IN3070/4070 – Logic – Autumn 2019

Lecture 3: LK: Soundness & Completeness

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Today's Plan

- Semantics for Sequents
- Soundness
- Completeness

Outline

- Semantics for Sequents
- Soundness
- Completeness

Semantics for Sequents

Definition 1.1 (Valid sequent).

A sequent $\Gamma \implies \Delta$ is valid if all interpretations that satisfy all formulas in Γ satisfy at least one formula in Δ.

Example.

The following sequents are valid:

- $p \implies p$
- $p \to q, r \implies p \to q, s$
- $p, p \to q \implies q$
- $p \to q \implies \neg q \to \neg p$

Definition 1.2 (Countermodel/falsifiable sequent).

- *An interpretation $\mathcal{I}$ is a countermodel for the sequent $\Gamma \Longrightarrow \Delta$ if $v_{\mathcal{I}}(A) = T$ for all formulae $A \in \Gamma$ and $v_{\mathcal{I}}(B) = F$ for all formulae $B \in \Delta$*
- *We say that a countermodel for a sequent falsifies the sequent.*
- *A sequent is falsifiable if it has a countermodel.*

Example.

The following sequents are falsifiable:

- $p \Longrightarrow q$ — *Countermodel:* $\mathcal{I}(p) = T, \mathcal{I}(q) = F$
- $p \vee q \Longrightarrow p \wedge q$ — *Countermodel: same, or* $\mathcal{I}(p) = F, \mathcal{I}(q) = T$
- $\Longrightarrow p$ — *Countermodel:* $\mathcal{I}(p) = F$
- $p \Longrightarrow$ — *Countermodel:* $\mathcal{I}(p) = T$
- $\Longrightarrow$ — *Countermodel: all interpretations!*

Summary

Valid

- $p, p \to q \Longrightarrow q$
- If $\mathcal{I} \models p$ and $\mathcal{I} \models p \to q$, then $\mathcal{I} \models q$.
- Validity is a semantic notion

Provable

$$\frac{p \Longrightarrow p, q \qquad q, q \Longrightarrow q}{p, p \to q \Longrightarrow q}$$

- Provability is a syntactic notion

Falsifiability

- $\neg p, p \to q \Longrightarrow \neg q$
- An interpretation $\mathcal{I}$ s.t. $\mathcal{I} \not\models p$ and $\mathcal{I} \models q$.

Not provable

$$\frac{\dfrac{q \Longrightarrow p, p}{\neg p \Longrightarrow p, \neg q} \qquad \dfrac{q, q \Longrightarrow p}{q, \neg p \Longrightarrow \neg q}}{\neg p, p \to q \Longrightarrow \neg q}$$

Outline

Soundness of LK

- We want all LK-provable sequents to be valid!
- If they are not, then LK would be incorrect or unsound . . .

Definition 2.1 (Soundness).

The sequent calculus LK *is sound if every* LK*-provable sequent is valid.*

Theorem 2.1.

The sequent calculus LK *is sound.*

How to show the Soundness Theorem?

We show the following lemmas:

- All LK-rules preserve falsifiability upwards.
- An LK-derivation with a falsifiable root sequent has at least one falsifiable leaf sequent
- All axioms are valid

Finally, we use these lemmas to show the soundness theorem.

Preservation of Falsifiability

Definition 2.2.

An LK-rule θ preserves falsifiability (upwards) if all interpretations that falsify the conclusion w of an instance $\frac{w_1 \cdots w_n}{w}$ of θ also falsify at least one of the premises w_i.

Lemma 2.1.

All LK-rules preserve falsifiability.

Proving Preservation of Falsifiability

- The proof has a separate case for each LK-rule.
- Consider for instance the $\to$-left-rule:

$$\frac{\Gamma \implies A, \Delta \qquad \Gamma, B \implies \Delta}{\Gamma, A \to B \implies \Delta}\ \to\text{-left}$$

- We have to show that all instances of $\to$-left preserve falsifiability upwards.
- We let Γ, Δ, A and B in the rule stand for arbitrary (sets of) propositional formulae

Proof for ¬-right

Proof for ¬-right.

$$\frac{\Gamma, A \implies \Delta}{\Gamma \implies \neg A, \Delta}\ \neg\text{-right}$$

- Assume that $\mathcal{I}$ falsifies the conclusion.
- Then $\mathcal{I} \models \Gamma$, $\mathcal{I} \not\models \neg A$ and $\mathcal{I}$ falsifies all formulae in Δ.
- Per model semantics, we have $\mathcal{I} \models A$.
- Therefore, $\mathcal{I} \models \Gamma \cup \{A\}$ and $\mathcal{I}$ falsifies all formlae in Δ.
- Thus, $\mathcal{I}$ falsifies the premisse.

□

Proof for →-left

Proof for →-left.

$$\frac{\Gamma \implies A, \Delta \qquad \Gamma, B \implies \Delta}{\Gamma, A \to B \implies \Delta} \to\text{-left}$$

- Assume that $\mathcal{I}$ falsifies the conclusion.
- Then $\mathcal{I}$ satisfies $\Gamma \cup \{A \to B\}$ and falsifies all formlae in Δ.
- Since $\mathcal{I}$ satisfies $A \to B$, by definition of model semantics,
  (1) $\mathcal{I} \not\models A$, or
  (2) $\mathcal{I} \models B$.
- In case (1), $\mathcal{I}$ falsifies the left premisse.
- In case (2) $\mathcal{I}$ falsifies the right premisse.

□

Proving "for all"-statements

- Consider the statement "for all $x \in S$: $P(x)$".
- We can show this by showing $P(a)$ for each element $a \in S$.
- What if S is very large, or infinite?
- We can generalise from an arbitrary element:
  - Choose an arbitrary element $a \in S$.
  - Show that $P(a)$ holds.
  - Since a was arbitrarily chosen, the original statement must hold.

How to show the Soundness Theorem?

We show the following lemmas:

- All LK-rules preserve falsifiability upwards.
- An LK-derivation with a falsifiable root sequent has at least one falsifiable leaf sequent
- All axioms are valid

Finally, we use these lemmas to show the soundness theorem.

Reminder: LK derivation

Definition 2.3 (LK Derivation).

- *Let $\Gamma \implies \Delta$ be a sequent. Then*

$$\Gamma \implies \Delta$$

  is an LK-derivation of $\Gamma \implies \Delta$.
- *Let* $\dfrac{w_1 \quad \cdots \quad w_n}{\Gamma \implies \Delta}$ *be an instance of an LK rule, and $\mathcal{D}_1, \ldots, \mathcal{D}_n$ derivations of $w_1, \ldots, w_n$. Then*

$$\frac{\mathcal{D}_1 \quad \cdots \quad \mathcal{D}_n}{\Gamma \implies \Delta}$$

  is an LK-derivation of $\Gamma \implies \Delta$.

Existence of a falsifiable leaf sequent

Lemma 2.2.

If an interpretation $\mathcal{I}$ falsifies the root sequent of an LK*-derivation δ, then $\mathcal{I}$ falsifies at least one of the leaf sequents of δ.*

Proof.

By structural induction on the LK-derivation δ.
Induction base: δ is a sequent $\Gamma \implies \Delta$:

$$\Gamma \implies \Delta$$

- Here, $\Gamma \implies \Delta$ is both root sequent and (only) leaf sequent.
- Assume $\mathcal{I}$ falsifies $\Gamma \implies \Delta$.
- Then $\mathcal{I}$ falsifies a leaf sequent in δ, namely $\Gamma \implies \Delta$.

□

Continued.

Induction step: δ is a derivation of the form

$$\cfrac{\begin{matrix}\mathcal{D}_1 \\ \vdots \\ \Gamma_1 \implies \Delta_1\end{matrix} \quad \cdots \quad \begin{matrix}\mathcal{D}_n \\ \vdots \\ \Gamma_n \implies \Delta_n\end{matrix}}{\Gamma \implies \Delta}\, r$$

for some smaller derivations $\mathcal{D}_i$ with roots $\Gamma_i \implies \Delta_i$.

- Assume $\mathcal{I}$ falsifies $\Gamma \implies \Delta$.
- Rule r preserves falsifiability upwards.
- Therefore $\mathcal{I}$ falsifies $\Gamma_i \implies \Delta_i$ for some $i \in \{1, \dots, n\}$.
- By induction, $\mathcal{I}$ falsifies one of the leaf sequents of $\mathcal{D}_i$.
- This is also a leaf sequent of δ

□

How to show the Soundness Theorem?

We show the following lemmas:

- All LK-rules preserve falsifiability upwards.
- An LK-derivation with a falsifiable root sequent has at least one falsifiable leaf sequent
- All axioms are valid

Finally, we use these lemmas to show the soundness theorem.

All axioms are valid

Lemma 2.3.

All axioms are valid.

Proof.

$$\Gamma, A \implies A, \Delta$$

- We will show that all interpretations that satisfy the antecedent also satisfy at least one formula of the succedent.
- Let $\mathcal{I}$ be an arbitrarily chosen interpretation that satisfies the antecedent.
- Then $\mathcal{I}$ satisfies the formula A in the succedent.

□

Proof of the Soundness Theorem for LK

Proof of soundness.

- Assume that $\mathcal{P}$ is an LK-proof for the sequent $\Gamma \implies \Delta$.
  - $\mathcal{P}$ is an LK-derivation where every leaf is an axiom.
- For the sake of contradiction, assume that $\Gamma \implies \Delta$ is not valid.
- Then there is a countermodel $\mathcal{I}$ that falsifies $\Gamma \implies \Delta$.
- We know from the previous Lemma that $\mathcal{I}$ falsifies at least one leaf sequent of $\mathcal{P}$.
- Then $\mathcal{P}$ has a leaf sequent that is not an axiom, since axioms are not falsifiable.
- So $\mathcal{P}$ cannot be an LK-proof.

□

Analysis

- An LK-derivation with a falsifiable root sequent has at least one falsifiable leaf sequent
- An axiom is never falsifiable
- Roots of LK-proofs are valid
- Most of this is independent of the actual rules.
- Central part is proving that every rule preserves falsifiability
- Shown individually for each rule
- Can add new rules, and just show "soundness" for those

Outline

- Semantics for Sequents
- Soundness
- Completeness

Completeness — Introduction

Definition 3.1 (Soundness).
The calculus LK *is sound if any* LK*-provable sequent is valid.*

Definition 3.2 (Completeness).
The calculus LK *is complete if every valid sequent is* LK*-provable.*

soundness ←

Validity
(semantic)
Universal statement:
"for all interpretations..."

Provability
(syntactic)
Existential statement:
"there exists a proof..."

completeness →

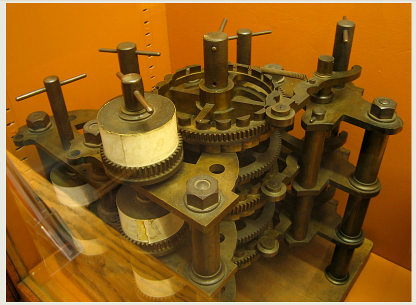
Completeness — Introduction

Soundnes: $\Gamma \Longrightarrow \Delta$ provable $\Rightarrow$ $\Gamma \Longrightarrow \Delta$ valid
Completeness: $\Gamma \Longrightarrow \Delta$ valid $\Rightarrow$ $\Gamma \Longrightarrow \Delta$ provable

- Soundness and Completeness are dual notions
- Soundness says that we cannot prove *more* than the valid sequents
- Completeness says that we can prove *all* valid sequents
- A sequent is valid if and only if it is not falsifiable
- We can therefore also express soundness and completeness as:

Soundness: $\Gamma \Longrightarrow \Delta$ falsifiable $\Rightarrow$ $\Gamma \Longrightarrow \Delta$ not provable
Completeness: $\Gamma \Longrightarrow \Delta$ not provable $\Rightarrow$ $\Gamma \Longrightarrow \Delta$ falsifiable

An LK-machine?



Soundness

All that is printed is valid.

Completeness

All that is valid will get printed.

- Something can be sound without being complete.
  - Then too little is shown.
  - Example with prime numbers: 2, 5, 7, 11, 17, 19, ...
- Something can be complete without being sound.
  - Then too much is shown
  - Example with prime numbers: 2, 3, 5, 7, 9, 11, 13, 15 ...
- We want both:
  - Not too much, not too little.
  - Example with prime numbers: 2, 3, 5, 7, 11, 13, 17, 19 ...

The Completeness Theorem

Theorem 3.1 (Completeness).

If $\Gamma \Longrightarrow \Delta$ is valid, then it is provable in LK.

To show completeness of our calculus, we show the equivalent statement:

Lemma 3.1 (Model existence).

If $\Gamma \Longrightarrow \Delta$ is not provable in LK, *then it is falsifiable.*

This means that there is an interpretation that makes all formulae in Γ true and all formulae in Δ false.

Proof of Completeness

Assume $\Gamma \Longrightarrow \Delta$ is not provable.

- Construct a derivation $\mathcal{D}$ from $\Gamma \Longrightarrow \Delta$ such that no further rule applications are possible. "A maximal derivation".
- Then there is (at least) one branch $\mathcal{B}$ that does not end in an axiom. We then have:
  - The leaf sequent of $\mathcal{B}$ contains only atomic formulae, and
  - the leaf sequent of $\mathcal{B}$ is not an axiom.
- We construct an interpretation that falsifies $\Gamma \Longrightarrow \Delta$. Let

  $\mathcal{B}^\top$ be the set of formulae that occur in an antecedent on $\mathcal{B}$, and

  $\mathcal{B}^\bot$ be the set of formulae that occur in an succedent on $\mathcal{B}$, and

  $\mathcal{I}_\mathcal{B}$ be the interpretation that makes all atomic formulae in $\mathcal{B}^\top$ true and all other atomic formulae (in particular those in $\mathcal{B}^\bot$) false.

Example

$$\dfrac{\dfrac{\dfrac{\overline{p \Longrightarrow q, p} \quad \overline{q, p \Longrightarrow q}}{p \to q, p \Longrightarrow q} \quad \dfrac{r \Longrightarrow q, p \quad \overline{q, r \Longrightarrow q}}{p \to q, r \Longrightarrow q}}{p \to q, p \lor r \Longrightarrow q}}{p \to q \Longrightarrow (p \lor r) \to q}$$

We see that the branch $\mathcal{B}$ with leaf sequent $r \Longrightarrow q, p$ is not closed.

$\mathcal{B}^\top = \{r, p \to q, p \lor r\}$

$\mathcal{B}^\perp = \{q, p, (p \lor r) \to q\}$

$\mathcal{I}_\mathcal{B} =$ interpretation with $\mathcal{I}_\mathcal{B}(r) = T$ og $\mathcal{I}_\mathcal{B}(q) = \mathcal{I}_\mathcal{B}(p) = F$

To show: this interpretation falsifies the root sequent.

Proof of Completeness, cont.

- We show by structural induction *on propositional formulae* that the interpretation $\mathcal{I}_\mathcal{B}$ makes all formulae in $\mathcal{B}^\top$ true, and all formulae in $\mathcal{B}^\perp$ false.
- We show for all propositional formulae A that
  If $A \in \mathcal{B}^\top$, then $\mathcal{I}_\mathcal{B} \models A$.
  If $A \in \mathcal{B}^\perp$, then $\mathcal{I}_\mathcal{B} \not\models A$.

Induction base: A is an atomic formula in $\mathcal{B}^\top/\mathcal{B}^\perp$.

- Our statment holds because that is how we defined $\mathcal{B}^\top/\mathcal{B}^\perp$.

Induction step: From the assumption (IH) that the statement holds for A and B, we must show that it holds for $\neg A$, $(A \land B)$, $(A \lor B)$ og $(A \to B)$. These are four cases, of which we show three here.

Case: Negation in antecedent/succedent

Assume that $\neg A \in \mathcal{B}^\top$.

- $\neg A$ appears in an antecedent, it can't 'go away' unless $\neg$-left is applied
- Since the derivation is maximal, $\neg$-left *is* eventually applied
- A appears in a succedent, so we have $A \in \mathcal{B}^\perp$.
- By the IH, we have $\mathcal{I}_\mathcal{B} \not\models A$.
- By definition of model semantics, $\mathcal{I}_\mathcal{B} \models \neg A$.

Assume that $\neg A \in \mathcal{B}^\perp$.

- $\neg A$ appears in a succedent, it can't 'go away' unless $\neg$-right is applied
- Since the derivation is maximal, $\neg$-right *is* eventually applied
- A appears in an antecedent, so we have $A \in \mathcal{B}^\top$.
- By the IH, we have $\mathcal{I}_\mathcal{B} \models A$.
- By definition of model semantics, $\mathcal{I}_\mathcal{B} \not\models \neg A$.

Case: Conjunction in antecedent/succedent

Assume that $(A \land B) \in \mathcal{B}^\top$.

- Since the derivation is maximal, we have $A \in \mathcal{B}^\top$ and $B \in \mathcal{B}^\top$.
- By the IH, we have $\mathcal{I}_\mathcal{B} \models A$ and $\mathcal{I}_\mathcal{B} \models B$.
- By definition of model semantics, $\mathcal{I}_\mathcal{B} \models (A \land B)$.

Assume that $(A \land B) \in \mathcal{B}^\perp$.

- Since the derivation is maximal, $\land$-right is eventually applied...
- ...introducing A in the succedent of one branch and B on the other.
- One of them is our branch $\mathcal{B}$, and therefore $A \in \mathcal{B}^\perp$ or $B \in \mathcal{B}^\perp$.
- By the IH, we have $\mathcal{I}_\mathcal{B} \not\models A$ or $\mathcal{I}_\mathcal{B} \not\models B$
- By definition of model semantics, $\mathcal{I}_\mathcal{B} \not\models (A \land B)$

Case: Implication in antecedent/succedent

Assume that $(A \to B) \in \mathcal{B}^{\top}$.

- Since the derivation is maximal, $\to$-left is eventually applied…
- …introducing A in the succedent of one branch and B in the antecedent of the other.
- One of them is our branch $\mathcal{B}$, and therefore $A \in \mathcal{B}^{\perp}$ or $B \in \mathcal{B}^{\top}$.
- By the IH, we have $\mathcal{I}_{\mathcal{B}} \not\models A$ or $\mathcal{I}_{\mathcal{B}} \models B$
- By definition of model semantics, $\mathcal{I}_{\mathcal{B}} \models (A \to B)$

Assume that $(A \to B) \in \mathcal{B}^{\perp}$.

- Since the derivation is maximal, we have $A \in \mathcal{B}^{\top}$ and $B \in \mathcal{B}^{\perp}$.
- By the IH, we have $\mathcal{I}_{\mathcal{B}} \models A$ and $\mathcal{I}_{\mathcal{B}} \not\models B$
- By definition of model semantics, $\mathcal{I}_{\mathcal{B}} \not\models (A \to B)$

Analysis

- If there is no proof for a sequent, there is a derivation…
  - Where all possible rules have been applied
  - At least one branch $\mathcal{B}$ has not been closed with an axiom
- We can use the atoms on $\mathcal{B}$ to construct an interpretation $\mathcal{I}_{\mathcal{B}}$
- $\mathcal{I}_{\mathcal{B}}$ makes atoms left true, and atoms right false
- $\mathcal{I}_{\mathcal{B}}$ also makes *all other* formulae left true and right false, because…
  - for every non-atomic formula, there is a rule that decomposes it
  - which must have been applied
  - and that guarantees that $\mathcal{I}_{\mathcal{B}}$ falsifies sequents, based on structural induction
- Structural induction on formulae, while soundness was by induction on derivations
- Not possible to prove completeness 'one rule at a time'

One-sided Sequent Calculus

- Only sequents with empty succedent: $\Gamma \Longrightarrow$
- To prove A, start with $\neg A \Longrightarrow$
- "Proof by contradiction" or "refutation"
- Negation rules combined with others:

$$\frac{\Gamma, \neg A, \neg B \Longrightarrow}{\Gamma, \neg(A \vee B) \Longrightarrow}\neg\vee \qquad \frac{\Gamma, \neg A \Longrightarrow \qquad \Gamma, \neg B \Longrightarrow}{\Gamma, \neg(A \wedge B) \Longrightarrow}\neg\wedge$$

- Double negation:

$$\frac{\Gamma, A \Longrightarrow}{\Gamma, \neg\neg A \Longrightarrow}\neg\neg$$

- Axiom:

$$\frac{}{\Gamma, A, \neg A \Longrightarrow}$$

- Can do the same with empty antecedents $\Longrightarrow \Delta$

Example with One-sided Sequents

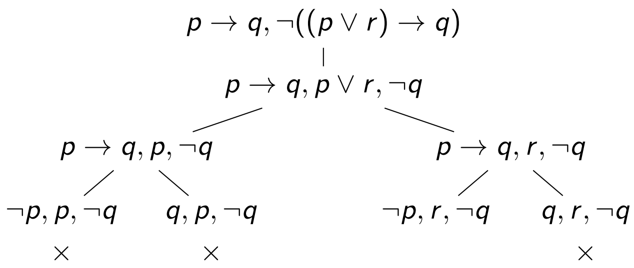
- Instead of $p \to q \Longrightarrow (p \vee r) \to q$
- Start with $p \to q, \neg((p \vee r) \to q) \Longrightarrow$

$$\dfrac{\dfrac{\dfrac{}{\neg p, p, \neg q \Longrightarrow} \quad \dfrac{}{q, p, \neg q \Longrightarrow}}{p \to q, p, \neg q \Longrightarrow} \quad \dfrac{\dfrac{}{\neg p, r, \neg q \Longrightarrow} \quad \dfrac{}{q, r, \neg q \Longrightarrow}}{p \to q, r, \neg q \Longrightarrow}}{\dfrac{p \to q, p \vee r, \neg q \Longrightarrow}{p \to q, \neg((p \vee r) \to q) \Longrightarrow}}$$

- Soundness and completeness very similar to two-sided LK.

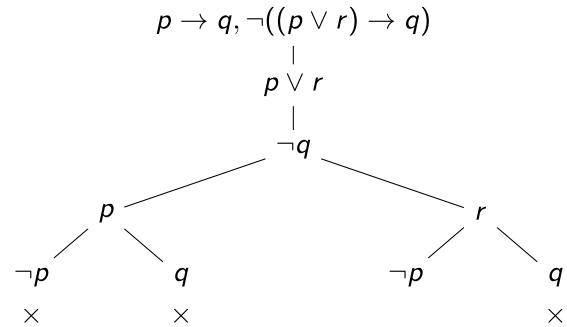
Semantic Tableaux (Ben-Ari 2.6)

- Others call these 'block tableaux'
- Sequent arrow $\Longrightarrow$ not needed for one-sided calculus
- More handy to write top-down, like everybody else
- Mark 'closed' branches (with axioms) with $\times$



Short Hand Notation for Tableaux

- Only write the new formula in every node.
- Even more handy to write
- Close branch using literals A and $\neg A$ anywhere on a branch.
- Have to make sure that all rules were used on every branch!



Summary and Outlook

Until now:

- Propositional logic and model semantics
- LK Calculus
- Soundness
- Completeness

Next three weeks:

- First-order logic and model semantics
- LK Calculus for first-order logic
- Soundness
- Completeness

After that: resolution, DPLL, Prolog,. . .