IN3070/4070 – Logic – Autumn 2019

Lecture 4: First-order Logic

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Today's Plan

- Motivation
- Syntax
- Variables
- Semantics
- The Substitution Lemma
- Satisfiability & Validity
- LK for First-order Logic
- Summary

Outline

Limitations of Propositional Logic

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First-order logic: extension of propositional logic

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Semantics:

- interpretation of constants, functions, variables
- interpretation of predicates
- value of terms
- truth value of (quantified) formulae
- satisfiability, validity, logical equivalence,...

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Example: a, x, $f(a, x)$, $f(g(x), b)$, and $g(f(a, g(y)))$ are terms.

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Let $\mathcal{P} = \{p, q, r, \ldots\}$ be a countable set of predicate symbols. If $p \in \mathcal{P}$ is an n-ary predicate (symbol) $n \geq 0$ and $t_1, \ldots, t_n$ are terms, then $p(t_1, \ldots, t_n)$, $\top$, and $\bot$ are atomic formulae (or atoms).

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- *Every atomic formula p is a formula.*
- *If A and B are formulae and $x \in \mathcal{V}$, then $(\neg A)$, $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$, $\forall x\, A$, and $\exists x\, A$ are formulae.*

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$\forall x\, (\neg \exists y\, p(x, y) \vee \neg \exists y\, p(y, x))$

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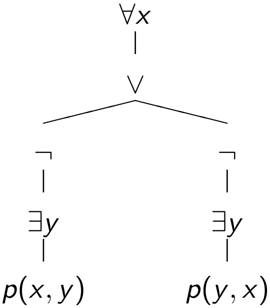

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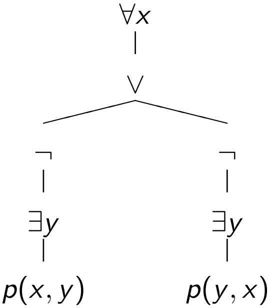

Definition 2.4 (Subformula, Main Operator).

Formula A is a (proper) subformula of formula B iff A is a (proper) subtree of B. If the root of a formula tree of A is a logical connective/quantifier, then it is called the main operator of A.

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A bound variable in a formula C is a variable that appears in $\forall x$ or $\exists x$ in some subformula of C. A formula/term is closed iff it has no free variables.

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If $\{x_1, \ldots, x_n\}$ are all the free variables of A, the universal closure of A is $\forall x_1 \ldots \forall x_n\, A$ and the existential closure of A is $\exists x_1 \ldots \exists x_n\, A$.

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- $p(x, y)$ has the two free variables x and y. Its universal closure is $\forall x\, \forall y\, p(x, y)$ and its existential closure is $\exists x\, \exists y\, p(x, y)$; $\exists y\, p(x, y)$ has the only free variable x; $\forall x\, \exists y\, p(x, y)$ is closed

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- In $\forall x\, p(x) \wedge q(x)$, the x occurs bound and free. The universal closure is $\forall x\, (\forall x\, p(x) \wedge q(x))$; renaming: $\forall y\, (\forall x\, p(x) \wedge q(y))$

Substitutions

Free variables in a first-order formula can be substituted by terms.

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Others: $[a/x, f(z, b)/y]$

Application of substitutions

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Let σ be a substitution. The application of σ to a term t or formula A, written $\sigma(t)$ or $\sigma(A)$, replaces every free variable in t or A according to its image under σ. Short hand: $A[x\backslash t]{=}\sigma(A)$ with $\sigma{=}\{x\backslash t\}$.

Example: Let $\sigma{=}\{x\backslash a, y\backslash f(z,b)\}$ be a substitution.
Then $\sigma(g(y)) = g(f(z,b))$
and $\sigma(p(x) \wedge \forall x\, q(x, g(y))) = p(a) \wedge \forall x\, q(x, g(f(z,b)))$

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Definition 3.6 (Capture-free substitution).

A substitution σ is capture-free for a formula A if for every free variable x in A, none of the variables in $\sigma(x)$ is bound in A.

Application of Substitutions

Definition 3.7 (Application of Substitutions, formally).

The application of a subtitution σ to a term or formula is defined by structural induction:

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- *…similarly for* $\neg A$, $A \lor B$, $A \to B$…

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- $\sigma(A \wedge B) = \sigma(A) \wedge \sigma(B)$ *for formulae* A, B
- *. . . similarly for* $\neg A$, $A \vee B$, $A \rightarrow B$*. . .*
- $\sigma(\exists x\, A) = \exists x\, \sigma_x(A), \quad \sigma(\forall x\, A) = \forall x\, \sigma_x(A)$

where we define σ_x *by:* $\sigma_x(x) = x$, *and* $\sigma_x(y) = \sigma(y)$ *for all* $y \neq x$

Outline

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- *Interpretation of propositional variables assigns each 0-ary predicate symbol $p \in \mathcal{P}$ a truth value $p^\iota \in \{T, F\}$*
- *Interpretation of predicate symbols assigns each n-ary predicate symbol $p \in \mathcal{P}$ with $n>0$ a relation $p^\iota \subseteq D^n$*

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Examples:

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- $+(1, *(4, 2))$ with $\mathcal{I} = (\mathbb{Z}, \iota)$ with $+^{\iota} = *$ (multiplication), $*^{\iota} = -$ (subtraction), $1^{\iota} = -20$, $2^{\iota} = 0$, $4^{\iota} = 10$;

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- $+(1, *(4, 2))$ with $\mathcal{I} = (\mathbb{Z}, \iota)$ with $+^{\iota} = *$ (multiplication), $*^{\iota} = -$ (subtraction), $1^{\iota} = -20$, $2^{\iota} = 0$, $4^{\iota} = 10$; then $v_{\mathcal{I}}(+(1, *(4, 2))) = -200$

Semantics — Variable Assignments, Value of Terms

The interpretation doesn't tell what to do about variables.
We need something additional.

Definition 4.3 (Variable Assignment).

Given the set of variables $\mathcal{V}$*, and an interpretation* $\mathcal{I} = (D, \iota)$*, a variable assignment* α *for* $\mathcal{I}$ *is a function* $\alpha : \mathcal{V} \to D$*.*

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Definition 4.4 (Term Value).

Let $\mathcal{I} = (D, \iota)$ be an interpretation, and α an variable assignment for $\mathcal{I}$. The term value $v_{\mathcal{I}}(\alpha, t)$ of a term $t \in \mathcal{T}$ under $\mathcal{I}$ and α is inductively defined:

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Let $\mathcal{I} = (D, \iota)$ be an interpretation, and α an variable assignment for $\mathcal{I}$. The term value $v_{\mathcal{I}}(\alpha, t)$ of a term $t \in \mathcal{T}$ under $\mathcal{I}$ and α is inductively defined:

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Semantics — Variable Assignments, Value of Terms

The interpretation doesn't tell what to do about variables.
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Definition 4.3 (Variable Assignment).

Given the set of variables $\mathcal{V}$, and an interpretation $\mathcal{I} = (D, \iota)$, a variable assignment α for $\mathcal{I}$ is a function $\alpha : \mathcal{V} \to D$.

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- $\mathcal{I} = (\mathbb{N}, \iota)$ with $f^\iota = +$, $a^\iota = 10$

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Semantics — Modification of an assignment

Definition 4.5 (Modification of a variable assignment).

Given an interpretation $\mathcal{I} = (D, \iota)$ and a variable assignment α for $\mathcal{I}$.
Given also a variable $y \in \mathcal{V}$ and a domain element $d \in D$.
The modified variable assignment $\alpha\{y \leftarrow d\}$ is defined as

$$\alpha\{y \leftarrow d\}(x) = \begin{cases} d & \text{if } x = y \\ \alpha(x) & \text{otherwise} \end{cases}$$

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- $\alpha\{y \leftarrow 7\}(x) = 3$

Semantics — Modification of an assignment

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The modified variable assignment $\alpha\{y \leftarrow d\}$ is defined as

$$\alpha\{y \leftarrow d\}(x) = \begin{cases} d & \text{if } x = y \\ \alpha(x) & \text{otherwise} \end{cases}$$

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- $\alpha\{y \leftarrow 7\}(x) = 3$ and $\alpha\{y \leftarrow 7\}(y) = 7$

Semantics — Truth Value

Definition 4.6 (Truth Value).

Let $\mathcal{I} = (D, \iota)$ *be an interpretation and* α *an assignment for* $\mathcal{I}$. *The truth value* $v_{\mathcal{I}}(\alpha, A) \in \{T, F\}$ *of a formula A under* $\mathcal{I}$ *and* α *is defined inductively as follows:*

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- $v_{\mathcal{I}}(\alpha, p) = T$ *for* 0*-ary* $p \in \mathcal{P}$ *iff* $p^{\iota} = T$*, otherwise* $v_{\mathcal{I}}(\alpha, p) = F$

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- $v_{\mathcal{I}}(\alpha, p) = T$ *for* 0*-ary* $p \in \mathcal{P}$ *iff* $p^{\iota} = T$, *otherwise* $v_{\mathcal{I}}(\alpha, p) = F$
- $v_{\mathcal{I}}(\alpha, p(t_1, \ldots, t_n)) = T$ *for* $p \in \mathcal{P}$, $n > 0$, *iff* $(v_{\mathcal{I}}(\alpha, t_1), \ldots, v_{\mathcal{I}}(\alpha, t_n)) \in p^{\iota}$, *otherwise* $v_{\mathcal{I}}(\alpha, p(t_1, \ldots, t_n)) = F$

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- $v_{\mathcal{I}}(\alpha, p(t_1, \ldots, t_n)) = T$ *for* $p \in \mathcal{P}$, $n > 0$, *iff* $(v_{\mathcal{I}}(\alpha, t_1), \ldots, v_{\mathcal{I}}(\alpha, t_n)) \in p^{\iota}$, *otherwise* $v_{\mathcal{I}}(\alpha, p(t_1, \ldots, t_n)) = F$
- $v_{\mathcal{I}}(\alpha, \neg A) = T$ *iff* $v_{\mathcal{I}}(\alpha, A) = F$, *otherwise* $v_{\mathcal{I}}(\alpha, \neg A) = F$

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- $v_{\mathcal{I}}(\alpha, \neg A) = T$ *iff* $v_{\mathcal{I}}(\alpha, A) = F$*, otherwise* $v_{\mathcal{I}}(\alpha, \neg A) = F$
- $v_{\mathcal{I}}(\alpha, A \wedge B) = T$ *iff* $v_{\mathcal{I}}(\alpha, A) = T$ *and* $v_{\mathcal{I}}(\alpha, B) = T$*, otherwise* $v_{\mathcal{I}}(\alpha, A \wedge B) = F$

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- $v_{\mathcal{I}}(\alpha, A \wedge B) = T$ iff $v_{\mathcal{I}}(\alpha, A) = T$ and $v_{\mathcal{I}}(\alpha, B) = T$, otherwise $v_{\mathcal{I}}(\alpha, A \wedge B) = F$
- $v_{\mathcal{I}}(\alpha, A \vee B) = T$ iff $v_{\mathcal{I}}(\alpha, A) = T$ or $v_{\mathcal{I}}(\alpha, B) = T$, otherwise $v_{\mathcal{I}}(\alpha, A \vee B) = F$

Semantics — Truth Value

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- $v_{\mathcal{I}}(\alpha, \forall x A) = T$ iff $v_{\mathcal{I}}(\alpha\{x \leftarrow d\}, A) = T$ for all $d \in D$, otherwise $v_{\mathcal{I}}(\alpha, \forall x A) = F$

Semantics — Truth Value

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- $v_{\mathcal{I}}(\alpha, \forall x A) = T$ iff $v_{\mathcal{I}}(\alpha\{x \leftarrow d\}, A) = T$ for all $d \in D$, otherwise $v_{\mathcal{I}}(\alpha, \forall x A) = F$
- $v_{\mathcal{I}}(\alpha, \exists x A) = T$ iff $v_{\mathcal{I}}(\alpha\{x \leftarrow d\}, A) = T$ for some $d \in D$, otherwise $v_{\mathcal{I}}(\alpha, \exists x A) = F$

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- $v_{\mathcal{I}}(\alpha, A \vee B) = T$ iff $v_{\mathcal{I}}(\alpha, A) = T$ or $v_{\mathcal{I}}(\alpha, B) = T$, otherwise $v_{\mathcal{I}}(\alpha, A \vee B) = F$
- $v_{\mathcal{I}}(\alpha, A \rightarrow B) = T$ iff $v_{\mathcal{I}}(\alpha, A) = F$ or $v_{\mathcal{I}}(\alpha, B) = T$, otherwise $v_{\mathcal{I}}(\alpha, A \rightarrow B) = F$
- $v_{\mathcal{I}}(\alpha, \forall x A) = T$ iff $v_{\mathcal{I}}(\alpha\{x \leftarrow d\}, A) = T$ for all $d \in D$, otherwise $v_{\mathcal{I}}(\alpha, \forall x A) = F$
- $v_{\mathcal{I}}(\alpha, \exists x A) = T$ iff $v_{\mathcal{I}}(\alpha\{x \leftarrow d\}, A) = T$ for some $d \in D$, otherwise $v_{\mathcal{I}}(\alpha, \exists x A) = F$
- $v_{\mathcal{I}}(\alpha, \top) = T$ and $v_{\mathcal{I}}(\alpha, \bot) = F$

Semantics — Truth Value

Theorem 4.1 (Value of closed formulae).

For a closed term or formula, the assignment has no influence on the term value or truth value. $v_{\mathcal{I}}$ *for* $v_{\mathcal{I},\alpha}$.

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Outline

The Substitution Lemma for Terms

Theorem 5.1 (Substitution Lemma for Terms).

Given an interpretation $\mathcal{I} = (D, \iota)$ and a variable assignment α for $\mathcal{I}$.
Given also a variable $y \in \mathcal{V}$, and terms $t, s \in \mathcal{T}$

$$v_{\mathcal{I}}(\alpha, t[y\backslash s]) = v_{\mathcal{I}}(\alpha\{y \leftarrow v_{\mathcal{I}}(\alpha, s)\}, t)$$

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Proof.

By structural induction on t.

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For a constant a, $a[y \backslash s] = a$, so $v_{\mathcal{I}}(\alpha, a[y \backslash s]) = v_{\mathcal{I}}(\alpha, a) = a^{\iota} = v_{\mathcal{I}}(\alpha', a)$

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For a variable $x \neq y$, $x[y \backslash s] = x$, so
$v_{\mathcal{I}}(\alpha, x[y \backslash s]) = v_{\mathcal{I}}(\alpha, x) = \alpha(x) = \alpha'(x) = v_{\mathcal{I}}(\alpha', x)$

Proof of substitution lemma, continued

Proof.

For the variable y, $y[y\backslash s] = s$, so
$v_{\mathcal{I}}(\alpha, y[y\backslash s]) = v_{\mathcal{I}}(\alpha, s) = v_{\mathcal{I}}(\alpha\{y{\leftarrow}v_{\mathcal{I}}(\alpha, s)\}, y)$

Proof of substitution lemma, continued

Proof.

For the variable y, $y[y\backslash s] = s$, so
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For a complex term, $f(\ldots t_i \ldots)[y\backslash s] = f(\ldots t_i[y\backslash s] \ldots)$, so

$$v_{\mathcal{I}}(\alpha, f(\ldots t_i \ldots)[y\backslash s])$$

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Proof of substitution lemma, continued

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□

The Substitution Lemma for Formulae

Theorem 5.2 (Substitution Lemma for Terms).

Given an interpretation $\mathcal{I} = (D, \iota)$ and a variable assignment α for $\mathcal{I}$. Given also a variable $y \in \mathcal{V}$, a formula A and a term $s \in \mathcal{T}$, such that $\{y \backslash s\}$ is collision-free for A.

$$v_{\mathcal{I}}(\alpha, A[y \backslash s]) = v_{\mathcal{I}}(\alpha', A)$$

Outline

- Motivation
- Syntax
- Variables
- Semantics
- The Substitution Lemma
- **Satisfiability & Validity**
- LK for First-order Logic
- Summary

Satisfiability and Validity

Definition 6.1 (Satisfiable,Model,Unsatisfiable,Valid,Invalid).

Let A be a closed (first-order) formula and $U=\{A_1, \ldots\}$ be a set of closed (first-order) formulae A_i.

- *A is satisfiable iff $v_{\mathcal{I}}(A) = T$ for some interpretation $\mathcal{I}$.*
- *A satisfying interpretation $\mathcal{I}$ for A is called a model for A.*

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Theorem 6.1 (Satisfiable, Valid, Unsatisfiable, Invalid).

A is valid iff $\neg A$ is unsatisfiable. A is satisfiable iff $\neg A$ is invalid.

Examples for Satisfiable and Invalid Formulae

Example: $A = \forall x\, p(a, x)$

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- $\mathcal{I} = (\mathbb{N}, \iota)$ with $p^\iota = \leq$ and $a^\iota = 3 \quad \rightsquigarrow v_{\mathcal{I}}(A) = F$
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Example: $\exists x\, \exists y\, (p(x) \wedge \neg p(y))$
$\leadsto$ only satisfiable for $|D| \geq 2$, invalid (e.g. $D = \mathbb{N}$, $p^\iota = even$)

Logical Equivalence

The concept of logical equivalence can be adapted to first-order logic, i.e. to closed first-order formulae.

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Definition 6.2 (Logical Equivalence).

Let A_1, A_2 be two closed formulae. A_1 is logically equivalent to A_2, denoted $A_1 \equiv A_2$ iff $v_{\mathcal{I}}(A_1) = v_{\mathcal{I}}(A_2)$ for all interpretations $\mathcal{I}$.

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Theorem 6.2 (Relation $\equiv$ and $\leftrightarrow$).

Let A, B be two closed formulae and $U = \{A_1, \ldots, A_n\}$ be a set of closed formulas. Then $A \equiv B$ iff $\models A \leftrightarrow B$.

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Important: even though $\equiv$ and $\leftrightarrow$ are closely related, they are different relations. Whereas $\leftrightarrow$ is part of the object language (i.e. the definition of formulae), $\equiv$ is used in the meta-language to talk about or relate formulae.

Logically Equivalent Formulae

Duality: $\forall$ can be expressed with $\exists$, and vice versa

- $\models \forall x\, A(x) \leftrightarrow \neg\exists x\, \neg A(x)$
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Commutativity:

- $\models \forall x\, \forall y\, A(x, y) \leftrightarrow \forall y\, \forall x\, A(x, y)$
- $\models \exists x\, \exists y\, A(x, y) \leftrightarrow \exists y\, \exists x\, A(x, y)$
- $\models \exists x\, \forall y\, A(x, y) \rightarrow \forall y\, \exists x\, A(x, y)$ (other direction is not valid!)

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Distributivity:

- $\models \exists x\, (A(x) \vee B(x)) \leftrightarrow \exists x\, A(x) \vee \exists x\, B(x)$
- $\models \forall x\, (A(x) \wedge B(x)) \leftrightarrow \forall x\, A(x) \wedge \forall x\, B(x)$
- $\models \forall x\, A(x) \vee \forall x\, B(x) \rightarrow \forall x\, (A(x) \vee B(x))$ (other direction not valid!)
- $\models \exists x\, (A(x) \wedge B(x)) \rightarrow \exists x\, A(x) \wedge \exists x\, B(x)$ (other direction not valid!)

See [Ben-Ari 2012] for more equivalences involving quantifiers.

Logical Consequence

Definition 6.3 (Logical Consequence).

Let A be a closed formula and U be a set of closed formulae. A is a logical consequence of U, denoted $U \models A$, iff every model of U is a model of A, i.e. $v_{\mathcal{I}}(A_i) = T$ for all $A_i \in U$ implies $v_{\mathcal{I}}(A) = T$.

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Theorem 6.3 (Logical Consequence and Validity).

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- again, we can reduce the problem of “logical consequence” to the problem of determining if a formula is valid
- hence, we need methods or proof search calculi that can deal with first-order formulae

Outline

- Motivation
- Syntax
- Variables
- Semantics
- The Substitution Lemma
- Satisfiability & Validity
- **LK for First-order Logic**
- Summary

LK — Axiom and Propositional Rules

- axiom

$$\dfrac{}{\Gamma, A \implies A, \Delta}\ \text{axiom}$$

- rules for $\land$ (conjunction)

$$\dfrac{\Gamma, A, B \implies \Delta}{\Gamma, A \land B \implies \Delta}\ \land\text{-left} \qquad \dfrac{\Gamma \implies A, \Delta \qquad \Gamma \implies B, \Delta}{\Gamma \implies A \land B, \Delta}\ \land\text{-right}$$

- rules for $\lor$ (disjunction)

$$\dfrac{\Gamma, A \implies \Delta \qquad \Gamma, B \implies \Delta}{\Gamma, A \lor B \implies \Delta}\ \lor\text{-left} \qquad \dfrac{\Gamma \implies A, B, \Delta}{\Gamma \implies A \lor B, \Delta}\ \lor\text{-right}$$

- rules for $\to$ (implication)

$$\dfrac{\Gamma \implies A, \Delta \qquad \Gamma, B \implies \Delta}{\Gamma, A \to B \implies \Delta}\ \to\text{-left} \qquad \dfrac{\Gamma, A \implies B, \Delta}{\Gamma \implies A \to B, \Delta}\ \to\text{-right}$$

- rules for $\neg$ (negation)

$$\dfrac{\Gamma \implies A, \Delta}{\Gamma, \neg A \implies \Delta}\ \neg\text{-left} \qquad \dfrac{\Gamma, A \implies \Delta}{\Gamma \implies \neg A, \Delta}\ \neg\text{-right}$$

LK — Rules for Universal and Existential Quantifier

- rules for $\forall$ (universal quantifier)

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LK — Rules for Universal and Existential Quantifier

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LK — Rules for Universal and Existential Quantifier

- rules for ∀ (universal quantifier)

$$\frac{\Gamma, A[x \backslash t], \forall x\, A \implies \Delta}{\Gamma, \forall x\, A \implies \Delta}\ \forall\text{-left} \qquad \frac{\Gamma \implies A[x \backslash a], \Delta}{\Gamma \implies \forall x\, A, \Delta}\ \forall\text{-right}^*$$

  - t is an arbitrary term
  - Eigenvariable condition for the rule ∀-right*: a must not occur in the conclusion, i.e. in Γ, Δ, or A
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Soundness and Completeness

Theorem 7.1 (Soundness and Completeness of LK).

The calculus of natural deduction LK is sound and complete, i.e.

- *if A is provable in LK, then A is valid (if $\vdash A$ then $\models A$)*
- *if A is valid, then A is provable in LK (if $\models A$ then $\vdash A$)*

Proof.

Next week. □

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$$\dfrac{}{\implies p(a) \to p(b)}\ (?)$$

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Example: $p(a) \to p(b)$

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Examples of LK Proofs

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rule $\exists$-left*with $p(x)[x \backslash a]$ cannot be applied as a occurs in the premise (Eigenvariable condition!)

Outline

- Motivation
- Syntax
- Variables
- Semantics
- The Substitution Lemma
- Satisfiability & Validity
- LK for First-order Logic
- **Summary**

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- Next week: Soundness and completeness