# IN3070/4070 – Logic – Autumn 2019

## Lecture 4: First-order Logic

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# Today's Plan

- Motivation
- Syntax
- Variables
- Semantics
- The Substitution Lemma
- Satisfiability & Validity
- LK for First-order Logic
- Summary

# Outline

# Limitations of Propositional Logic

Propositional logic: atomic formula $(p, q, r)$, $\wedge$, $\vee$, $\neg$, $\rightarrow$, (, )

Problem: How do we represent the following statements?

- "all men are mortal" $\forall x(man(x) \rightarrow mortal(x))$
- "there exist prime numbers that are even" $\exists y(prime(y) \wedge even(y))$
- "1 is smaller than 3" $1 < 3$ or $<(1, 3)$
- "transitivity of smaller" $\forall x \, \forall y \, \forall z \, (x{<}y \wedge y{<}z \rightarrow x{<}z)$
- $2 * 8 = 16$ $= (*(2, 8), 16)$
- "if $x$ is even than $x + 2$ is even" $\forall x \, (even(x) \rightarrow even(x + 2))$
- "if $x$ is prime than $x + 2$ is prime" $\forall x \, (prime(x) \rightarrow prime(x + 2))$

First-order logic: extension of propositional logic

# First-Order Logic — Overview

Extending propositional logic by...

Syntax:

- constants $(a, b, c)$, functions $(f, g, h)$, variables $(x, y, z)$
- predicates $(p, q, r)$
- terms $(t, u, v)$
- quantifiers $(\forall, \exists)$
- scope of variables, free variables, variable assignment/substitution

Semantics:

- interpretation of constants, functions, variables
- interpretation of predicates
- value of terms
- truth value of (quantified) formulae
- satisfiability, validity, logical equivalence,...

# Outline

# Syntax — Terms

Terms are built up of constant (symbols), variable (symbols), and function (symbols).

## Definition 2.1 (Terms).

*Let* $\mathcal{A} = \{a, b, \dots\}$ *be a countable set of constant symbols,*
$\mathcal{V} = \{x, y, z, \dots\}$ *be a countable set of variable symbols, and*
$\mathcal{F} = \{f, g, h, \dots\}$ *be a countable set of function symbols.*

*Terms, denoted* $t, u, v$*, are inductively defined as follows:*

- *Every variable* $x \in \mathcal{V}$ *is a term.*
- *Every constant* $a \in \mathcal{A}$ *is a term.*
- *If* $f \in \mathcal{F}$ *is an n-ary function (symbol)* $n>0$ *and* $t_1, \dots, t_n$ *are terms, then* $f(t_1, \dots, t_n)$ *is a term.*

Example: $a$, $x$, $f(a, x)$, $f(g(x), b)$, and $g(f(a, g(y)))$ are terms.

# Syntax — First-Order Formulae

Formulae are built up of atomic formulae and the logical connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$, and $\forall$ (universal quantifier), $\exists$ (existential quantifier).

**Definition 2.2 (Atomic Formulae).**

*Let $\mathcal{P} = \{p, q, r, \ldots\}$ be a countable set of predicate symbols. If $p \in \mathcal{P}$ is an n-ary predicate (symbol) $n \geq 0$ and $t_1, \ldots, t_n$ are terms, then $p(t_1, \ldots, t_n)$, $\top$, and $\bot$ are atomic formulae (or atoms).*

**Definition 2.3 ((First-Order) Formulae).**

*(First-order) formulae, denoted $A, B, C, F, G, H$, are inductively defined as follows:*

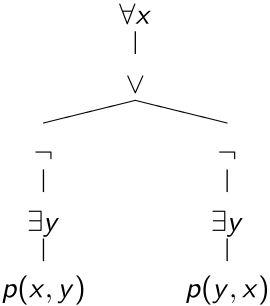
- *Every atomic formula $p$ is a formula.*
- *If $A$ and $B$ are formulae and $x \in \mathcal{V}$, then $(\neg A)$, $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$, $\forall x\, A$, and $\exists x\, A$ are formulae.*

# Formula Trees

A formula can be presented as formula tree.

Example:

$\forall x\,(\neg\exists y\, p(x,y) \vee \neg\exists y\, p(y,x))$




**Definition 2.4 (Subformula, Main Operator).**

*Formula $A$ is a (proper) subformula of formula $B$ iff $A$ is a (proper) subtree of $B$. If the root of a formula tree of $A$ is a logical connective/quantifier, then it is called the main operator of $A$.*

# Outline

# Free Variables

A free variable is a variable that is not in the scope of a quantifier.

**Definition 3.1 (Free/Bound Variables, Closed Formula/Term).**

*Free variables in a formula $A$ are inductively defined:*

- *If $A$ is an atomic formula, then all variables in $A$ are free.*
- *If $A = \neg B$, then the free variables of $A$ are exactly those of $B$.*
- *If $A = B \wedge C$, $A = B \vee C$, or $A = B \rightarrow C$, then the free variables of $A$ are those of $B$ together with those of $C$.*
- *If $A = \forall x\, B$ or $A = \exists x\, B$, then the free variables of $A$ are those of $B$ without the variable $x$.*

*A bound variable in a formula $C$ is a variable that appears in $\forall x$ or $\exists x$ in some subformula of $C$. A formula/term is closed iff it has no free variables.*

# Scope, Universal and Existential Closure

## Definition 3.2 (Scope of Variables).

*Let $\forall x\, A$ or $\exists x\, A$ be a universally or existentially quantified formula. Then $x$ is the quantified variable and its scope is the formula $A$.*

Remark: It is not required that $x$ actually appears in the scope of its quantification, e.g. $\forall x\, \exists y\, p(y, y)$.

## Definition 3.3 (Universal and Existential Closure).

*If $\{x_1, \ldots, x_n\}$ are all the free variables of $A$, the universal closure of $A$ is $\forall x_1 \ldots \forall x_n\, A$ and the existential closure of $A$ is $\exists x_1 \ldots \exists x_n\, A$.*

- $p(x, y)$ has the two free variables $x$ and $y$. Its universal closure is $\forall x\, \forall y\, p(x, y)$ and its existential closure is $\exists x\, \exists y\, p(x, y)$; $\exists y\, p(x, y)$ has the only free variable $x$; $\forall x\, \exists y\, p(x, y)$ is closed
- In $\forall x\, p(x) \wedge q(x)$, the $x$ occurs bound and free. The universal closure is $\forall x\, (\forall x\, p(x) \wedge q(x))$; renaming: $\forall y\, (\forall x\, p(x) \wedge q(y))$

# Substitutions

Free variables in a first-order formula can be substituted by terms.

**Definition 3.4 (Substitution).**

*Let $\mathcal{V}$ be a set of variables, $\mathcal{T}$ be the set of terms. A substitution $\sigma : \mathcal{V} \to \mathcal{T}$ assigns each variable a term.*

Remark: The substitution $\sigma$ is often represented as set $\{x \backslash t \mid \sigma(x) = t\}$.

Example: For the variable set $\{x, y\}$, $\sigma(x) = a$, $\sigma(y) = f(z, b)$ is a substitution and can also be represented as $\{x \backslash a, y \backslash f(z, b)\}$.

Ben-Ari: $\{x \leftarrow a, y \leftarrow f(z, b)\}$.

Others: $[a/x, f(z, b)/y]$

# Application of substitutions

## Definition 3.5 (Application of Substitutions, informally).

*Let $\sigma$ be a substitution. The application of $\sigma$ to a term $t$ or formula $A$, written $\sigma(t)$ or $\sigma(A)$, replaces every free variable in $t$ or $A$ according to its image under $\sigma$. Short hand: $A[x\backslash t]{=}\sigma(A)$ with $\sigma{=}\{x\backslash t\}$.*

Example: Let $\sigma{=}\{x\backslash a, y\backslash f(z,b)\}$ be a substitution.
Then $\sigma(g(y)) = g(f(z,b))$
and $\sigma(p(x) \wedge \forall x\, q(x, g(y))) = p(a) \wedge \forall x\, q(x, g(f(z,b)))$

Problem: $\sigma(\forall z\, p(z,y)) \;= \forall z\, p(z, f(z,b))$
The free variable $z$ in $\sigma$ is captured by the quantifier.
This is bad because the effect depends on the choice of variable names

## Definition 3.6 (Capture-free substitution).

*A substitution $\sigma$ is capture-free for a formula $A$ if for every free variable $x$ in $A$, none of the variables in $\sigma(x)$ is bound in $A$.*

# Application of Substitutions

**Definition 3.7 (Application of Substitutions, formally).**

*The application of a subtitution $\sigma$ to a term or formula is defined by structural induction:*

- $\sigma(x) = \sigma(x)$ *for variables* $x$ *in the range of* $\sigma$
- $\sigma(y) = y$ *for variables* $y$ *not in the range of* $\sigma$
- $\sigma(a) = a$ *for constants* $a \in \mathcal{A}$
- $\sigma(f(t_1, \ldots, t_n)) = f(\sigma(t_1), \ldots, \sigma(t_n))$ *for a function symbol* $f \in \mathcal{F}$
- $\sigma(p(t_1, \ldots, t_n)) = p(\sigma(t_1), \ldots, \sigma(t_n))$ *for a predicate symbol* $p \in \mathcal{P}$
- $\sigma(A \wedge B) = \sigma(A) \wedge \sigma(B)$ *for formulae* $A$, $B$
- *...similarly for* $\neg A$, $A \vee B$, $A \rightarrow B \ldots$
- $\sigma(\exists x\, A) = \exists x\, \sigma_x(A), \quad \sigma(\forall x\, A) = \forall x\, \sigma_x(A)$

*where we define* $\sigma_x$ *by:* $\sigma_x(x) = x$, *and* $\sigma_x(y) = \sigma(y)$ *for all* $y \neq x$

# Outline

# Semantics — Interpretation

An interpretation assigns concrete objects, functions and relations to constant symbols, function symbols, and predicate symbols.

**Definition 4.1 (Interpretation/Structure).**

*An interpretation (or structure) $\mathcal{I} = (D, \iota)$ consists of the following elements:*

- *Domain $D$ is a non-empty set*
- *Interpretation of constant symbols assigns each constant $a \in \mathcal{A}$ an element $a^\iota \in D$*
- *Interpretation of function symbols assigns each n-ary function symbol $f \in \mathcal{F}$ with $n{>}0$ a function $f^\iota : D^n \to D$*
- *Interpretation of propositional variables assigns each 0-ary predicate symbol $p \in \mathcal{P}$ a truth value $p^\iota{\in}\{T, F\}$*
- *Interpretation of predicate symbols assigns each n-ary predicate symbol $p \in \mathcal{P}$ with $n{>}0$ a relation $p^\iota \subseteq D^n$*

# Semantics — Examples

Example: $\forall x\, p(a, x)$ with the interpretations

- $\mathcal{I} = (\mathbb{N}, \iota)$ with $p^\iota = \leq$ and $a^\iota = 0$
- $\mathcal{I} = (\mathbb{N}, \iota)$ with $p^\iota = \leq$ and $a^\iota = 3$
- $\mathcal{I} = (\mathbb{Z}, \iota)$ with $p^\iota = \leq$ and $a^\iota = 0$
- $\mathcal{I} = (\{c, d, e, f\}, \iota)$ with $p^\iota = \leq_{lexi}$ and $a^\iota = c$

Remark: In Ben-Ari: $(\mathbb{N}, \{\leq\}, \{0\})$, $(\mathbb{N}, \{\leq\}, \{3\})$, $(\mathbb{Z}, \{\leq\}, \{0\})$

Example: $\forall x \forall y\, (p(x, y) \rightarrow p(f(x, a), f(y, a)))$ with interpretations

- $\mathcal{I} = (\mathbb{Z}, \iota)$ with $p^\iota = \leq$, $f^\iota = +$, and $a^\iota = 1$
- $\mathcal{I} = (\mathbb{Z}, \iota)$ with $p^\iota = >$, $f^\iota = *$, and $a^\iota = -1$

Remark: In Ben-Ari: $(\mathbb{Z}, \{\leq\}, \{+\}, \{1\})$, $(\mathbb{Z}, \{>\}, \{*\}, \{-1\})$.

# Semantics — Value of Closed Terms

Terms are evaluated according to the interpretation of their constant and function symbols.

## Definition 4.2 (Term Value for Closed Terms).

*Let $\mathcal{I} = (D, \iota)$ be an interpretation. The term value $v_{\mathcal{I}}(t)$ of a closed term $t \in \mathcal{T}$ under the interpretation $\mathcal{I}$ is inductively defined:*

- *For a constant symbol $a \in \mathcal{A}$ the term value is $v_{\mathcal{I}}(a) = a^{\iota}$;*
- *Let $f \in \mathcal{F}$ be an n-ary function, $n>0$, and $t_1, \ldots, t_n$ be terms; the term value of $f(t_1, \ldots, t_n)$ is $v_{\mathcal{I}}(f(t_1, \ldots, t_n)) = f^{\iota}(v_{\mathcal{I}}(t_1), \ldots, v_{\mathcal{I}}(t_n))$*

Examples:

- $f(a, f(a, b))$ with $\mathcal{I} = (\mathbb{N}, \iota)$ with $f^{\iota} = +$, $a^{\iota} = 20$, $b^{\iota} = 2$; then $v_{\mathcal{I}}(f(a, f(a, b))) = 42$
- $+(1, *(4, 2))$ with $\mathcal{I} = (\mathbb{Z}, \iota)$ with $+^{\iota} = *$ (multiplication), $*^{\iota} = -$ (subtraction), $1^{\iota} = -20$, $2^{\iota} = 0$, $4^{\iota} = 10$; then $v_{\mathcal{I}}(+(1, *(4, 2))) = -200$

# Semantics — Variable Assignments, Value of Terms

The interpretation doesn't tell what to do about variables.
We need something additional.

**Definition 4.3 (Variable Assignment).**

*Given the set of variables $\mathcal{V}$, and an interpretation $\mathcal{I} = (D, \iota)$, a variable assignment $\alpha$ for $\mathcal{I}$ is a function $\alpha : \mathcal{V} \to D$.*

Ben-Ari (7.18) writes this $\sigma_{\mathcal{I}_A}$

**Definition 4.4 (Term Value).**

*Let $\mathcal{I} = (D, \iota)$ be an interpretation, and $\alpha$ an variable assignment for $\mathcal{I}$. The term value $v_{\mathcal{I}}(\alpha, t)$ of a term $t \in \mathcal{T}$ under $\mathcal{I}$ and $\alpha$ is inductively defined:*

- $v_{\mathcal{I}}(\alpha, x) = \alpha(x)$ *for a variable $v \in \mathcal{V}$*
- $v_{\mathcal{I}}(\alpha, a) = a^{\iota}$ *for a constant symbol $a \in \mathcal{A}$*
- $v_{\mathcal{I}}(\alpha, f(t_1, \ldots, t_n)) = f^{\iota}(v_{\mathcal{I}}(\alpha, t_1), \ldots, v_{\mathcal{I}}(\alpha, t_n))$ *for an n-ary $f \in \mathcal{F}$*

# Semantics — Term value Examples

- $\mathcal{I} = (\mathbb{N}, \iota)$ with $f^\iota = +$, $a^\iota = 10$
  - $\mathcal{V} = \{x, y\}$
  - $\alpha(x) = 3 \in \mathbb{N}$ and $\alpha(y) = 5 \in \mathbb{N}$ is an assignment for $\mathcal{I}$
  - $v_\mathcal{I}(\alpha, f(a, f(a, x))) = 23$
- $\mathcal{I} = (\text{Strings}, \iota)$ with $g^\iota =$ concatenation, $a^\iota =$"Hello"
  - $\mathcal{V} = \{y\}$
  - $\alpha(y) =$"students"
  - $v_\mathcal{I}(\alpha, f(a, f(x, a))) =$ "HellostudentsHello"

# Semantics — Modification of an assignment

## Definition 4.5 (Modification of a variable assignment).

*Given an interpretation $\mathcal{I} = (D, \iota)$ and a variable assignment $\alpha$ for $\mathcal{I}$. Given also a variable $y \in \mathcal{V}$ and a domain element $d \in D$. The modified variable assignment $\alpha\{y \leftarrow d\}$ is defined as*

$$\alpha\{y \leftarrow d\}(x) = \begin{cases} d & \text{if } x = y \\ \alpha(x) & \text{otherwise} \end{cases}$$

- $\mathcal{I} = (\mathbb{N}, \iota)$
- $\mathcal{V} = \{x, y\}$
- $\alpha(x) = 3 \in \mathbb{N}$ and $\alpha(y) = 5 \in \mathbb{N}$ is an assignment for $\mathcal{I}$
- $\alpha\{y \leftarrow 7\}(x) = 3$ and $\alpha\{y \leftarrow 7\}(y) = 7$

# Semantics — Truth Value

## Definition 4.6 (Truth Value).

*Let $\mathcal{I} = (D, \iota)$ be an interpretation and $\alpha$ an assignment for $\mathcal{I}$. The truth value $v_{\mathcal{I}}(\alpha, A) \in \{T, F\}$ of a formula $A$ under $\mathcal{I}$ and $\alpha$ is defined inductively as follows:*

- $v_{\mathcal{I}}(\alpha, p) = T$ for 0-ary $p \in \mathcal{P}$ iff $p^{\iota} = T$, otherwise $v_{\mathcal{I}}(\alpha, p) = F$
- $v_{\mathcal{I}}(\alpha, p(t_1, \ldots, t_n)) = T$ for $p \in \mathcal{P}$, $n > 0$, iff $(v_{\mathcal{I}}(\alpha, t_1), \ldots, v_{\mathcal{I}}(\alpha, t_n)) \in p^{\iota}$, otherwise $v_{\mathcal{I}}(\alpha, p(t_1, \ldots, t_n)) = F$
- $v_{\mathcal{I}}(\alpha, \neg A) = T$ iff $v_{\mathcal{I}}(\alpha, A) = F$, otherwise $v_{\mathcal{I}}(\alpha, \neg A) = F$
- $v_{\mathcal{I}}(\alpha, A \wedge B) = T$ iff $v_{\mathcal{I}}(\alpha, A) = T$ and $v_{\mathcal{I}}(\alpha, B) = T$, otherwise $v_{\mathcal{I}}(\alpha, A \wedge B) = F$
- $v_{\mathcal{I}}(\alpha, A \vee B) = T$ iff $v_{\mathcal{I}}(\alpha, A) = T$ or $v_{\mathcal{I}}(\alpha, B) = T$, otherwise $v_{\mathcal{I}}(\alpha, A \vee B) = F$
- $v_{\mathcal{I}}(\alpha, A \rightarrow B) = T$ iff $v_{\mathcal{I}}(\alpha, A) = F$ or $v_{\mathcal{I}}(\alpha, B) = T$, otherwise $v_{\mathcal{I}}(\alpha, A \rightarrow B) = F$
- $v_{\mathcal{I}}(\alpha, \forall x A) = T$ iff $v_{\mathcal{I}}(\alpha\{x \leftarrow d\}, A) = T$ for all $d \in D$, otherwise $v_{\mathcal{I}}(\alpha, \forall x A) = F$
- $v_{\mathcal{I}}(\alpha, \exists x A) = T$ iff $v_{\mathcal{I}}(\alpha\{x \leftarrow d\}, A) = T$ for some $d \in D$, otherwise $v_{\mathcal{I}}(\alpha, \exists x A) = F$
- $v_{\mathcal{I}}(\alpha, \top) = T$ and $v_{\mathcal{I}}(\alpha, \bot) = F$

# Semantics — Truth Value

**Theorem 4.1 (Value of closed formulae).**

*For a closed term or formula, the assignment has no influence on the term value or truth value.* $v_{\mathcal{I}}$ *for* $v_{\mathcal{I},\alpha}$.

Example: $A = \forall x\, p(a, x)$ with the interpretations

- $\mathcal{I} = (\mathbb{N}, \iota)$ with $p^\iota = \leq$ and $a^\iota = 0$ $\leadsto v_{\mathcal{I}}(A) = T$
- $\mathcal{I} = (\mathbb{N}, \iota)$ with $p^\iota = \leq$ and $a^\iota = 3$ $\leadsto v_{\mathcal{I}}(A) = F$
- $\mathcal{I} = (\mathbb{Z}, \iota)$ with $p^\iota = \leq$ and $a^\iota = 0$ $\leadsto v_{\mathcal{I}}(A) = F$
- $\mathcal{I} = (\{c, d, e, f\}, \iota)$ with $p^\iota = \leq_{lexi}$ and $a^\iota = c$ $\leadsto v_{\mathcal{I}}(A) = T$

Example: $B = \forall x\, \forall y\, (p(x, y) \to p(f(x, a), f(y, a)))$ with interpretations

- $\mathcal{I} = (\mathbb{Z}, \iota)$ with $p^\iota = \leq$, $f^\iota = +$, and $a^\iota = 1$
  $\leadsto v_{\mathcal{I}}(B) = T$
- $\mathcal{I} = (\mathbb{Z}, \iota)$ with $p^\iota = >$, $f^\iota = *$, and $a^\iota = -1$
  $\leadsto v_{\mathcal{I}}(B) = F$

# Outline

# The Substitution Lemma for Terms

**Theorem 5.1 (Substitution Lemma for Terms).**

*Given an interpretation $\mathcal{I} = (D, \iota)$ and a variable assignment $\alpha$ for $\mathcal{I}$. Given also a variable $y \in \mathcal{V}$, and terms $t, s \in \mathcal{T}$*

$$v_{\mathcal{I}}(\alpha, t[y \backslash s]) = v_{\mathcal{I}}(\alpha\{y \leftarrow v_{\mathcal{I}}(\alpha, s)\}, t)$$

Proof.

*By structural induction on $t$. We abbreviate:* $\alpha' := \alpha\{y \leftarrow v_{\mathcal{I}}(\alpha, s)\}$

*For a constant $a$, $a[y \backslash s] = a$, so* $v_{\mathcal{I}}(\alpha, a[y \backslash s]) = v_{\mathcal{I}}(\alpha, a) = a^{\iota} = v_{\mathcal{I}}(\alpha', a)$

*For a variable $x \neq y$, $x[y \backslash s] = x$, so*
$v_{\mathcal{I}}(\alpha, x[y \backslash s]) = v_{\mathcal{I}}(\alpha, x) = \alpha(x) = \alpha'(x) = v_{\mathcal{I}}(\alpha', x)$ □

# Proof of substitution lemma, continued

**Proof.**

For the variable $y$, $y[y\backslash s] = s$, so
$v_{\mathcal{I}}(\alpha, y[y\backslash s]) = v_{\mathcal{I}}(\alpha, s) = v_{\mathcal{I}}(\alpha\{y{\leftarrow}v_{\mathcal{I}}(\alpha, s)\}, y)$

For a complex term, $f(\ldots t_i \ldots)[y\backslash s] = f(\ldots t_i[y\backslash s] \ldots)$, so

$$
\begin{aligned}
& v_{\mathcal{I}}(\alpha, f(\ldots t_i \ldots)[y\backslash s]) \\
= \;& v_{\mathcal{I}}(\alpha, f(\ldots t_i[y\backslash s] \ldots)) \quad \text{by def. of substitution} \\
= \;& f^{\iota}(\ldots v_{\mathcal{I}}(\alpha, t_i[y\backslash s]) \ldots) \quad \text{by model semantics} \\
= \;& f^{\iota}(\ldots v_{\mathcal{I}}(\alpha', t_i) \ldots) \quad \text{by the induction hypothesis} \\
= \;& v_{\mathcal{I}}(\alpha', f(\ldots t_i \ldots)) \quad \text{by model semantics}
\end{aligned}
$$

□

# The Substitution Lemma for Formulae

**Theorem 5.2 (Substitution Lemma for Terms).**

*Given an interpretation $\mathcal{I} = (D, \iota)$ and a variable assignment $\alpha$ for $\mathcal{I}$. Given also a variable $y \in \mathcal{V}$, a formula $A$ and a term $s \in \mathcal{T}$, such that $\{y \backslash s\}$ is collision-free for $A$.*

$$v_{\mathcal{I}}(\alpha, A[y \backslash s]) = v_{\mathcal{I}}(\alpha', A)$$

# Outline

# Satisfiability and Validity

**Definition 6.1 (Satisfiable,Model,Unsatisfiable,Valid,Invalid).**

*Let $A$ be a closed (first-order) formula and $U=\{A_1, \dots\}$ be a set of closed (first-order) formulae $A_i$.*

- *$A$ is satisfiable iff $v_{\mathcal{I}}(A) = T$ for some interpretation $\mathcal{I}$.*
- *A satisfying interpretation $\mathcal{I}$ for $A$ is called a model for $A$.*
- *$U=\{A_1, \dots\}$ is satisfiable iff there is (common) model for all $A_i$.*
- *$A/U$ is unsatisfiable iff $A/U$ is not satisfiable.*
- *$A$ is valid, denoted $\models A$, iff $v_{\mathcal{I}}(A) = T$ for all interpretations $\mathcal{I}$.*
- *$A$ is invalid/falsifiable iff $A$ is not valid.*

**Theorem 6.1 (Satisfiable, Valid, Unsatisfiable, Invalid).**

*$A$ is valid iff $\neg A$ is unsatisfiable. $A$ is satisfiable iff $\neg A$ is invalid.*

# Examples for Satisfiable and Invalid Formulae

Example: $A = \forall x\, p(a, x)$

- $\mathcal{I} = (\mathbb{N}, \iota)$ with $p^\iota = \leq$ and $a^\iota = 3$ $\rightsquigarrow v_\mathcal{I}(A) = F$
  $\rightsquigarrow$ $A$ is invalid
- $\mathcal{I} = (\{c, d, e, f\}, \iota)$ with $p^\iota = \leq_{lexi}$ and $a^\iota = c$ $\rightsquigarrow v_\mathcal{I}(A) = T$
  $\rightsquigarrow$ $A$ is satisfiable ($\mathcal{I}$ is a model)

Example: $B = \forall x\, \forall y\, (p(x, y) \rightarrow p(f(x, a), f(y, a)))$

- $\mathcal{I} = (\mathbb{Z}, \iota)$ with $p^\iota = \leq$, $f^\iota = +$, and $a^\iota = 1$ $\rightsquigarrow v_\mathcal{I}(B) = T$
  $\rightsquigarrow$ satisfiable ($\mathcal{I}$ is a model)
- $\mathcal{I} = (\mathbb{Z}, \iota)$ with $p^\iota = >$, $f^\iota = *$, and $a^\iota = -1$ $\rightsquigarrow v_\mathcal{I}(B) = F$
  $\rightsquigarrow$ invalid ($\mathcal{I}$ is a "counter-model")

Example: $\forall x\, \forall y\, (p(x, y) \rightarrow p(y, x))$
$\rightsquigarrow$ satisfiable (e.g. $p^\iota =$ "$=$"), but invalid (e.g. $p^\iota =$ "$<$")

Example: $\exists x\, \exists y\, (p(x) \land \neg p(y))$
$\rightsquigarrow$ only satisfiable for $|D| \geq 2$, invalid (e.g. $D = \mathbb{N}$, $p^\iota = even$)

# Logical Equivalence

The concept of logical equivalence can be adapted to first-order logic, i.e. to closed first-order formulae.

**Definition 6.2 (Logical Equivalence).**

*Let $A_1$, $A_2$ be two closed formulae. $A_1$ is logically equivalent to $A_2$, denoted $A_1 \equiv A_2$ iff $v_{\mathcal{I}}(A_1) = v_{\mathcal{I}}(A_2)$ for all interpretations $\mathcal{I}$.*

**Theorem 6.2 (Relation $\equiv$ and $\leftrightarrow$).**

*Let $A, B$ be two closed formulae and $U = \{A_1, \dots, A_n\}$ be a set of closed formulas. Then $A \equiv B$ iff $\models A \leftrightarrow B$.*

Remark: $A \leftrightarrow B := (A \rightarrow B) \wedge (B \rightarrow A)$

Important: even though $\equiv$ and $\leftrightarrow$ are closely related, they are different relations. Whereas $\leftrightarrow$ is part of the object language (i.e. the definition of formulae), $\equiv$ is used in the meta-language to talk about or relate formulae.

# Logically Equivalent Formulae

Duality: $\forall$ can be expressed with $\exists$, and vice versa

- $\models \forall x\, A(x) \leftrightarrow \neg\exists x\, \neg A(x)$
- $\models \exists x\, A(x) \leftrightarrow \neg\forall x\, \neg A(x)$

Commutativity:

- $\models \forall x\, \forall y\, A(x,y) \leftrightarrow \forall y\, \forall x\, A(x,y)$
- $\models \exists x\, \exists y\, A(x,y) \leftrightarrow \exists y\, \exists x\, A(x,y)$
- $\models \exists x\, \forall y\, A(x,y) \rightarrow \forall y\, \exists x\, A(x,y)$ (other direction is not valid!)

Distributivity:

- $\models \exists x\, (A(x) \vee B(x)) \leftrightarrow \exists x\, A(x) \vee \exists x\, B(x)$
- $\models \forall x\, (A(x) \wedge B(x)) \leftrightarrow \forall x\, A(x) \wedge \forall x\, B(x)$
- $\models \forall x\, A(x) \vee \forall x\, B(x) \rightarrow \forall x\, (A(x) \vee B(x))$ (other direction not valid!)
- $\models \exists x\, (A(x) \wedge B(x)) \rightarrow \exists x\, A(x) \wedge \exists x\, B(x)$ (other direction not valid!)

See [Ben-Ari 2012] for more equivalences involving quantifiers.

# Logical Consequence

**Definition 6.3 (Logical Consequence).**

*Let $A$ be a closed formula and $U$ be a set of closed formulae. $A$ is a logical consequence of $U$, denoted $U \models A$, iff every model of $U$ is a model of $A$, i.e. $v_{\mathcal{I}}(A_i) = T$ for all $A_i \in U$ implies $v_{\mathcal{I}}(A) = T$.*

**Theorem 6.3 (Logical Consequence and Validity).**

*Let $A$ be a closed formula and $U = \{A_1, \ldots, A_n\}$ be a set of closed formulae. Then $U \models A$ iff $\models (A_1 \wedge \cdots \wedge A_n) \rightarrow A$.*

- again, we can reduce the problem of "logical consequence" to the problem of determining if a formula is valid
- hence, we need methods or proof search calculi that can deal with first-order formulae

# Outline

- Motivation
- Syntax
- Variables
- Semantics
- The Substitution Lemma
- Satisfiability & Validity
- LK for First-order Logic
- Summary

# LK — Axiom and Propositional Rules

- axiom

$$\frac{}{\Gamma, A \implies A, \Delta}\text{ axiom}$$

- rules for $\wedge$ (conjunction)

$$\frac{\Gamma, A, B \implies \Delta}{\Gamma, A \wedge B \implies \Delta}\ \wedge\text{-left} \qquad \frac{\Gamma \implies A, \Delta \qquad \Gamma \implies B, \Delta}{\Gamma \implies A \wedge B, \Delta}\ \wedge\text{-right}$$

- rules for $\vee$ (disjunction)

$$\frac{\Gamma, A \implies \Delta \qquad \Gamma, B \implies \Delta}{\Gamma, A \vee B \implies \Delta}\ \vee\text{-left} \qquad \frac{\Gamma \implies A, B, \Delta}{\Gamma \implies A \vee B, \Delta}\ \vee\text{-right}$$

- rules for $\rightarrow$ (implication)

$$\frac{\Gamma \implies A, \Delta \qquad \Gamma, B \implies \Delta}{\Gamma, A \rightarrow B \implies \Delta}\ \rightarrow\text{-left} \qquad \frac{\Gamma, A \implies B, \Delta}{\Gamma \implies A \rightarrow B, \Delta}\ \rightarrow\text{-right}$$

- rules for $\neg$ (negation)

$$\frac{\Gamma \implies A, \Delta}{\Gamma, \neg A \implies \Delta}\ \neg\text{-left} \qquad \frac{\Gamma, A \implies \Delta}{\Gamma \implies \neg A, \Delta}\ \neg\text{-right}$$

# LK — Rules for Universal and Existential Quantifier

- rules for $\forall$ (universal quantifier)

$$\frac{\Gamma, A[x\backslash t], \forall x\, A \implies \Delta}{\Gamma, \forall x\, A \implies \Delta}\ \forall\text{-left} \qquad \frac{\Gamma \implies A[x\backslash a], \Delta}{\Gamma \implies \forall x\, A, \Delta}\ \forall\text{-right}^*$$

  - $t$ is an arbitrary term
  - Eigenvariable condition for the rule $\forall$-right$^*$: $a$ must not occur in the conclusion, i.e. in $\Gamma$, $\Delta$, or $A$
  - the formula $\forall x\, A$ is preserved in the premise of the rule $\forall$-left

- rules for $\exists$ (existential quantifier)

$$\frac{\Gamma, A[x\backslash a] \implies \Delta}{\Gamma, \exists x\, A \implies \Delta}\ \exists\text{-left}^* \qquad \frac{\Gamma \implies \exists x\, A, A[x\backslash t], \Delta}{\Gamma \implies \exists x\, A, \Delta}\ \exists\text{-right}$$

  - $t$ is an arbitrary term
  - Eigenvariable condition for the rule $\exists$-left$^*$: $a$ must not occur in the conclusion, i.e. in $\Gamma$, $\Delta$, or $A$
  - the formula $\exists x\, A$ is preserved in the premise of the rule $\exists$-right

# Soundness and Completeness

**Theorem 7.1 (Soundness and Completeness of LK).**

*The calculus of natural deduction LK is sound and complete, i.e.*

- *if A is provable in LK, then A is valid (if* $\vdash A$ *then* $\models A$*)*
- *if A is valid, then A is provable in LK (if* $\models A$ *then* $\vdash A$*)*

Proof.

*Next week.* □

# Examples of LK Proofs

Example: $\forall x\, p(x) \to \exists x\, p(x)$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{p(c), \forall x\, p(x) \Longrightarrow p(c), \exists x\, p(x)}\ \text{axiom}}{p(c), \forall x\, p(x) \Longrightarrow \exists x\, p(x)}\ \exists\text{-right}}{\forall x\, p(x) \Longrightarrow \exists x\, p(x)}\ \forall\text{-left}}{\Longrightarrow \forall x\, p(x) \to \exists x\, p(x)}\ \to\text{-right}$$

Example: $p(a) \to p(b)$

$$\dfrac{}{\Longrightarrow p(a) \to p(b)}\ (?)$$

Example: $p(a) \to \exists x\, p(x)$

$$\dfrac{\dfrac{\dfrac{}{p(a) \Longrightarrow p(a), \exists x\, p(x)}\ \text{axiom}}{p(a) \Longrightarrow \exists x\, p(x)}\ \exists\text{-right}}{\Longrightarrow p(a) \to \exists x\, p(x)}\ \to\text{-right}$$

Example: $\exists x\, p(x) \to p(a)$

$$\dfrac{\dfrac{}{\exists x\, p(x) \Longrightarrow p(a)}\ \exists\text{-left}^*}{\Longrightarrow \exists x\, p(x) \to p(a)}\ \to\text{-right}$$

rule $\exists$-left$^*$ with $p(x)[x \backslash a]$ cannot be applied as $a$ occurs in the premise (Eigenvariable condition!)

# Outline

# Summary

- first-order logic extends the syntax of propositional logic by: constants, functions, variables, predicates, and the quantifiers $\forall/\exists$
- the semantics consists of a domain $D$ and an interpretation $\iota$
- the interpretation $\iota$ relates constants to elements of the domain, function symbols to actual functions, and predicates to relations
- variables are interpreted by a variable assignment $\alpha$
- the formula $\forall x\, p(x) / \exists x\, p(x)$ evaluates to $T$ iff $p(x)$ evaluates to $T$ for all/some element(s) in $D$
- the truth value of formulae is inductively evaluated, and takes the value of terms into account
- most concepts from propositional logic can be adapted
- four semantical concepts: satisfiable, valid, unsatisfiable, invalid
- logical consequence in first-order logic can be reduced to validity
- Next week: Soundness and completeness