. The intuitionistic truth value $v_{\mathcal{I}_J}(w, G)$ of a formula G in the world w under the structure $\mathcal{I}_J$ is T (true) if "w forces G under $\mathcal{I}_J$", denoted $w \Vdash G$, and F (false), otherwise. $v_{\mathcal{I}_J}(w, t)$ is the (classic) evaluation of the term t in world w.

The forcing relation $w \Vdash G$ is defined as follows:

- $w \Vdash p$ for $p \in \mathcal{P}^0$ iff $p^{\iota^w} = T$
- $w \Vdash p(t_1, ..., t_n)$ for $p \in \mathcal{P}^n$, $n>0$, iff $(v_{\mathcal{I}_J}(w, t_1), ..., v_{\mathcal{I}_J}(w, t_n)) \in P^{\iota^w}$
- $w \Vdash \neg A$ iff $v \nVdash A$ for all $v \in W$ with $(w, v) \in R$
- $w \Vdash A \wedge B$ iff $w \Vdash A$ and $w \Vdash B$
- $w \Vdash A \vee B$ iff $w \Vdash A$ or $w \Vdash B$
- $w \Vdash A \rightarrow B$ iff $v \Vdash A$ implies $v \Vdash B$ for all $v \in W$ with $(w, v) \in R$
- $w \Vdash \exists x A$ iff $w \Vdash A[x \backslash d]$ for some $d \in D^w$
- $w \Vdash \forall x A$ iff $v \Vdash A[x \backslash d]$ for all $d \in D^v$ for all $v \in W$ with $(w, v) \in R$

Satisfiability and Validity

In intuitionistic logic a formula F is valid, if it evaluates to *true* in all worlds and for all intuitionistic interpretations.

Definition 4.5 (Satisfiable,Model,Unsatisfiable,Valid,Invalid).

Let F be a closed (first-order) formula.

- *Let $\mathcal{I}_J$ be an intuitionistic interpretation. $\mathcal{I}_J$ is an intuitionistic model for a F, denoted $\mathcal{I}_J \models F$, iff $v_{\mathcal{I}}(w, F) = T$ for all $w \in W$.*
- *F is intuitionistically satisfiable iff $\mathcal{I}_J \models F$ for some intuitionistic interpretation $\mathcal{I}_J$.*
- *F is intuitionistically unsatisfiable iff F is not intuit. satisfiable.*
- *F is intuitionistically valid, denoted $\models F$, iff $\mathcal{I}_J \models F$ for all intuitionistic interpretations $\mathcal{I}_J$.*
- *F is intuitionistically invalid/falsifiable iff F is not intuit. valid.*

LJ – Rules for Conjunction and Disjunction

- rules for $\wedge$ (conjunction)

$$\frac{\Gamma, A, B \;\Rightarrow\; D}{\Gamma, A \wedge B \;\Rightarrow\; D}\ \wedge\text{-left} \qquad \frac{\Gamma \;\Rightarrow\; A \qquad \Gamma \;\Rightarrow\; B}{\Gamma \;\Rightarrow\; A \wedge B}\ \wedge\text{-right}$$

- rules for $\vee$ (disjunction)

$$\frac{\Gamma, A \;\Rightarrow\; D \qquad \Gamma, B \;\Rightarrow\; D}{\Gamma, A \vee B \;\Rightarrow\; D}\ \vee\text{-left}$$

$$\frac{\Gamma \;\Rightarrow\; A}{\Gamma \;\Rightarrow\; A \vee B}\ \vee\text{-right}_1 \qquad \frac{\Gamma \;\Rightarrow\; B}{\Gamma \;\Rightarrow\; A \vee B}\ \vee\text{-right}_2$$

LJ – Rules for Implication and Negation, Axiom

- rules for $\rightarrow$ (implication)

$$\frac{\Gamma, A \rightarrow B \;\Rightarrow\; A \qquad \Gamma, B \;\Rightarrow\; D}{\Gamma, A \rightarrow B \;\Rightarrow\; D}\ \rightarrow\text{-left} \qquad \frac{\Gamma, A \;\Rightarrow\; B}{\Gamma \;\Rightarrow\; A \rightarrow B}\ \rightarrow\text{-right}$$

- rules for $\neg$ (negation)

$$\frac{\Gamma, \neg A \;\Rightarrow\; A}{\Gamma, \neg A \;\Rightarrow\; D}\ \neg\text{-left} \qquad \frac{\Gamma, A \;\Rightarrow}{\Gamma \;\Rightarrow\; \neg A}\ \neg\text{-right}$$

- the axiom

$$\frac{}{\Gamma, A \;\Rightarrow\; A}\ \text{axiom}$$

LJ – Rules for Universal and Existential Quantifier

- rules for $\forall$ (universal quantifier)

$$\frac{\Gamma, A[x\backslash t], \forall x\, A \;\Rightarrow\; D}{\Gamma, \forall x\, A \;\Rightarrow\; D}\ \forall\text{-left} \qquad \frac{\Gamma \;\Rightarrow\; A[x\backslash a]}{\Gamma \;\Rightarrow\; \forall x\, A}\ \forall\text{-right}$$

  - t is an arbitrary closed term
  - Eigenvariable condition for the rule $\forall$-right: a must not occur in the conclusion, i.e. in Γ or A
  - the formula $\forall x\, A$ is preserved in the premise of the rule $\forall$-left

- rules for $\exists$ (existential quantifier)

$$\frac{\Gamma, A[x\backslash a] \;\Rightarrow\; D}{\Gamma, \exists x\, A \;\Rightarrow\; D}\ \exists\text{-left} \qquad \frac{\Gamma \;\Rightarrow\; A[x\backslash t]}{\Gamma \;\Rightarrow\; \exists x\, A}\ \exists\text{-right}$$

  - t is an arbitrary closed term
  - Eigenvariable condition for the rule $\exists$-left: a must not occur in the conclusion, i.e. in Γ, D, or A
  - the formula $\exists x\, A$ is not preserved in the premise of the rule $\exists$-right