

IN3070/4070 – Logic – Autumn 2019

Lecture 2: Propositional Logic & Sequent Calculus

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DEPARTMENT OF
INFORMATICS



UNIVERSITY OF
OSLO

Today's Plan

- ▶ Motivation
- ▶ Syntax
- ▶ Semantics
- ▶ Logical Equivalence
- ▶ Satisfiability & Validity
- ▶ Summary
- ▶ Motivation
- ▶ Sequent Calculus
- ▶ Decision Procedure
- ▶ Summary

Outline

- ▶ Motivation
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- ▶ logical systems **formalize reasoning** similar to programming languages that formalize computation

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- ▶ logical systems **formalize reasoning** similar to programming languages that formalize computation
- ▶ consequent separation of **syntactical** notions (formulae, proofs) and **semantical** notions (truth values, models)
- ▶ **syntax** defines what strings of symbols are “legal” formulae
- ▶ **semantics** assign meanings to legal formulae (through an interpretation of its symbols)

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Syntax — Formulae

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Definition 2.2 (Propositional Formulae).

The **propositional formulae**, denoted A, B, C, F, G, H , are inductively defined as follows:

- ▶ Every atom $A \in \mathcal{P}$ is a formula.
- ▶ If A and B are formulae, then $(\neg A)$, $(A \wedge B)$, $(A \vee B)$ and $(A \rightarrow B)$ are formulae.

Let \mathcal{F} be the set of all (legal) formulae.

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In order to make formulae easier to read, parentheses can be omitted:

- ▶ the order of **precedence** of the logical connectives is as follows (from high to low): \neg , \wedge , \vee , \rightarrow , \leftrightarrow
- ▶ connectives are assumed to be **right-associative**, i.e., $A \vee B \vee C$ means $(A \vee (B \vee C))$

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Examples:

$((p \rightarrow q) \leftrightarrow ((\neg p) \rightarrow (\neg q)))$ is a (legal) formula, identical to $(p \rightarrow q) \leftrightarrow (\neg p \rightarrow \neg q)$ and $p \rightarrow q \leftrightarrow \neg p \rightarrow \neg q$

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Alternative connectives: \Rightarrow and \supset (for \rightarrow), \Leftrightarrow (for \leftrightarrow), $\&$ (for \wedge)

Formula Trees

Definition 2.4 (Formula Tree).

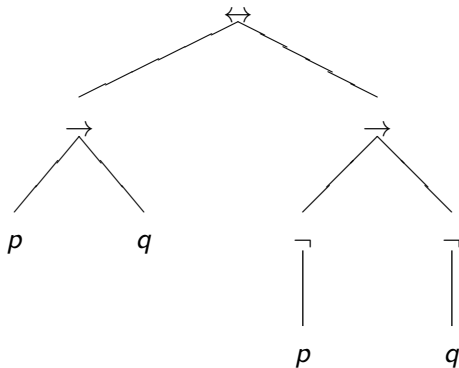
*A formula can be presented as **formula tree**.*

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A formula can be presented as *formula tree*.

Example: $(p \rightarrow q) \leftrightarrow (\neg p \rightarrow \neg q)$



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Semantics — Truth Value

Truth values are assigned to the atoms of a formula in order to evaluate the truth value of the formula.

Definition 3.1 (Interpretation).

Let A be a formula and \mathcal{P}_A the set of atoms in A .

An **interpretation** for A is a total function $\mathcal{I}_A : \mathcal{P}_A \rightarrow \{T, F\}$ that assigns one of the truth values T or F to every atom in \mathcal{P}_A .

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Definition 3.2 (Truth Value).

Let \mathcal{I}_A be an interpretation for $A \in \mathcal{F}$. The **truth value** $v_{\mathcal{I}_A}(A)$ (shortly $v(A)$) of A under \mathcal{I}_A is defined inductively as follows.

A	$v(A_1)$	$v(A_2)$	$v(A)$
$\neg A_1$	T		F
$\neg A_1$	F		T
$A_1 \vee A_2$	F	F	F
$A_1 \vee A_2$	otherwise		T

A	$v(A_1)$	$v(A_2)$	$v(A)$
$A_1 \wedge A_2$	T	T	T
$A_1 \wedge A_2$	otherwise		F
$A_1 \rightarrow A_2$	T	F	F
$A_1 \rightarrow A_2$	otherwise		T

Semantics — Truth Value

Definition 3.3 (Truth Value (continued)).

For the equivalence connective the *truth value* is as follows.

A	$v(A_1)$	$v(A_2)$	$v(A)$
$A_1 \leftrightarrow A_2$	$v(A_1) = v(A_2)$		T

A	$v(A_1)$	$v(A_2)$	$v(A)$
$A_1 \leftrightarrow A_2$	$v(A_1) \neq v(A_2)$		F

Semantics — Truth Value

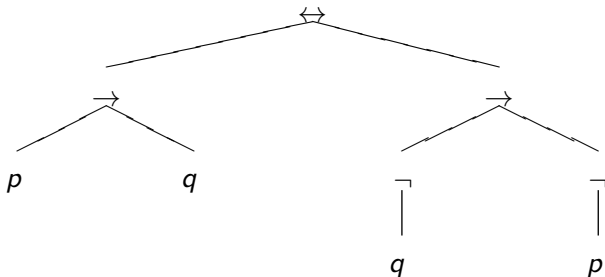
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Example: Let $A = (p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p)$
with $\mathcal{P}_A = \{p, q\}$, $\mathcal{I}_A(p) = F$ and $\mathcal{I}_A(q) = T$.



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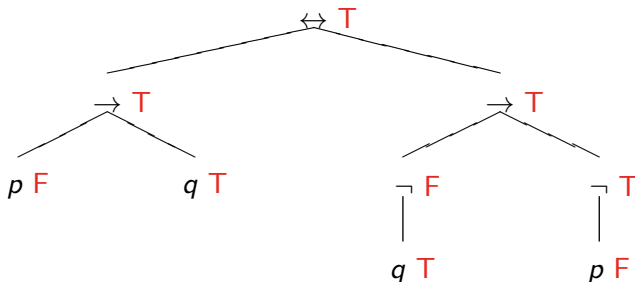
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Truth Tables

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Definition 3.4 (Truth Table).

Let $A \in \mathcal{F}$ and $|\mathcal{P}_A| = n$. A **truth table** has $n + 1$ columns and 2^n rows. There is a column for each atom in \mathcal{P}_A , plus a column for the formula A . The first n columns specify all possible interpretations \mathcal{I} that map atoms in \mathcal{P}_A to $\{T, F\}$. The last column shows $v_{\mathcal{I}}(A)$, the truth value of A for each interpretation \mathcal{I} .

p_1	p_2	\dots	p_n	A
T	T	\dots	T	$v_{\mathcal{I}}(A)$
T	T	\dots	F	$v_{\mathcal{I}}(A)$
\vdots	\vdots	\vdots	\vdots	\vdots
F	F	\dots	F	$v_{\mathcal{I}}(A)$

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p	q	$p \rightarrow q$	$\neg p$	$\neg q$	$\neg q \rightarrow \neg p$	$(p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p)$
T	T					
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p	q	$p \rightarrow q$	$\neg p$	$\neg q$	$\neg q \rightarrow \neg p$	$(p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p)$
T	T	T	F	F	T	T
T	F	F	F	T	F	T
F	T	T	T	F	T	T
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Material Implication

The operator of $p \rightarrow q$ is called **material implication**.

- ▶ p is the antecedent and q is the consequent
- ▶ it does **not** claim **causation**; i.e., it does not assert that the antecedent causes the consequent (or is even related to the consequent in any way)

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is **false** since the antecedent is true and the consequent is false, but:

“Earth is farther from the sun than Mars” \rightarrow “ $1 + 1 = 3$ ”

is **true(!)** as the falsity of the antecedent by itself is sufficient to ensure the truth of the implication

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Definition 4.1 (Logical Equivalence).

Let $A_1, A_2 \in \mathcal{F}$. If $v_{\mathcal{I}}(A_1) = v_{\mathcal{I}}(A_2)$ for all interpretations \mathcal{I} , then A_1 is *logically equivalent* to A_2 , denoted $A_1 \equiv A_2$.

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Theorem 4.1 (Logical Equivalence “Commutativity”).

Let $A, B \in \mathcal{F}$. Then $A \vee B \equiv B \vee A$.

Proof.

Let \mathcal{I} be an arbitrary interpretation for $A \vee B$. Then \mathcal{I} is also an interpretation for $B \vee A$. Since $\mathcal{P}_A \subseteq \mathcal{P}_A \cup \mathcal{P}_B$, \mathcal{I} is an interpretation for A . Similarly, \mathcal{I} is an interpretation for B . Now $v_{\mathcal{I}}(A \vee B) = T$ iff $v_{\mathcal{I}}(A) = T$ or $v_{\mathcal{I}}(B) = T$, and $v_{\mathcal{I}}(B \vee A) = T$ iff $v_{\mathcal{I}}(B) = T$ or $v_{\mathcal{I}}(A) = T$. If $v_{\mathcal{I}}(A) = T$, then $v_{\mathcal{I}}(A \vee B) = T = v_{\mathcal{I}}(B \vee A)$, and similarly if $v_{\mathcal{I}}(B) = T$. Since \mathcal{I} was arbitrary, $A \vee B \equiv B \vee A$. □

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Theorem 4.2 (Relation between \equiv and \leftrightarrow).

$A \equiv B$ iff $v_{\mathcal{I}}(A \leftrightarrow B) = T$ for every interpretation \mathcal{I} .

Proof.

Suppose that $A \equiv B$ and let \mathcal{I} be an arbitrary interpretation; then $v_{\mathcal{I}}(A) = v_{\mathcal{I}}(B)$ by definition of logical equivalence. From Definition 6, $v_{\mathcal{I}}(A \leftrightarrow B) = T$. Since \mathcal{I} was arbitrary, $v_{\mathcal{I}}(A \leftrightarrow B) = T$ for all interpretations \mathcal{I} . The proof of the converse is similar. \square

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The following formulae are **logical equivalent** (more in [Ben-Ari, 2.3.3]):

$$A \vee \text{true} \equiv \text{true}$$

$$A \vee \text{false} \equiv A$$

$$A \rightarrow \text{true} \equiv \text{true}$$

$$A \rightarrow \text{false} \equiv \neg A$$

$$A \equiv A \wedge A$$

$$A \vee B \equiv B \vee A$$

$$A \vee (B \vee C) \equiv (A \vee B) \vee C$$

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$$A \wedge \text{true} \equiv A$$

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$$\text{true} \rightarrow A \equiv A$$

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Contrapositive: $A \rightarrow B \equiv \neg B \rightarrow \neg A$

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Satisfiability and Validity

Definition 5.1 (Satisfiable, Model, Valid, Unsatisfiable, Invalid).

Let $A \in \mathcal{F}$.

- ▶ A is *satisfiable* iff $v_{\mathcal{I}}(A) = T$ for some interpretation \mathcal{I} .
A satisfying interpretation \mathcal{I} is a *model* for A .

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- ▶ A is *valid*, denoted $\models A$, iff $v_{\mathcal{I}}(A) = T$ for all interpretations \mathcal{I} . A valid propositional formula is also called a *tautology*.

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- ▶ A is **invalid** (or **falsifiable**), denoted $\not\models A$, iff it is not valid, that is, if $v_{\mathcal{I}}(A) = F$ for some interpretation \mathcal{I} .
- ▶ A **set** of formulae $U = \{A_1, \dots\}$ is (**simultaneously**) **satisfiable** iff there exists an interpretation \mathcal{I} such that $v_{\mathcal{I}}(A_i) = T$ for all i ; otherwise U is **unsatisfiable**. The satisfying interpretation is a **model** of U .

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There is a close relation between these four semantical concepts.

Theorem 5.1 (Satisfiable, Valid, Unsatisfiable, Invalid).

Let $A \in F$. A is *valid* iff $\neg A$ is *unsatisfiable*. A is *satisfiable* iff $\neg A$ is *invalid*.

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Proof.

Let \mathcal{I} be an arbitrary interpretation. $v_{\mathcal{I}}(A) = T$ if and only if $v_{\mathcal{I}}(\neg A) = F$ by definition of the truth value of negation. Since \mathcal{I} was arbitrary, $v_{\mathcal{I}}(A) = T$ for all interpretations if and only if $v_{\mathcal{I}}(\neg A) = F$ for all interpretations, that is, iff $\neg A$ is unsatisfiable.

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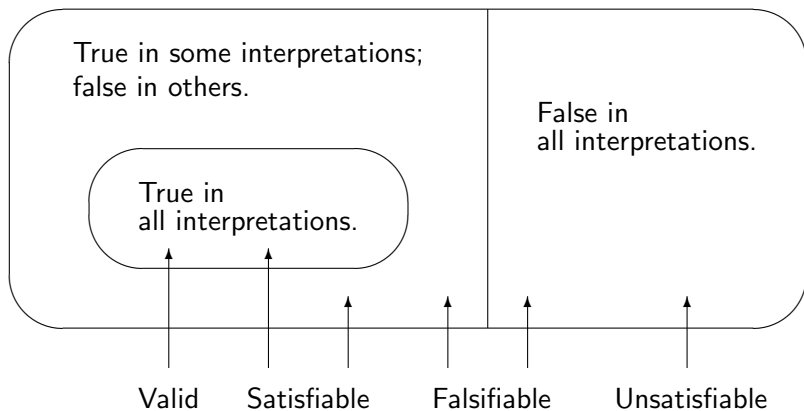
Let $A \in F$. A is *valid* iff $\neg A$ is *unsatisfiable*. A is *satisfiable* iff $\neg A$ is *invalid*.

Proof.

Let \mathcal{I} be an arbitrary interpretation. $v_{\mathcal{I}}(A) = T$ if and only if $v_{\mathcal{I}}(\neg A) = F$ by definition of the truth value of negation. Since \mathcal{I} was arbitrary, $v_{\mathcal{I}}(A) = T$ for all interpretations if and only if $v_{\mathcal{I}}(\neg A) = F$ for all interpretations, that is, iff $\neg A$ is unsatisfiable.

If A is satisfiable then for some interpretation \mathcal{I} , $v_{\mathcal{I}}(A) = T$. By definition of the truth value of negation, $v_{\mathcal{I}}(\neg A) = F$ so that $\neg A$ is invalid. Conversely, if $v_{\mathcal{I}}(\neg A) = F$ then $v_{\mathcal{I}}(A) = T$. □

Satisfiability and Validity



Decidability

Definition 5.2 (Decision Procedure).

Let $\mathcal{U} \subseteq \mathcal{F}$ be a set of (propositional) formulae. An algorithm is a *decision procedure* for \mathcal{U} if given a formula $A \in \mathcal{F}$, it terminates and returns the answer “yes” if $A \in \mathcal{U}$ and the answer “no” if $A \notin \mathcal{U}$.

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For a given formula A with $n = |\mathcal{P}_A|$, use truth tables to evaluate truth values for A . If $v_{\mathcal{I}}(A) = T$ for all 2^n possible interpretations \mathcal{I} , then answer “yes”; otherwise answer “no”. □

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This method is **not very efficient**; more efficient procedures will be introduced later.

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Let U be a set of formulas, A be a formula. A is a *logical consequence* of U , denoted $U \models A$, iff every model of U is a model of A .

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Example: Let $A = (p \vee r) \wedge (\neg q \vee \neg r)$. A is a logical consequence of $\{p, \neg q\}$, denoted $\{p, \neg q\} \models A$, as $v_{\mathcal{I}}(A) = T$ for all interpretations \mathcal{I} such that $\mathcal{I}(p) = T$ and $\mathcal{I}(q) = F$. But A is not valid, as $v_{\mathcal{I}}(A) = F$ for the interpretation \mathcal{I} where $\mathcal{I}(p) = F$, $\mathcal{I}(q) = T$, $\mathcal{I}(r) = T$.

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Proof. Left as an exercise.

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- ▶ Motivation
- ▶ Syntax
- ▶ Semantics
- ▶ Logical Equivalence
- ▶ Satisfiability & Validity
- ▶ **Summary**
- ▶ Motivation
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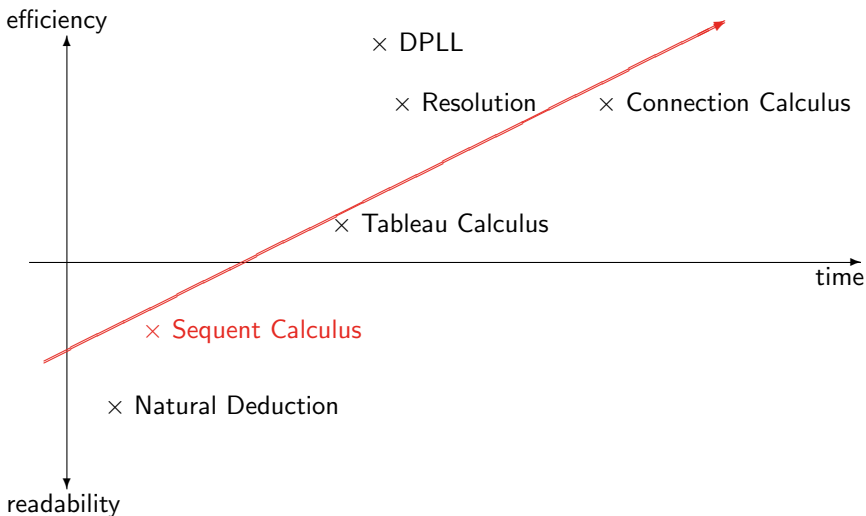
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Proof Search Calculi



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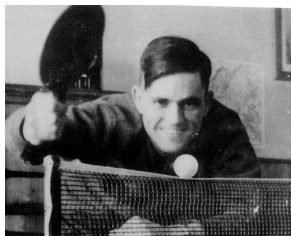


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- ▶ Natural Deduction and Sequent calculus was developed by **Gerhard Gentzen** in the 1930's
- ▶ Tools for investigating mathematical reasoning.

Sequents

Definition 8.1 (Sequent).

A *sequent* has the form $\Gamma \Longrightarrow \Delta$ in which Γ and Δ are finite (possibly empty) multisets of formulae. The left side of the sequent is called the *antecedent*, the right side is called the *succedent*.

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The Sequent Calculus LK

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$$\frac{\frac{\frac{p \Rightarrow p, q \text{ axiom} \quad p, q \Rightarrow q \text{ axiom}}{p, p \rightarrow q \Rightarrow q} \rightarrow\text{-left}}{p \wedge (p \rightarrow q) \Rightarrow q} \wedge\text{-left}}{\Rightarrow p \wedge (p \rightarrow q) \rightarrow q} \rightarrow\text{-right}$$

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- ▶ So let's look at the rules in detail!

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Examples of LK Proofs

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Calculus and Proof — General Definitions

Definition 8.2 (Calculus/Deductive System).

A *calculus* consists of *axioms* and *inference rules*.

Axioms have the form $\frac{}{w}$; *rules* have the form $\frac{w_1 \cdots w_n}{w}$
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Definition 8.3 (Proof, Derivation).

Let $\mathcal{A} = \{A_1, \dots\}$ be axioms and $\mathcal{R} = \{R_1, \dots\}$ be rules of a calculus.

- ▶ Let $\frac{}{w}$ be an instance of an axiom $A_i \in \mathcal{A}$. Then $\frac{}{w}$ is a *proof* of w .
- ▶ Let $\frac{w_1 \cdots w_n}{w}$ be an instance of a rule $R_i \in \mathcal{R}$ and $\mathcal{D}_1, \dots, \mathcal{D}_n$ proofs of w_1, \dots, w_n . Then $\frac{\mathcal{D}_1 \cdots \mathcal{D}_n}{w}$ is a *proof* of w .

A *derivation* is defined similarly, but leaves do not need to be axioms.

The Sequent Calculus LK

Definition 8.4 (Proofs in LK).

A *proof* of a formula A in the LK calculus is a proof of the sequent $\Rightarrow A$ using the rules and axiom of LK. A formula A is *provable*, written $\vdash A$, iff there is a proof for A .

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Theorem 8.1 (Soundness and Completeness of LK).

The calculus of natural deduction LK is sound and complete, i.e.

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Proof.

Next week!



Outline

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- ▶ Syntax
- ▶ Semantics
- ▶ Logical Equivalence
- ▶ Satisfiability & Validity
- ▶ Summary
- ▶ Motivation
- ▶ Sequent Calculus
- ▶ **Decision Procedure**
- ▶ Summary

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$$\begin{array}{c}
 \frac{}{p, \Longrightarrow p, r} \text{ axiom} \quad \frac{}{\bar{p}, \bar{q} \Longrightarrow \bar{r}} ? \\
 \hline
 \frac{p, p \rightarrow q \Longrightarrow r}{p \wedge (p \rightarrow q) \Longrightarrow r} \wedge\text{-left} \quad \rightarrow\text{-left} \\
 \hline
 \frac{}{\Longrightarrow p \wedge (p \rightarrow q) \rightarrow r} \rightarrow\text{-right}
 \end{array}$$

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- ▶ **Next week**: Soundness and Completeness proofs