

IN3070/4070 – Logic – Autumn 2019

Lecture 3: LK: Soundness & Completeness

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Today's Plan

- ▶ Semantics for Sequents
- ▶ Soundness
- ▶ Completeness

Outline

- ▶ Semantics for Sequents
- ▶ Soundness
- ▶ Completeness

Semantics for Sequents

Definition 1.1 (Valid sequent).

A sequent $\Gamma \Longrightarrow \Delta$ is *valid* if all interpretations that satisfy all formulas in Γ satisfy at least one formula in Δ .

Example.

The following sequents are valid:

- ▶ $p \Longrightarrow p$
- ▶ $p \rightarrow q, r \Longrightarrow p \rightarrow q, s$
- ▶ $p, p \rightarrow q \Longrightarrow q$
- ▶ $p \rightarrow q \Longrightarrow \neg q \rightarrow \neg p$

Definition 1.2 (Countermodel/falsifiable sequent).

- ▶ An interpretation \mathcal{I} is a *countermodel* for the sequent $\Gamma \Longrightarrow \Delta$ if $v_{\mathcal{I}}(A) = T$ for all formulae $A \in \Gamma$ and $v_{\mathcal{I}}(B) = F$ for all formulae $B \in \Delta$
- ▶ We say that a countermodel for a sequent *falsifies* the sequent.
- ▶ A sequent is *falsifiable* if it has a countermodel.

Example.

The following sequents are falsifiable:

- ▶ $p \Longrightarrow q$ Countermodel: $\mathcal{I}(p) = T, \mathcal{I}(q) = F$
- ▶ $p \vee q \Longrightarrow p \wedge q$ Countermodel: same, or $\mathcal{I}(p) = F, \mathcal{I}(q) = T$
- ▶ $\Longrightarrow p$ Countermodel: $\mathcal{I}(p) = F$
- ▶ $p \Longrightarrow$ Countermodel: $\mathcal{I}(p) = T$
- ▶ \Longrightarrow Countermodel: *all interpretations!*

Summary

Valid

- ▶ $p, p \rightarrow q \implies q$
- ▶ If $\mathcal{I} \models p$ and $\mathcal{I} \models p \rightarrow q$, then $\mathcal{I} \models q$.
- ▶ Validity is a semantic notion

Falsifiability

- ▶ $\neg p, p \rightarrow q \implies \neg q$
- ▶ An interpretation \mathcal{I} s.t. $\mathcal{I} \not\models p$ and $\mathcal{I} \models q$.

Provable

$$\frac{p \implies p, q \quad q, q \implies q}{p, p \rightarrow q \implies q}$$

- ▶ Provability is a syntactic notion

Not provable

$$\frac{\frac{q \implies p, p}{\neg p \implies p, \neg q} \quad \frac{q, q \implies p}{q, \neg p \implies \neg q}}{\neg p, p \rightarrow q \implies \neg q}$$

Outline

- ▶ Semantics for Sequents
- ▶ **Soundness**
- ▶ Completeness

Soundness of LK

- ▶ We want all LK-provable sequents to be valid!
- ▶ If they are not, then LK would be **incorrect** or **unsound** ...

Definition 2.1 (Soundness).

*The sequent calculus LK is **sound** if every LK-provable sequent is valid.*

Theorem 2.1.

The sequent calculus LK is sound.

How to show the Soundness Theorem?

We show the following lemmas:

- ▶ All LK-rules preserve falsifiability upwards.
- ▶ An LK-derivation with a falsifiable root sequent has at least one falsifiable leaf sequent
- ▶ All axioms are valid

Finally, we use these lemmas to show the soundness theorem.

Preservation of Falsifiability

Definition 2.2.

An LK-rule θ *preserves falsifiability (upwards)* if all interpretations that falsify the conclusion w of an instance $\frac{w_1 \cdots w_n}{w}$ of θ also falsify at least one of the premises w_i .

Lemma 2.1.

All LK-rules preserve falsifiability.

Proving Preservation of Falsifiability

- ▶ The proof has a separate case for each LK-rule.
- ▶ Consider for instance the \rightarrow -left-rule:

$$\frac{\Gamma \Longrightarrow A, \Delta \quad \Gamma, B \Longrightarrow \Delta}{\Gamma, A \rightarrow B \Longrightarrow \Delta} \rightarrow\text{-left}$$

- ▶ We have to show that all instances of \rightarrow -left preserve falsifiability upwards.
- ▶ We let Γ , Δ , A and B in the rule stand for arbitrary (sets of) propositional formulae

Proof for \neg -right

Proof for \neg -right.

$$\frac{\Gamma, A \implies \Delta}{\Gamma \implies \neg A, \Delta} \neg\text{-right}$$

- ▶ Assume that \mathcal{I} falsifies the conclusion.
- ▶ Then $\mathcal{I} \models \Gamma$, $\mathcal{I} \not\models \neg A$ and \mathcal{I} falsifies all formulae in Δ .
- ▶ Per model semantics, we have $\mathcal{I} \models A$.
- ▶ Therefore, $\mathcal{I} \models \Gamma \cup \{A\}$ and \mathcal{I} falsifies all formulae in Δ .
- ▶ Thus, \mathcal{I} falsifies the premise.



Proof for \rightarrow -leftProof for \rightarrow -left.

$$\frac{\Gamma \Longrightarrow A, \Delta \quad \Gamma, B \Longrightarrow \Delta}{\Gamma, A \rightarrow B \Longrightarrow \Delta} \rightarrow\text{-left}$$

- ▶ Assume that \mathcal{I} falsifies the conclusion.
- ▶ Then \mathcal{I} satisfies $\Gamma \cup \{A \rightarrow B\}$ and falsifies all formulae in Δ .
- ▶ Since \mathcal{I} satisfies $A \rightarrow B$, by definition of model semantics,
 - (1) $\mathcal{I} \not\models A$, or
 - (2) $\mathcal{I} \models B$.
- ▶ In case (1), \mathcal{I} falsifies the left premiss.
- ▶ In case (2) \mathcal{I} falsifies the right premiss.



Proving “for all”-statements

- ▶ Consider the statement “for all $x \in S$: $P(x)$ ”.
- ▶ We can show this by showing $P(a)$ for each element $a \in S$.
- ▶ What if S is very large, or infinite?
- ▶ We can **generalise from an arbitrary element**:
 - ▶ Choose an **arbitrary** element $a \in S$.
 - ▶ Show that $P(a)$ holds.
 - ▶ Since a was arbitrarily chosen, the original statement must hold.

How to show the Soundness Theorem?

We show the following lemmas:

- ▶ All LK-rules preserve falsifiability upwards.
- ▶ An LK-derivation with a falsifiable root sequent has at least one falsifiable leaf sequent
- ▶ All axioms are valid

Finally, we use these lemmas to show the soundness theorem.

Reminder: LK derivation

Definition 2.3 (LK Derivation).

- Let $\Gamma \Longrightarrow \Delta$ be a sequent. Then

$$\Gamma \Longrightarrow \Delta$$

is an **LK-derivation** of $\Gamma \Longrightarrow \Delta$.

- Let $\frac{w_1 \quad \dots \quad w_n}{\Gamma \Longrightarrow \Delta}$ be an instance of an LK rule, and $\mathcal{D}_1, \dots, \mathcal{D}_n$ derivations of w_1, \dots, w_n . Then

$$\frac{\mathcal{D}_1 \quad \dots \quad \mathcal{D}_n}{\Gamma \Longrightarrow \Delta}$$

is an **LK-derivation** of $\Gamma \Longrightarrow \Delta$.

Existence of a falsifiable leaf sequent

Lemma 2.2.

If an interpretation \mathcal{I} falsifies the root sequent of an LK-derivation δ , then \mathcal{I} falsifies at least one of the leaf sequents of δ .

Proof.

By structural induction on the LK-derivation δ .

Induction base: δ is a sequent $\Gamma \Longrightarrow \Delta$:

$$\Gamma \Longrightarrow \Delta$$

- ▶ Here, $\Gamma \Longrightarrow \Delta$ is both root sequent and (only) leaf sequent.
- ▶ Assume \mathcal{I} falsifies $\Gamma \Longrightarrow \Delta$.
- ▶ Then \mathcal{I} falsifies a leaf sequent in δ , namely $\Gamma \Longrightarrow \Delta$.



Continued.

Induction step: δ is a derivation of the form

$$\frac{\begin{array}{c} \mathcal{D}_1 \\ \vdots \\ \Gamma_1 \implies \Delta_1 \end{array} \quad \cdots \quad \begin{array}{c} \mathcal{D}_n \\ \vdots \\ \Gamma_n \implies \Delta_n \end{array}}{\Gamma \implies \Delta} r$$

for some smaller derivations \mathcal{D}_i with roots $\Gamma_i \implies \Delta_i$.

- ▶ Assume \mathcal{I} falsifies $\Gamma \implies \Delta$.
- ▶ Rule r preserves falsifiability upwards.
- ▶ Therefore \mathcal{I} falsifies $\Gamma_i \implies \Delta_i$ for some $i \in \{1, \dots, n\}$.
- ▶ By induction, \mathcal{I} falsifies one of the leaf sequents of \mathcal{D}_i .
- ▶ This is also a leaf sequent of δ



How to show the Soundness Theorem?

We show the following lemmas:

- ▶ All LK-rules preserve falsifiability upwards.
- ▶ An LK-derivation with a falsifiable root sequent has at least one falsifiable leaf sequent
- ▶ All axioms are valid

Finally, we use these lemmas to show the soundness theorem.

All axioms are valid

Lemma 2.3.

All axioms are valid.

Proof.

$$\Gamma, A \implies A, \Delta$$

- ▶ We will show that all interpretations that satisfy the antecedent also satisfy at least one formula of the succedent.
- ▶ Let \mathcal{I} be an arbitrarily chosen interpretation that satisfies the antecedent.
- ▶ Then \mathcal{I} satisfies the formula A in the succedent.



Proof of the Soundness Theorem for LK

Proof of soundness.

- ▶ Assume that \mathcal{P} is an LK-proof for the sequent $\Gamma \Longrightarrow \Delta$.
 - ▶ \mathcal{P} is an LK-derivation where every leaf is an axiom.
- ▶ For the sake of contradiction, assume that $\Gamma \Longrightarrow \Delta$ is **not** valid.
- ▶ Then there is a countermodel \mathcal{I} that falsifies $\Gamma \Longrightarrow \Delta$.
- ▶ We know from the previous Lemma that \mathcal{I} falsifies at least one leaf sequent of \mathcal{P} .
- ▶ Then \mathcal{P} has a leaf sequent that is not an axiom, since axioms are not falsifiable.
- ▶ So \mathcal{P} cannot be an LK-proof.



Analysis

- ▶ An LK-derivation with a falsifiable root sequent has at least one falsifiable leaf sequent
- ▶ An axiom is never falsifiable
- ▶ Roots of LK-proofs are valid
- ▶ Most of this is independent of the actual rules.
- ▶ Central part is proving that **every rule preserves falsifiability**
- ▶ Shown individually for each rule
- ▶ Can add new rules, and just show “soundness” for those

Outline

- ▶ Semantics for Sequents
- ▶ Soundness
- ▶ Completeness

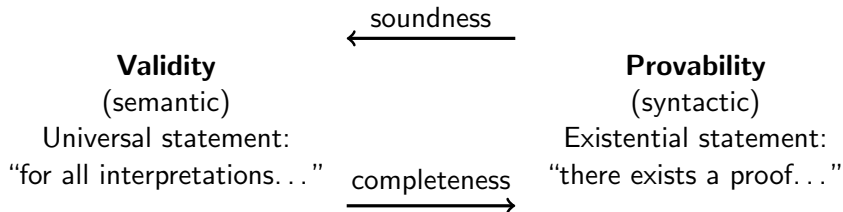
Completeness — Introduction

Definition 3.1 (Soundness).

The calculus LK is *sound* if any LK-provable sequent is valid.

Definition 3.2 (Completeness).

The calculus LK is *complete* if every valid sequent is LK-provable.



Completeness — Introduction

Soundness: $\Gamma \Longrightarrow \Delta$ provable $\Rightarrow \Gamma \Longrightarrow \Delta$ valid

Completeness: $\Gamma \Longrightarrow \Delta$ valid $\Rightarrow \Gamma \Longrightarrow \Delta$ provable

- ▶ Soundness and Completeness are dual notions
- ▶ Soundness says that we cannot prove *more* than the valid sequents
- ▶ Completeness says that we can prove *all* valid sequents
- ▶ A sequent is valid if and only if it is not falsifiable
- ▶ We can therefore also express soundness and completeness as:

Soundness: $\Gamma \Longrightarrow \Delta$ falsifiable $\Rightarrow \Gamma \Longrightarrow \Delta$ not provable

Completeness: $\Gamma \Longrightarrow \Delta$ not provable $\Rightarrow \Gamma \Longrightarrow \Delta$ falsifiable

An LK-machine?



Soundness

All that is printed is valid.

Completeness

All that is valid will get printed.

- ▶ Something can be sound without being complete.
 - ▶ Then too little is shown.
 - ▶ Example with prime numbers: 2, 5, 7, 11, 17, 19, ...
- ▶ Something can be complete without being sound.
 - ▶ Then too much is shown
 - ▶ Example with prime numbers: 2, 3, 5, 7, 9, 11, 13, 15 ...
- ▶ We want both:
 - ▶ Not too much, not too little.
 - ▶ Example with prime numbers: 2, 3, 5, 7, 11, 13, 17, 19 ...

The Completeness Theorem

Theorem 3.1 (Completeness).

If $\Gamma \implies \Delta$ is valid, then it is provable in LK.

To show completeness of our calculus, we show the equivalent statement:

Lemma 3.1 (Model existence).

If $\Gamma \implies \Delta$ is not provable in LK, then it is falsifiable.

This means that there is an interpretation that makes all formulae in Γ true and all formulae in Δ false.

Proof of Completeness

Assume $\Gamma \implies \Delta$ is not provable.

- ▶ Construct a derivation \mathcal{D} from $\Gamma \implies \Delta$ such that no further rule applications are possible. “A maximal derivation”.
- ▶ Then there is (at least) one branch \mathcal{B} that does not end in an axiom. We then have:
 - ▶ The leaf sequent of \mathcal{B} contains only atomic formulae, and
 - ▶ the leaf sequent of \mathcal{B} is not an axiom.
- ▶ We construct an interpretation that falsifies $\Gamma \implies \Delta$. Let

\mathcal{B}^\top be the set of formulae that occur in an antecedent on \mathcal{B} , and

\mathcal{B}^\perp be the set of formulae that occur in a succedent on \mathcal{B} , and

$\mathcal{I}_{\mathcal{B}}$ be the interpretation that makes all atomic formulae in \mathcal{B}^\top true and all other atomic formulae (in particular those in \mathcal{B}^\perp) false.

Example

$$\frac{\frac{p \implies q, p}{p \rightarrow q, p \implies q} \quad \frac{q, p \implies q}{p \rightarrow q, p \implies q}}{\frac{r \implies q, p \quad q, r \implies q}{p \rightarrow q, r \implies q}}}{p \rightarrow q, p \vee r \implies q} \\
 \frac{p \rightarrow q, p \vee r \implies q}{p \rightarrow q \implies (p \vee r) \rightarrow q}$$

We see that the branch \mathcal{B} with leaf sequent $r \implies q, p$ is not closed.

$$\mathcal{B}^\top = \{r, p \rightarrow q, p \vee r\}$$

$$\mathcal{B}^\perp = \{q, p, (p \vee r) \rightarrow q\}$$

$$\mathcal{I}_{\mathcal{B}} = \text{interpretation with } \mathcal{I}_{\mathcal{B}}(r) = T \text{ og } \mathcal{I}_{\mathcal{B}}(q) = \mathcal{I}_{\mathcal{B}}(p) = F$$

To show: this interpretation falsifies the root sequent.

Proof of Completeness, cont.

- ▶ We show by structural induction *on propositional formulae* that the interpretation $\mathcal{I}_{\mathcal{B}}$ makes all formulae in \mathcal{B}^{\top} true, and all formulae in \mathcal{B}^{\perp} false.
- ▶ We show for all propositional formulae A that
 - If $A \in \mathcal{B}^{\top}$, then $\mathcal{I}_{\mathcal{B}} \models A$.
 - If $A \in \mathcal{B}^{\perp}$, then $\mathcal{I}_{\mathcal{B}} \not\models A$.

Induction base: A is an atomic formula in $\mathcal{B}^{\top}/\mathcal{B}^{\perp}$.

- ▶ Our statement holds because that is how we defined $\mathcal{B}^{\top}/\mathcal{B}^{\perp}$.

Induction step: From the assumption (IH) that the statement holds for A and B , we must show that it holds for $\neg A$, $(A \wedge B)$, $(A \vee B)$ og $(A \rightarrow B)$. These are four cases, of which we show three here.

Case: Negation in antecedent/succedent

Assume that $\neg A \in \mathcal{B}^\top$.

- ▶ $\neg A$ appears in an antecedent, it can't 'go away' unless \neg -left is applied
- ▶ Since the derivation is maximal, \neg -left *is* eventually applied
- ▶ A appears in a succedent, so we have $A \in \mathcal{B}^\perp$.
- ▶ By the IH, we have $\mathcal{I}_\mathcal{B} \not\models A$.
- ▶ By definition of model semantics, $\mathcal{I}_\mathcal{B} \models \neg A$.

Assume that $\neg A \in \mathcal{B}^\perp$.

- ▶ $\neg A$ appears in a succedent, it can't 'go away' unless \neg -right is applied
- ▶ Since the derivation is maximal, \neg -right *is* eventually applied
- ▶ A appears in an antecedent, so we have $A \in \mathcal{B}^\top$.
- ▶ By the IH, we have $\mathcal{I}_\mathcal{B} \models A$.
- ▶ By definition of model semantics, $\mathcal{I}_\mathcal{B} \not\models \neg A$.

Case: Conjunction in antecedent/succedent

Assume that $(A \wedge B) \in \mathcal{B}^\top$.

- ▶ Since the derivation is maximal, we have $A \in \mathcal{B}^\top$ and $B \in \mathcal{B}^\top$.
- ▶ By the IH, we have $\mathcal{I}_\mathcal{B} \models A$ and $\mathcal{I}_\mathcal{B} \models B$.
- ▶ By definition of model semantics, $\mathcal{I}_\mathcal{B} \models (A \wedge B)$.

Assume that $(A \wedge B) \in \mathcal{B}^\perp$.

- ▶ Since the derivation is maximal, \wedge -right is eventually applied...
- ▶ ...introducing A in the succedent of one branch and B on the other.
- ▶ One of them is our branch \mathcal{B} , and therefore $A \in \mathcal{B}^\perp$ or $B \in \mathcal{B}^\perp$.
- ▶ By the IH, we have $\mathcal{I}_\mathcal{B} \not\models A$ or $\mathcal{I}_\mathcal{B} \not\models B$
- ▶ By definition of model semantics, $\mathcal{I}_\mathcal{B} \not\models (A \wedge B)$

Case: Implication in antecedent/succedent

Assume that $(A \rightarrow B) \in \mathcal{B}^\top$.

- ▶ Since the derivation is maximal, \rightarrow -left is eventually applied...
- ▶ ...introducing A in the succedent of one branch and B in the antecedent of the other.
- ▶ One of them is our branch \mathcal{B} , and therefore $A \in \mathcal{B}^\perp$ or $B \in \mathcal{B}^\top$.
- ▶ By the IH, we have $\mathcal{I}_\mathcal{B} \not\models A$ or $\mathcal{I}_\mathcal{B} \models B$
- ▶ By definition of model semantics, $\mathcal{I}_\mathcal{B} \models (A \rightarrow B)$

Assume that $(A \rightarrow B) \in \mathcal{B}^\perp$.

- ▶ Since the derivation is maximal, we have $A \in \mathcal{B}^\top$ and $B \in \mathcal{B}^\perp$.
- ▶ By the IH, we have $\mathcal{I}_\mathcal{B} \models A$ and $\mathcal{I}_\mathcal{B} \not\models B$
- ▶ By definition of model semantics, $\mathcal{I}_\mathcal{B} \not\models (A \rightarrow B)$

Analysis

- ▶ If there is no proof for a sequent, there is a derivation. . .
 - ▶ Where all possible rules have been applied
 - ▶ At least one branch \mathcal{B} has not been closed with an axiom
- ▶ We can use the atoms on \mathcal{B} to construct an interpretation $\mathcal{I}_{\mathcal{B}}$
- ▶ $\mathcal{I}_{\mathcal{B}}$ makes atoms left true, and atoms right false
- ▶ $\mathcal{I}_{\mathcal{B}}$ also makes *all other* formulae left true and right false, because. . .
 - ▶ for every non-atomic formula, there is a rule that decomposes it
 - ▶ which must have been applied
 - ▶ and that guarantees that $\mathcal{I}_{\mathcal{B}}$ falsifies sequents, based on structural induction
- ▶ Structural induction on formulae, while soundness was by induction on derivations
- ▶ Not possible to prove completeness ‘one rule at a time’

One-sided Sequent Calculus

- ▶ Only sequents with empty succedent: $\Gamma \Longrightarrow$
- ▶ To prove A , start with $\neg A \Longrightarrow$
- ▶ “Proof by contradiction” or “refutation”
- ▶ Negation rules combined with others:

$$\frac{\Gamma, \neg A, \neg B \Longrightarrow}{\Gamma, \neg(A \vee B) \Longrightarrow} \neg\vee$$

$$\frac{\Gamma, \neg A \Longrightarrow \quad \Gamma, \neg B \Longrightarrow}{\Gamma, \neg(A \wedge B) \Longrightarrow} \neg\wedge$$

- ▶ Double negation:

$$\frac{\Gamma, A \Longrightarrow}{\Gamma, \neg\neg A \Longrightarrow} \neg\neg$$

- ▶ Axiom:

$$\overline{\Gamma, A, \neg A \Longrightarrow}$$

- ▶ Can do the same with empty antecedents $\Longrightarrow \Delta$

Example with One-sided Sequents

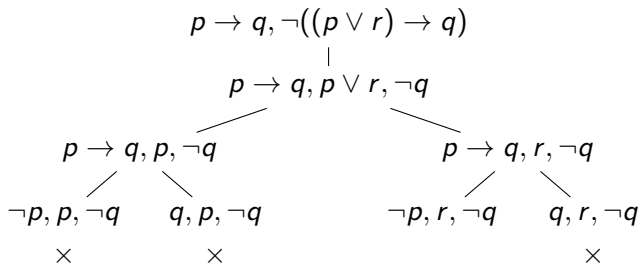
- ▶ Instead of $p \rightarrow q \Rightarrow (p \vee r) \rightarrow q$
- ▶ Start with $p \rightarrow q, \neg((p \vee r) \rightarrow q) \Rightarrow$

$$\frac{\frac{\overline{\neg p, p, \neg q \Rightarrow}}{p \rightarrow q, p, \neg q \Rightarrow} \quad \frac{\overline{q, p, \neg q \Rightarrow}}{p \rightarrow q, p \vee r, \neg q \Rightarrow}}{\frac{\overline{\neg p, r, \neg q \Rightarrow} \quad \overline{q, r, \neg q \Rightarrow}}{p \rightarrow q, r, \neg q \Rightarrow}}$$

- ▶ Soundness and completeness very similar to two-sided LK.

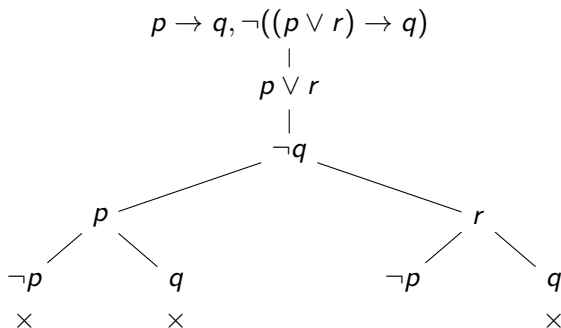
Semantic Tableaux (Ben-Ari 2.6)

- ▶ Others call these 'block tableaux'
- ▶ Sequent arrow \implies not needed for one-sided calculus
- ▶ More handy to write top-down, like everybody else
- ▶ Mark 'closed' branches (with axioms) with \times



Short Hand Notation for Tableaux

- ▶ Only write the new formula in every node.
- ▶ Even more handy to write
- ▶ Close branch using literals A and $\neg A$ anywhere on a branch.
- ▶ Have to make sure that all rules were used on every branch!



Summary and Outlook

Until now:

- ▶ Propositional logic and model semantics
- ▶ LK Calculus
- ▶ Soundness
- ▶ Completeness

Next three weeks:

- ▶ First-order logic and model semantics
- ▶ LK Calculus for first-order logic
- ▶ Soundness
- ▶ Completeness

After that: resolution, DPLL, Prolog,...