IN3070/4070 – Logic – Autumn 2019

Lecture 3: LK: Soundness & Completeness

Martin Giese

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Department of Informatics

University of Oslo

Today's Plan

- Semantics for Sequents
- Soundness
- Completeness

Outline

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- Soundness
- Completeness

Semantics for Sequents

Definition 1.1 (Valid sequent).

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- $p, p \to q \implies q$
- $p \to q \implies \neg q \to \neg p$

Definition 1.2 (Countermodel/falsifiable sequent).

- *An interpretation* $\mathcal{I}$ *is a countermodel for the sequent* $\Gamma \implies \Delta$ *if* $v_{\mathcal{I}}(A) = T$ *for all formulae* $A \in \Gamma$ *and* $v_{\mathcal{I}}(B) = F$ *for all formulae* $B \in \Delta$

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- $p \implies$ *Countermodel:* $\mathcal{I}(p) = T$
- $\implies$ *Countermodel: all interpretations!*

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Not provable

$$\dfrac{\dfrac{q \implies p, p}{\neg p \implies p, \neg q} \qquad \dfrac{q, q \implies p}{q, \neg p \implies \neg q}}{\neg p, p \to q \implies \neg q}$$

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Theorem 2.1.

The sequent calculus LK *is sound.*

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Preservation of Falsifiability

Definition 2.2.

An LK*-rule* θ *preserves falsifiability (upwards) if all interpretations that falsify the conclusion* w *of an instance* $\frac{w_1 \cdots w_n}{w}$ *of* θ *also falsify at least one of the premises* w_i*.*

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- We let Γ, Δ, A and B in the rule stand for arbitrary (sets of) propositional formulae

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- Since $\mathcal{I}$ satisfies $A \to B$

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- We can generalise from an arbitrary element:
  - Choose an arbitrary element $a \in S$.
  - Show that $P(a)$ holds.
  - Since a was arbitrarily chosen, the original statement must hold.

How to show the Soundness Theorem?

We show the following lemmas:

- All LK-rules preserve falsifiability upwards.
- An LK-derivation with a falsifiable root sequent has at least one falsifiable leaf sequent
- All axioms are valid

Finally, we use these lemmas to show the soundness theorem.

Reminder: LK derivation

Definition 2.3 (LK Derivation).

- *Let* $\Gamma \implies \Delta$ *be a sequent. Then*

$$\Gamma \implies \Delta$$

  is an LK-derivation of $\Gamma \implies \Delta$.

- *Let* $\dfrac{w_1 \quad \cdots \quad w_n}{\Gamma \implies \Delta}$ *be an instance of an* LK *rule, and* $\mathcal{D}_1, \ldots, \mathcal{D}_n$ *derivations of* $w_1, \ldots, w_n$. *Then*

$$\dfrac{\mathcal{D}_1 \quad \cdots \quad \mathcal{D}_n}{\Gamma \implies \Delta}$$

  is an LK-derivation of $\Gamma \implies \Delta$.

Existence of a falsifiable leaf sequent

Lemma 2.2.

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- This is also a leaf sequent of δ

$\square$

How to show the Soundness Theorem?

We show the following lemmas:

- All LK-rules preserve falsifiability upwards.
- An LK-derivation with a falsifiable root sequent has at least one falsifiable leaf sequent
- All axioms are valid

Finally, we use these lemmas to show the soundness theorem.

All axioms are valid

Lemma 2.3.

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$$\Gamma, A \implies A, \Delta$$

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- Then $\mathcal{I}$ satisfies the formula A in the succedent.

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Proof of the Soundness Theorem for LK

Proof of soundness.

- Assume that $\mathcal{P}$ is an LK-proof for the sequent $\Gamma \implies \Delta$.

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- Then $\mathcal{P}$ has a leaf sequent that is not an axiom, since axioms are not falsifiable.
- So $\mathcal{P}$ cannot be an LK-proof.

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- Shown individually for each rule
- Can add new rules, and just show “soundness” for those

Outline

- Semantics for Sequents
- Soundness
- Completeness

Completeness — Introduction

Definition 3.1 (Soundness).

The calculus LK *is sound if any* LK*-provable sequent is valid.*

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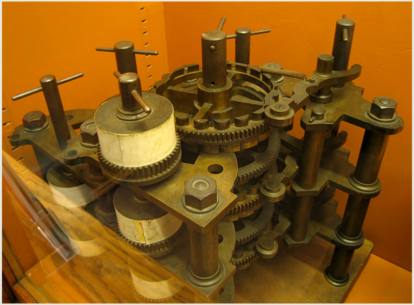
Completeness: $\Gamma \Longrightarrow \Delta$ not provable $\Rightarrow$ $\Gamma \Longrightarrow \Delta$ falsifiable

An LK-machine?



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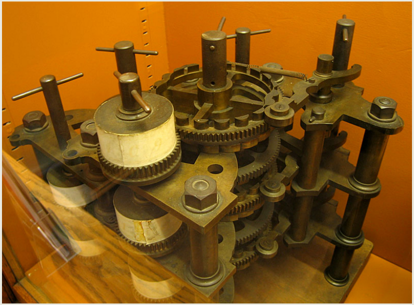

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All that is printed is valid.

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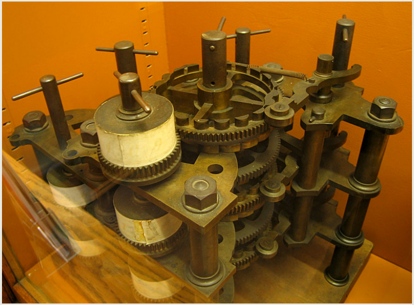

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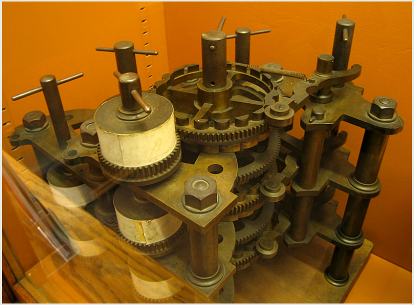

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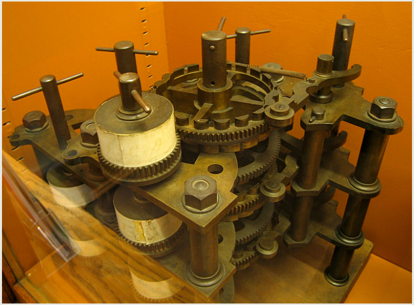

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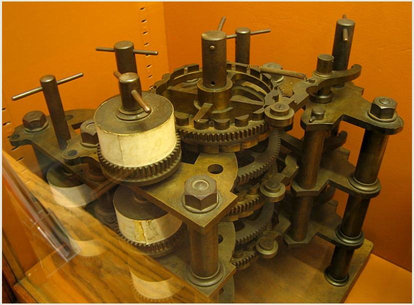

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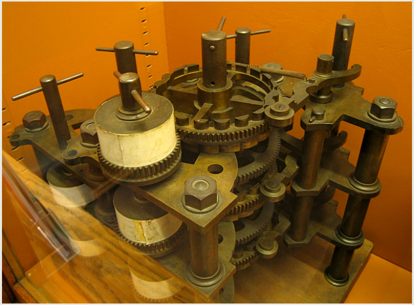

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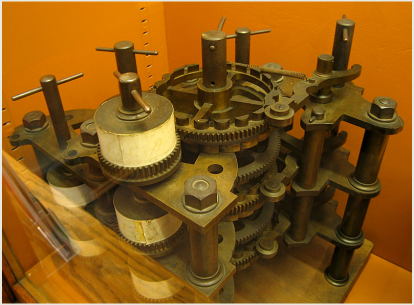

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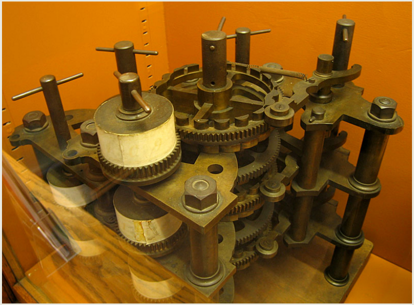

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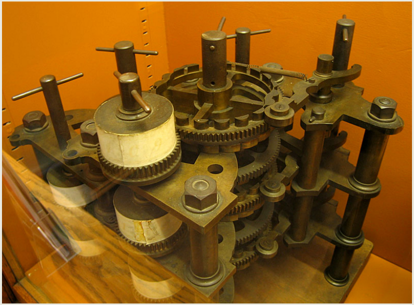

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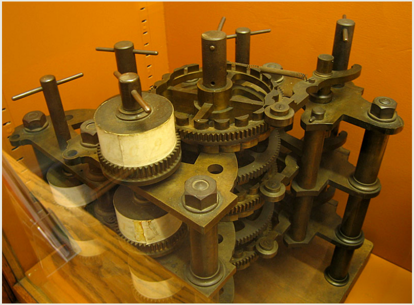

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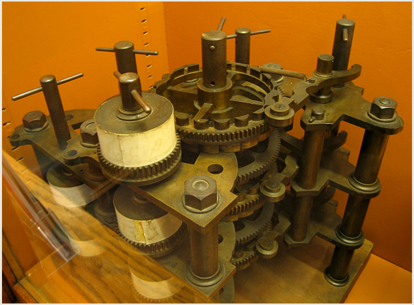

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Theorem 3.1 (Completeness).

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This means that there is an interpretation that makes all formulae in Γ true and all formulae in Δ false.

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- Then there is (at least) one branch $\mathcal{B}$ that does not end in an axiom. We then have:
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  $\mathcal{B}^\top$ be the set of formulae that occur in an antecedent on B, and

  $\mathcal{B}^\perp$ be the set of formulae that occur in an succedent on $\mathcal{B}$, and

  $\mathcal{I}_\mathcal{B}$ be the interpretation that makes all atomic formulae in $\mathcal{B}^\top$ true and all other atomic formulae (in particular those in $\mathcal{B}^\perp$) false.

Example

$$
\dfrac{\dfrac{\dfrac{}{p \implies q, p} \qquad \dfrac{}{q, p \implies q}}{p \to q, p \implies q} \qquad \dfrac{\dfrac{}{r \implies q, p} \qquad \dfrac{}{q, r \implies q}}{p \to q, r \implies q}}{\dfrac{p \to q, p \lor r \implies q}{p \to q \implies (p \lor r) \to q}}
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Example

$$\dfrac{\dfrac{\dfrac{}{p \Longrightarrow q, p} \quad \dfrac{}{q, p \Longrightarrow q}}{p \to q, p \Longrightarrow q} \qquad \dfrac{\dfrac{}{r \Longrightarrow q, p} \quad \dfrac{}{q, r \Longrightarrow q}}{p \to q, r \Longrightarrow q}}{\dfrac{p \to q, p \lor r \Longrightarrow q}{p \to q \Longrightarrow (p \lor r) \to q}}$$

Example

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We see that the branch $\mathcal{B}$ with leaf sequent $r \implies q, p$ is not closed.

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$$
\begin{aligned}
\mathcal{B}^{\top} &= \{r, p \to q, p \lor r\} \\
\mathcal{B}^{\bot} &= \{q, p, (p \lor r) \to q\} \\
\mathcal{I}_{\mathcal{B}} &= \text{interpretation with } \mathcal{I}_{\mathcal{B}}(r) = T \text{ og } \mathcal{I}_{\mathcal{B}}(q) = \mathcal{I}_{\mathcal{B}}(p) = F
\end{aligned}
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& \text{To show: this interpretation falsifies the root sequent.}
\end{aligned}$$

Proof of Completeness, cont.

- We show by structural induction *on propositional formulae* that the interpretation $\mathcal{I}_{\mathcal{B}}$ makes all formulae in $\mathcal{B}^{\top}$ true, and all formulae in $\mathcal{B}^{\perp}$ false.

Proof of Completeness, cont.

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Induction base: A is an atomic formula in $\mathcal{B}^{\top}/\mathcal{B}^{\perp}$.

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- Our statment holds because that is how we defined $\mathcal{B}^{\top}/\mathcal{B}^{\perp}$.

Proof of Completeness, cont.

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- We show for all propositional formulae A that

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Induction base: A is an atomic formula in $\mathcal{B}^{\top}/\mathcal{B}^{\perp}$.

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Induction step: From the assumption (IH) that the statement holds for A and B, we must show that it holds for $\neg A$, $(A \wedge B)$, $(A \vee B)$ og $(A \rightarrow B)$. These are four cases, of which we show three here.

Case: Negation in antecedent/succedent

Assume that $\neg A \in \mathcal{B}^{\top}$.

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- By the IH, we have $\mathcal{I}_{\mathcal{B}} \not\models A$.

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- By the IH, we have $\mathcal{I}_{\mathcal{B}} \not\models A$.
- By definition of model semantics, $\mathcal{I}_{\mathcal{B}} \models \neg A$.

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Assume that $\neg A \in \mathcal{B}^\perp$.

- $\neg A$ appears in a succedent, it can't 'go away' unless $\neg$-right is applied
- Since the derivation is maximal, $\neg$-right *is* eventually applied
- A appears in an antecedent, so we have $A \in \mathcal{B}^\top$.
- By the IH, we have $\mathcal{I}_\mathcal{B} \models A$.
- By definition of model semantics, $\mathcal{I}_\mathcal{B} \not\models \neg A$.

Case: Conjunction in antecedent/succedent

Assume that $(A \wedge B) \in \mathcal{B}^\top$.

Case: Conjunction in antecedent/succedent

Assume that $(A \wedge B) \in \mathcal{B}^\top$.

- Since the derivation is maximal, we have $A \in \mathcal{B}^\top$ and $B \in \mathcal{B}^\top$.

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- By definition of model semantics, $\mathcal{I}_\mathcal{B} \models (A \wedge B)$.

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Assume that $(A \wedge B) \in \mathcal{B}^{\perp}$.

Case: Conjunction in antecedent/succedent

Assume that $(A \wedge B) \in \mathcal{B}^{\top}$.

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- By the IH, we have $\mathcal{I}_{\mathcal{B}} \models A$ and $\mathcal{I}_{\mathcal{B}} \models B$.
- By definition of model semantics, $\mathcal{I}_{\mathcal{B}} \models (A \wedge B)$.

Assume that $(A \wedge B) \in \mathcal{B}^{\bot}$.

- Since the derivation is maximal, $\wedge$-right is eventually applied...

Case: Conjunction in antecedent/succedent

Assume that $(A \wedge B) \in \mathcal{B}^\top$.

- Since the derivation is maximal, we have $A \in \mathcal{B}^\top$ and $B \in \mathcal{B}^\top$.
- By the IH, we have $\mathcal{I}_\mathcal{B} \models A$ and $\mathcal{I}_\mathcal{B} \models B$.
- By definition of model semantics, $\mathcal{I}_\mathcal{B} \models (A \wedge B)$.

Assume that $(A \wedge B) \in \mathcal{B}^\perp$.

- Since the derivation is maximal, $\wedge$-right is eventually applied...
- ...introducing A in the succedent of one branch and B on the other.

Case: Conjunction in antecedent/succedent

Assume that $(A \land B) \in \mathcal{B}^\top$.

- Since the derivation is maximal, we have $A \in \mathcal{B}^\top$ and $B \in \mathcal{B}^\top$.
- By the IH, we have $\mathcal{I}_\mathcal{B} \models A$ and $\mathcal{I}_\mathcal{B} \models B$.
- By definition of model semantics, $\mathcal{I}_\mathcal{B} \models (A \land B)$.

Assume that $(A \land B) \in \mathcal{B}^\bot$.

- Since the derivation is maximal, $\land$-right is eventually applied...
- ...introducing A in the succedent of one branch and B on the other.
- One of them is our branch $\mathcal{B}$, and therefore $A \in \mathcal{B}^\bot$ or $B \in \mathcal{B}^\bot$.

Case: Conjunction in antecedent/succedent

Assume that $(A \land B) \in \mathcal{B}^\top$.

- Since the derivation is maximal, we have $A \in \mathcal{B}^\top$ and $B \in \mathcal{B}^\top$.
- By the IH, we have $\mathcal{I}_\mathcal{B} \models A$ and $\mathcal{I}_\mathcal{B} \models B$.
- By definition of model semantics, $\mathcal{I}_\mathcal{B} \models (A \land B)$.

Assume that $(A \land B) \in \mathcal{B}^\perp$.

- Since the derivation is maximal, $\land$-right is eventually applied...
- ...introducing A in the succedent of one branch and B on the other.
- One of them is our branch $\mathcal{B}$, and therefore $A \in \mathcal{B}^\perp$ or $B \in \mathcal{B}^\perp$.
- By the IH, we have $\mathcal{I}_\mathcal{B} \not\models A$ or $\mathcal{I}_\mathcal{B} \not\models B$

Case: Conjunction in antecedent/succedent

Assume that $(A \wedge B) \in \mathcal{B}^\top$.

- Since the derivation is maximal, we have $A \in \mathcal{B}^\top$ and $B \in \mathcal{B}^\top$.
- By the IH, we have $\mathcal{I}_\mathcal{B} \models A$ and $\mathcal{I}_\mathcal{B} \models B$.
- By definition of model semantics, $\mathcal{I}_\mathcal{B} \models (A \wedge B)$.

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- By definition of model semantics, $\mathcal{I}_\mathcal{B} \not\models (A \wedge B)$

Case: Implication in antecedent/succedent

Assume that $(A \to B) \in \mathcal{B}^\top$.

Case: Implication in antecedent/succedent

Assume that $(A \rightarrow B) \in \mathcal{B}^{\top}$.

- Since the derivation is maximal, $\rightarrow$-left is eventually applied...

Case: Implication in antecedent/succedent

Assume that $(A \rightarrow B) \in \mathcal{B}^{\top}$.

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- By the IH, we have $\mathcal{I}_{\mathcal{B}} \not\models A$ or $\mathcal{I}_{\mathcal{B}} \models B$

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- By definition of model semantics, $\mathcal{I}_\mathcal{B} \models (A \to B)$

Assume that $(A \to B) \in \mathcal{B}^\perp$.

Case: Implication in antecedent/succedent

Assume that $(A \rightarrow B) \in \mathcal{B}^{\top}$.

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- One of them is our branch $\mathcal{B}$, and therefore $A \in \mathcal{B}^{\perp}$ or $B \in \mathcal{B}^{\top}$.
- By the IH, we have $\mathcal{I}_{\mathcal{B}} \not\models A$ or $\mathcal{I}_{\mathcal{B}} \models B$
- By definition of model semantics, $\mathcal{I}_{\mathcal{B}} \models (A \rightarrow B)$

Assume that $(A \rightarrow B) \in \mathcal{B}^{\perp}$.

- Since the derivation is maximal, we have $A \in \mathcal{B}^{\top}$ and $B \in \mathcal{B}^{\perp}$.

Case: Implication in antecedent/succedent

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- Structural induction on formulae, while soundness was by induction on derivations
- Not possible to prove completeness 'one rule at a time'

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- Can do the same with empty antecedents $\implies \Delta$

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- Start with $p \to q, \neg((p \lor r) \to q) \implies$

$$\dfrac{\dfrac{\dfrac{}{\neg p, p, \neg q \implies} \qquad \dfrac{}{q, p, \neg q \implies}}{p \to q, p, \neg q \implies} \qquad \dfrac{\dfrac{}{\neg p, r, \neg q \implies} \qquad \dfrac{}{q, r, \neg q \implies}}{p \to q, r, \neg q \implies}}{\dfrac{p \to q, p \lor r, \neg q \implies}{p \to q, \neg((p \lor r) \to q) \implies}}$$

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$$
\cfrac{
  \cfrac{
    \cfrac{}{\neg p, p, \neg q \implies} \qquad \cfrac{}{q, p, \neg q \implies}
  }{p \to q, p, \neg q \implies}
  \qquad
  \cfrac{
    \cfrac{}{\neg p, r, \neg q \implies} \qquad \cfrac{}{q, r, \neg q \implies}
  }{p \to q, r, \neg q \implies}
}{
  \cfrac{p \to q, p \lor r, \neg q \implies}{p \to q, \color{red}{\neg((p \lor r) \to q) \implies}}
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  }{
    {\color{red} p \to q}, p, \neg q \implies
  }
  \qquad
  \dfrac{
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  }{
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\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{}{{\color{red}\neg p, p}, \neg q \implies}
      \quad
      \dfrac{}{q, p, \neg q \implies}
    }{p \to q, p, \neg q \implies}
    \qquad
    \dfrac{
      \dfrac{}{\neg p, r, \neg q \implies}
      \quad
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Example with One-sided Sequents

- Instead of $p \rightarrow q \implies (p \vee r) \rightarrow q$
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$$
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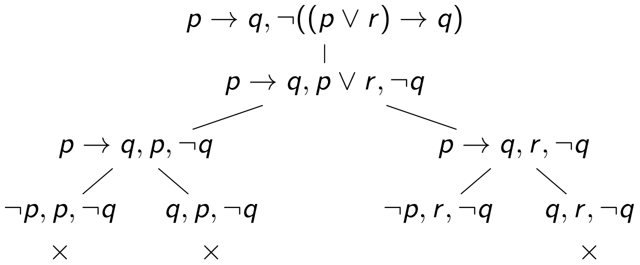
- Soundness and completeness very similar to two-sided LK.

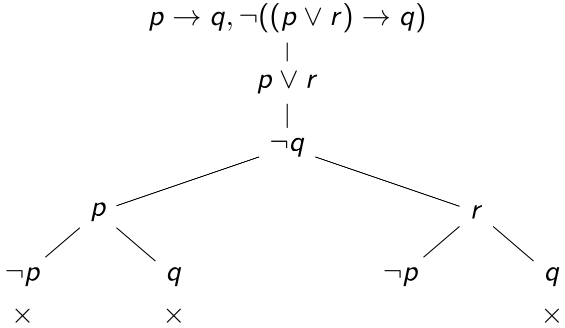
Semantic Tableaux (Ben-Ari 2.6)

- Others call these 'block tableaux'
- Sequent arrow $\Longrightarrow$ not needed for one-sided calculus
- More handy to write top-down, like everybody else
- Mark 'closed' branches (with axioms) with $\times$



Short Hand Notation for Tableaux

- Only write the new formula in every node.
- Even more handy to write
- Close branch using literals A and $\neg A$ anywhere on a branch.
- Have to make sure that all rules were used on every branch!



Summary and Outlook

Until now:

- Propositional logic and model semantics
- LK Calculus
- Soundness
- Completeness

Next three weeks:

- First-order logic and model semantics
- LK Calculus for first-order logic
- Soundness
- Completeness

After that: resolution, DPLL, Prolog,...