IN3070/4070 – Logic – Autumn 2019

Lecture 5: Soundness & Completeness for 1st-order LK

Martin Giese

18th September 2019

Department of Informatics

University of Oslo

Today's Plan

- Preliminaries and Reminders
- Soundness Proof
- Completeness: Preliminaries
- Proof of Completeness
- Examples of Counter-model Construction

Outline

- Preliminaries and Reminders
- Soundness Proof
- Completeness: Preliminaries
- Proof of Completeness
- Examples of Counter-model Construction

Reminder Soundness of LK

- We want all LK-provable sequents to be valid!

Reminder Soundness of LK

- We want all LK-provable sequents to be valid!
- If they are not, then LK would be incorrect or unsound . . .

Reminder Soundness of LK

- We want all LK-provable sequents to be valid!
- If they are not, then LK would be incorrect or unsound . . .

Definition 1.1 (Soundness).

The sequent calculus LK *is sound if every* LK*-provable sequent is valid.*

Reminder Soundness of LK

- We want all LK-provable sequents to be valid!
- If they are not, then LK would be incorrect or unsound . . .

Definition 1.1 (Soundness).

The sequent calculus LK *is sound if every* LK*-provable sequent is valid.*

Theorem 1.1.

The sequent calculus LK *is sound.*

Assumptions about the first order language

- We assume that a first-order language is given, by sets of constants, function symbols, and predicates.

Assumptions about the first order language

- We assume that a first-order language is given, by sets of constants, function symbols, and predicates.
- Some rules require “fresh” constants, so we assume that the set of constant symbols $\mathcal{A}$ is (countably) infinite.

Assumptions about the first order language

- We assume that a first-order language is given, by sets of constants, function symbols, and predicates.
- Some rules require "fresh" constants, so we assume that the set of constant symbols $\mathcal{A}$ is (countably) infinite.
- A root sequent $\Gamma \implies \Delta$ consists of *closed* formulae.

Assumptions about the first order language

- We assume that a first-order language is given, by sets of constants, function symbols, and predicates.
- Some rules require "fresh" constants, so we assume that the set of constant symbols $\mathcal{A}$ is (countably) infinite.
- A root sequent $\Gamma \implies \Delta$ consists of *closed* formulae.
- We show that if $\Gamma \implies \Delta$ is provable, then $\Gamma \implies \Delta$ is valid

Reminer: Semantics for Sequents

Definition 1.2 (Valid sequent).

A sequent $\Gamma \implies \Delta$ is valid if all interpretations that satisfy all formulae in Γ satisfy at least one formula in Δ.

Definition 1.3 (Countermodel/falsifiable sequent).

- *An interpretation $\mathcal{I}$ is a countermodel for the sequent $\Gamma \implies \Delta$ if $v_{\mathcal{I}}(A) = T$ for all formulae $A \in \Gamma$ and $v_{\mathcal{I}}(B) = F$ for all formulae $B \in \Delta$*

Reminer: Semantics for Sequents

Definition 1.2 (Valid sequent).

A sequent $\Gamma \implies \Delta$ is valid if all interpretations that satisfy all formulae in Γ satisfy at least one formula in Δ.

Definition 1.3 (Countermodel/falsifiable sequent).

- *An interpretation $\mathcal{I}$ is a countermodel for the sequent $\Gamma \implies \Delta$ if $v_{\mathcal{I}}(A) = T$ for all formulae $A \in \Gamma$ and $v_{\mathcal{I}}(B) = F$ for all formulae $B \in \Delta$*
- *We say that a countermodel for a sequent falsifies the sequent.*

Reminer: Semantics for Sequents

Definition 1.2 (Valid sequent).

A sequent $\Gamma \implies \Delta$ *is valid if all interpretations that satisfy all formulae in* Γ *satisfy at least one formula in* Δ.

Definition 1.3 (Countermodel/falsifiable sequent).

- *An interpretation* $\mathcal{I}$ *is a countermodel for the sequent* $\Gamma \implies \Delta$ *if* $v_{\mathcal{I}}(A) = T$ *for all formulae* $A \in \Gamma$ *and* $v_{\mathcal{I}}(B) = F$ *for all formulae* $B \in \Delta$
- *We say that a countermodel for a sequent falsifies the sequent.*
- *A sequent is falsifiable if it has a countermodel.*

Syntax vs. Semantics for Quantifiers

- Soundness and Completeness give the connection between

Syntax vs. Semantics for Quantifiers

- Soundness and Completeness give the connection between
  - syntax (= calculus)

Syntax vs. Semantics for Quantifiers

- Soundness and Completeness give the connection between
  - syntax (= calculus)
  - semantics ($\mathcal{I} \models \varphi$)

Syntax vs. Semantics for Quantifiers

- Soundness and Completeness give the connection between
  - syntax (= calculus)
  - semantics ($\mathcal{I} \models \varphi$)
- Quantifier rules use substitutions

Syntax vs. Semantics for Quantifiers

- Soundness and Completeness give the connection between
  - syntax (= calculus)
  - semantics ($\mathcal{I} \models \varphi$)
- Quantifier rules use substitutions
- The semantics of quantifiers use variable assignments

Syntax vs. Semantics for Quantifiers

- Soundness and Completeness give the connection between
  - syntax (= calculus)
  - semantics ($\mathcal{I} \models \varphi$)
- Quantifier rules use substitutions
- The semantics of quantifiers use variable assignments
- We therefore need a connection between

Syntax vs. Semantics for Quantifiers

- Soundness and Completeness give the connection between
  - syntax (= calculus)
  - semantics ($\mathcal{I} \models \varphi$)
- Quantifier rules use substitutions
- The semantics of quantifiers use variable assignments
- We therefore need a connection between
  - substitutions (= syntactic operations)

Syntax vs. Semantics for Quantifiers

- Soundness and Completeness give the connection between
  - syntax (= calculus)
  - semantics ($\mathcal{I} \models \varphi$)
- Quantifier rules use substitutions
- The semantics of quantifiers use variable assignments
- We therefore need a connection between
  - substitutions (= syntactic operations)
  - variable assignments (= semantic objects)

Syntax vs. Semantics for Quantifiers

- Soundness and Completeness give the connection between
  - syntax (= calculus)
  - semantics ($\mathcal{I} \models \varphi$)
- Quantifier rules use substitutions
- The semantics of quantifiers use variable assignments
- We therefore need a connection between
  - substitutions (= syntactic operations)
  - variable assignments (= semantic objects)
- This connection is given by the substitution emma

Reminder: Substitution Lemma

Theorem 1.2 (Substitution Lemma for Formulae).

Given an interpretation $\mathcal{I} = (D, \iota)$ and a variable assignment α for $\mathcal{I}$. Given also a variable $y \in \mathcal{V}$, a formula A and a term $s \in \mathcal{T}$, such that $\{y \backslash s\}$ is capture-free for A.

$$v_{\mathcal{I}}(\alpha, A[y \backslash s]) = v_{\mathcal{I}}(\alpha\{y \leftarrow v_{\mathcal{I}}(\alpha, s)\}, A)$$

Reminder: Substitution Lemma

Theorem 1.2 (Substitution Lemma for Formulae).

Given an interpretation $\mathcal{I} = (D, \iota)$ and a variable assignment α for $\mathcal{I}$. Given also a variable $y \in \mathcal{V}$, a formula A and a term $s \in \mathcal{T}$, such that $\{y \backslash s\}$ is capture-free for A.

$$v_{\mathcal{I}}(\alpha, A[y \backslash s]) = v_{\mathcal{I}}(\alpha\{y \leftarrow v_{\mathcal{I}}(\alpha, s)\}, A)$$

Definition 1.4 (Capture-free substitution).

A substitution σ is capture-free for a formula A if for every free variable x in A, none of the variables in $\sigma(x)$ is bound in A.

Reminder: Substitution Lemma

Theorem 1.2 (Substitution Lemma for Formulae).

Given an interpretation $\mathcal{I} = (D, \iota)$ and a variable assignment α for $\mathcal{I}$. Given also a variable $y \in \mathcal{V}$, a formula A and a term $s \in \mathcal{T}$, such that $\{y \backslash s\}$ is capture-free for A.

$$v_{\mathcal{I}}(\alpha, A[y \backslash s]) = v_{\mathcal{I}}(\alpha\{y \leftarrow v_{\mathcal{I}}(\alpha, s)\}, A)$$

Definition 1.4 (Capture-free substitution).

A substitution σ is capture-free for a formula A if for every free variable x in A, none of the variables in $\sigma(x)$ is bound in A.

Note: if $t \in \mathcal{T}$ is a *closed* term, then $\{y \backslash t\}$ is capture-free for any A.

Outline

How to show the Soundness Theorem?

How to show the Soundness Theorem?

As for propositional logic, we show the following lemmas:

How to show the Soundness Theorem?

As for propositional logic, we show the following lemmas:

1. All LK-rules preserve falsifiability upwards.

How to show the Soundness Theorem?

As for propositional logic, we show the following lemmas:

1. All LK-rules preserve falsifiability upwards.
2. An LK-derivation with a falsifiable root sequent has at least one falsifiable leaf sequent

How to show the Soundness Theorem?

As for propositional logic, we show the following lemmas:

1. All LK-rules preserve falsifiability upwards.
2. An LK-derivation with a falsifiable root sequent has at least one falsifiable leaf sequent
3. All axioms are valid

How to show the Soundness Theorem?

As for propositional logic, we show the following lemmas:

1. All LK-rules preserve falsifiability upwards.
2. An LK-derivation with a falsifiable root sequent has at least one falsifiable leaf sequent
3. All axioms are valid

Finally, we use these lemmas to show the soundness theorem.

How to show the Soundness Theorem?

As for propositional logic, we show the following lemmas:

1. All LK-rules preserve falsifiability upwards.
2. An LK-derivation with a falsifiable root sequent has at least one falsifiable leaf sequent
3. All axioms are valid

Finally, we use these lemmas to show the soundness theorem.

Preservation of Falsifiability

Definition 2.1.

An LK*-rule* θ *preserves falsifiability (upwards) if whenever the conclusion* w *of an instance* $\dfrac{w_1 \cdots w_n}{w}$ *of* θ *is falsifiabile, then also at least one of the premises* w_i *is falsifiable*

Preservation of Falsifiability

Definition 2.1.

An LK*-rule* θ *preserves falsifiability (upwards) if whenever the conclusion* w *of an instance* $\frac{w_1 \cdots w_n}{w}$ *of* θ *is falsifiabile, then also at least one of the premises* w_i *is falsifiable*

NEW: the falsifying interpretation for the conclusion does not need to be the same as for the conclusion.

Preservation of Falsifiability

Definition 2.1.

An LK*-rule* θ *preserves falsifiability (upwards) if whenever the conclusion* w *of an instance* $\dfrac{w_1 \cdots w_n}{w}$ *of* θ *is falsifiabile, then also at least one of the premises* w_i *is falsifiable*

NEW: the falsifying interpretation for the conclusion does not need to be the same as for the conclusion.

Lemma 2.1.

All LK*-rules preserve falsifiability.*

- We have shown that the rules for propositional connectives ($\wedge$, $\vee$, $\rightarrow$, $\neg$) have this property.
- It remains to show that also the $\forall$ and $\exists$ rules preserve falsifiability.

Proof: ∀-left preserves falsifiability

Proof: ∀-left preserves falsifiability

$$\frac{\Gamma, \forall x\, A, A[x \backslash t] \implies \Delta}{\Gamma, \forall x\, A \implies \Delta}\ \forall\text{-left}$$

t is a closed term

Proof: ∀-left preserves falsifiability

$$\frac{\Gamma, \forall x\, A, A[x \backslash t] \implies \Delta}{\Gamma, \forall x\, A \implies \Delta}\ \forall\text{-left} \qquad t \text{ is a closed term}$$

- Assume that $\mathcal{I} = (D, \iota)$ falsifies the conclusion $\Gamma, \forall x\, A \implies \Delta$.

Proof: ∀-left preserves falsifiability

$$\frac{\Gamma, \forall x\, A, A[x \backslash t] \implies \Delta}{\Gamma, \forall x\, A \implies \Delta}\ \forall\text{-left} \qquad t \text{ is a closed term}$$

- Assume that $\mathcal{I} = (D, \iota)$ falsifies the conclusion $\Gamma, \forall x\, A \implies \Delta$.
- $\mathcal{I}$ makes all formulae in $\Gamma \cup \{\forall x A\}$ true and all formulae in Δ false.

Proof: ∀-left preserves falsifiability

$$\frac{\Gamma, \forall x\, A, A[x \backslash t] \implies \Delta}{\Gamma, \forall x\, A \implies \Delta}\ \forall\text{-left} \qquad t \text{ is a closed term}$$

- Assume that $\mathcal{I} = (D, \iota)$ falsifies the conclusion $\Gamma, \forall x\, A \implies \Delta$.
- $\mathcal{I}$ makes all formulae in $\Gamma \cup \{\forall x A\}$ true and all formulae in Δ false.
- It suffices to show that $\mathcal{I} \models A[x \backslash t]$.

Proof: $\forall$-left preserves falsifiability

$$\frac{\Gamma, \forall x\, A, A[x \backslash t] \implies \Delta}{\Gamma, \forall x\, A \implies \Delta}\ \forall\text{-left} \qquad t \text{ is a closed term}$$

- Assume that $\mathcal{I} = (D, \iota)$ falsifies the conclusion $\Gamma, \forall x\, A \implies \Delta$.
- $\mathcal{I}$ makes all formulae in $\Gamma \cup \{\forall x A\}$ true and all formulae in Δ false.
- It suffices to show that $\mathcal{I} \models A[x \backslash t]$. Then, the premiss is falsified by $\mathcal{I}$.

Proof: ∀-left preserves falsifiability

$$\frac{\Gamma, \forall x\, A, A[x \backslash t] \implies \Delta}{\Gamma, \forall x\, A \implies \Delta}\ \forall\text{-left} \qquad t \text{ is a closed term}$$

- Assume that $\mathcal{I} = (D, \iota)$ falsifies the conclusion $\Gamma, \forall x\, A \implies \Delta$.
- $\mathcal{I}$ makes all formulae in $\Gamma \cup \{\forall x A\}$ true and all formulae in Δ false.
- It suffices to show that $\mathcal{I} \models A[x \backslash t]$. Then, the premiss is falsified by $\mathcal{I}$.
- Since $\mathcal{I} \models \forall x\, A$, we know that $v_{\mathcal{I}}(\alpha\{x \leftarrow d\}, A) = T$ for all $d \in D$ and any α.

Proof: $\forall$-left preserves falsifiability

$$\frac{\Gamma, \forall x\, A, A[x \backslash t] \implies \Delta}{\Gamma, \forall x\, A \implies \Delta}\ \forall\text{-left} \qquad t \text{ is a closed term}$$

- Assume that $\mathcal{I} = (D, \iota)$ falsifies the conclusion $\Gamma, \forall x\, A \implies \Delta$.
- $\mathcal{I}$ makes all formulae in $\Gamma \cup \{\forall x A\}$ true and all formulae in Δ false.
- It suffices to show that $\mathcal{I} \models A[x \backslash t]$. Then, the premiss is falsified by $\mathcal{I}$.
- Since $\mathcal{I} \models \forall x\, A$, we know that $v_{\mathcal{I}}(\alpha\{x \leftarrow d\}, A) = T$ for all $d \in D$ and any α. (Using the semantics of $\forall$)

Proof: $\forall$-left preserves falsifiability

$$\frac{\Gamma, \forall x\, A, A[x \backslash t] \implies \Delta}{\Gamma, \forall x\, A \implies \Delta}\ \forall\text{-left} \qquad t \text{ is a closed term}$$

- Assume that $\mathcal{I} = (D, \iota)$ falsifies the conclusion $\Gamma, \forall x\, A \implies \Delta$.
- $\mathcal{I}$ makes all formulae in $\Gamma \cup \{\forall x A\}$ true and all formulae in Δ false.
- It suffices to show that $\mathcal{I} \models A[x \backslash t]$. Then, the premiss is falsified by $\mathcal{I}$.
- Since $\mathcal{I} \models \forall x\, A$, we know that $v_{\mathcal{I}}(\alpha\{x \leftarrow d\}, A) = T$ for all $d \in D$ and any α. (Using the semantics of $\forall$)
- In particular, $v_{\mathcal{I}}(\alpha\{x \leftarrow v_{\mathcal{I}}(\alpha, t)\}, A) = T$

Proof: $\forall$-left preserves falsifiability

$$\frac{\Gamma, \forall x\, A, A[x\backslash t] \implies \Delta}{\Gamma, \forall x\, A \implies \Delta}\ \forall\text{-left} \qquad t \text{ is a closed term}$$

- Assume that $\mathcal{I} = (D, \iota)$ falsifies the conclusion $\Gamma, \forall x\, A \implies \Delta$.
- $\mathcal{I}$ makes all formulae in $\Gamma \cup \{\forall x A\}$ true and all formulae in Δ false.
- It suffices to show that $\mathcal{I} \models A[x\backslash t]$. Then, the premiss is falsified by $\mathcal{I}$.
- Since $\mathcal{I} \models \forall x\, A$, we know that $v_{\mathcal{I}}(\alpha\{x \leftarrow d\}, A) = T$ for all $d \in D$ and any α. (Using the semantics of $\forall$)
- In particular, $v_{\mathcal{I}}(\alpha\{x \leftarrow v_{\mathcal{I}}(\alpha, t)\}, A) = T$
- By the substitution lemma: $v_{\mathcal{I}}(\alpha, A[x\backslash t]) = T$

Proof: $\forall$-left preserves falsifiability

$$\frac{\Gamma, \forall x\, A, A[x\backslash t] \implies \Delta}{\Gamma, \forall x\, A \implies \Delta}\ \forall\text{-left} \qquad t \text{ is a closed term}$$

- Assume that $\mathcal{I} = (D, \iota)$ falsifies the conclusion $\Gamma, \forall x\, A \implies \Delta$.
- $\mathcal{I}$ makes all formulae in $\Gamma \cup \{\forall x A\}$ true and all formulae in Δ false.
- It suffices to show that $\mathcal{I} \models A[x\backslash t]$. Then, the premiss is falsified by $\mathcal{I}$.
- Since $\mathcal{I} \models \forall x\, A$, we know that $v_{\mathcal{I}}(\alpha\{x \leftarrow d\}, A) = T$ for all $d \in D$ and any α. (Using the semantics of $\forall$)
- In particular, $v_{\mathcal{I}}(\alpha\{x \leftarrow v_{\mathcal{I}}(\alpha, t)\}, A) = T$
- By the substitution lemma: $v_{\mathcal{I}}(\alpha, A[x\backslash t]) = T$
- And therefore: $\mathcal{I} \models A[x\backslash t]$.

Proof: ∃-left preserves falsifiability

Proof: ∃-left preserves falsifiability

$$\frac{\Gamma, A[x\backslash a] \implies \Delta}{\Gamma, \exists x\, A \implies \Delta}\ \exists\text{-left}$$

a is a constant that does not occur in the conculsion

Proof: ∃-left preserves falsifiability

$$\frac{\Gamma, A[x \backslash a] \implies \Delta}{\Gamma, \exists x\, A \implies \Delta}\ \exists\text{-left}$$

a is a constant that does not occur in the conculsion

- Assume that $\mathcal{I} = (D, \iota)$ falsifies the conclusion $\Gamma, \exists x\, A \implies \Delta$.

Proof: ∃-left preserves falsifiability

$$\frac{\Gamma, A[x\backslash a] \implies \Delta}{\Gamma, \exists x\, A \implies \Delta}\ \exists\text{-left}$$

a is a constant that does not occur in the conculsion

- Assume that $\mathcal{I} = (D, \iota)$ falsifies the conclusion $\Gamma, \exists x\, A \implies \Delta$.
- $\mathcal{I}$ makes all formulae in $\Gamma \cup \{\exists x A\}$ true and alle formulae in Δ false.

Proof: ∃-left preserves falsifiability

$$\frac{\Gamma, A[x\backslash a] \implies \Delta}{\Gamma, \exists x\, A \implies \Delta}\ \exists\text{-left}$$

a is a constant that does not occur in the conculsion

- Assume that $\mathcal{I} = (D, \iota)$ falsifies the conclusion $\Gamma, \exists x\, A \implies \Delta$.
- $\mathcal{I}$ makes all formulae in $\Gamma \cup \{\exists x A\}$ true and alle formulae in Δ false.
- We have to find an interpretation that falsifies the premisse.

Proof: ∃-left preserves falsifiability

$$\frac{\Gamma, A[x\backslash a] \implies \Delta}{\Gamma, \exists x\, A \implies \Delta}\ \exists\text{-left}$$

a is a constant that does not occur in the conculsion

- Assume that $\mathcal{I} = (D, \iota)$ falsifies the conclusion $\Gamma, \exists x\, A \implies \Delta$.
- $\mathcal{I}$ makes all formulae in $\Gamma \cup \{\exists x A\}$ true and alle formulae in Δ false.
- We have to find an interpretation that falsifies the premisse.
- But we can not simply assume that $\mathcal{I} \models A[x\backslash a]$.

Proof: ∃-left preserves falsifiability

$$\frac{\Gamma, A[x \backslash a] \implies \Delta}{\Gamma, \exists x\, A \implies \Delta} \;\exists\text{-left}$$

a is a constant that does not occur in the conculsion

- Assume that $\mathcal{I} = (D, \iota)$ falsifies the conclusion $\Gamma, \exists x\, A \implies \Delta$.
- $\mathcal{I}$ makes all formulae in $\Gamma \cup \{\exists x A\}$ true and alle formulae in Δ false.
- We have to find an interpretation that falsifies the premisse.
- But we can not simply assume that $\mathcal{I} \models A[x \backslash a]$.
- Since $\mathcal{I} \models \exists x\, A$ we know that $v_{\mathcal{I}}(\alpha\{x \leftarrow d\}, A) = T$ for some $d \in D$.

Proof: ∃-left preserves falsifiability

$$\frac{\Gamma, A[x\backslash a] \implies \Delta}{\Gamma, \exists x\, A \implies \Delta}\ \exists\text{-left}$$

a is a constant that does not occur in the conculsion

- Assume that $\mathcal{I} = (D, \iota)$ falsifies the conclusion $\Gamma, \exists x\, A \implies \Delta$.
- $\mathcal{I}$ makes all formulae in $\Gamma \cup \{\exists x A\}$ true and alle formulae in Δ false.
- We have to find an interpretation that falsifies the premisse.
- But we can not simply assume that $\mathcal{I} \models A[x\backslash a]$.
- Since $\mathcal{I} \models \exists x\, A$ we know that $v_{\mathcal{I}}(\alpha\{x{\leftarrow}d\}, A) = T$ for some $d \in D$.
- Based on $\mathcal{I}$ and d, we define a new model $\mathcal{I}'$ as follows:

Proof: ∃-left preserves falsifiability

$$\frac{\Gamma, A[x\backslash a] \implies \Delta}{\Gamma, \exists x\, A \implies \Delta}\ \exists\text{-left}$$

a is a constant that does not occur in the conculsion

- Assume that $\mathcal{I} = (D, \iota)$ falsifies the conclusion $\Gamma, \exists x\, A \implies \Delta$.
- $\mathcal{I}$ makes all formulae in $\Gamma \cup \{\exists x A\}$ true and alle formulae in Δ false.
- We have to find an interpretation that falsifies the premisse.
- But we can not simply assume that $\mathcal{I} \models A[x\backslash a]$.
- Since $\mathcal{I} \models \exists x\, A$ we know that $v_{\mathcal{I}}(\alpha\{x{\leftarrow}d\}, A) = T$ for some $d \in D$.
- Based on $\mathcal{I}$ and d, we define a new model $\mathcal{I}'$ as follows:
  - $\mathcal{I}' = (D, \iota')$ is identical to $\mathcal{I}$ except for the interpretation of the constant a.

Proof: ∃-left preserves falsifiability

$$\frac{\Gamma, A[x\backslash a] \implies \Delta}{\Gamma, \exists x\, A \implies \Delta}\ \exists\text{-left}$$

a is a constant that does not occur in the conculsion

- Assume that $\mathcal{I} = (D, \iota)$ falsifies the conclusion $\Gamma, \exists x\, A \implies \Delta$.
- $\mathcal{I}$ makes all formulae in $\Gamma \cup \{\exists x A\}$ true and alle formulae in Δ false.
- We have to find an interpretation that falsifies the premisse.
- But we can not simply assume that $\mathcal{I} \models A[x\backslash a]$.
- Since $\mathcal{I} \models \exists x\, A$ we know that $v_{\mathcal{I}}(\alpha\{x \leftarrow d\}, A) = T$ for some $d \in D$.
- Based on $\mathcal{I}$ and d, we define a new model $\mathcal{I}'$ as follows:
  - $\mathcal{I}' = (D, \iota')$ is identical to $\mathcal{I}$ except for the interpretation of the constant a.
  - a is interpreted as d, ie. $a^{\iota'} = d$.

Proof: ∃-left preserves falsifiability

$$\frac{\Gamma, A[x\backslash a] \implies \Delta}{\Gamma, \exists x\, A \implies \Delta}\ \exists\text{-left}$$

a is a constant that does not occur in the conculsion

- Assume that $\mathcal{I} = (D, \iota)$ falsifies the conclusion $\Gamma, \exists x\, A \implies \Delta$.
- $\mathcal{I}$ makes all formulae in $\Gamma \cup \{\exists x A\}$ true and alle formulae in Δ false.
- We have to find an interpretation that falsifies the premisse.
- But we can not simply assume that $\mathcal{I} \models A[x\backslash a]$.
- Since $\mathcal{I} \models \exists x\, A$ we know that $v_{\mathcal{I}}(\alpha\{x \leftarrow d\}, A) = T$ for some $d \in D$.
- Based on $\mathcal{I}$ and d, we define a new model $\mathcal{I}'$ as follows:
  - $\mathcal{I}' = (D, \iota')$ is identical to $\mathcal{I}$ except for the interpretation of the constant a.
  - a is interpreted as d, ie. $a^{\iota'} = d$.
- Then $\mathcal{I}'$ falsifies the premisse:

Proof: $\exists$-left preserves falsifiability

$$\frac{\Gamma, A[x\backslash a] \implies \Delta}{\Gamma, \exists x\, A \implies \Delta}\ \exists\text{-left}$$

a is a constant that does not occur in the conculsion

- Assume that $\mathcal{I} = (D, \iota)$ falsifies the conclusion $\Gamma, \exists x\, A \implies \Delta$.
- $\mathcal{I}$ makes all formulae in $\Gamma \cup \{\exists x A\}$ true and alle formulae in Δ false.
- We have to find an interpretation that falsifies the premisse.
- But we can not simply assume that $\mathcal{I} \models A[x\backslash a]$.
- Since $\mathcal{I} \models \exists x\, A$ we know that $v_{\mathcal{I}}(\alpha\{x \leftarrow d\}, A) = T$ for some $d \in D$.
- Based on $\mathcal{I}$ and d, we define a new model $\mathcal{I}'$ as follows:
  - $\mathcal{I}' = (D, \iota')$ is identical to $\mathcal{I}$ except for the interpretation of the constant a.
  - a is interpreted as d, ie. $a^{\iota'} = d$.
- Then $\mathcal{I}'$ falsifies the premisse:
  - Since a does not occur in the conclusion, $\mathcal{I}'$ and $\mathcal{I}$ interpret the fml. in Γ og Δ equally.

Proof: ∃-left preserves falsifiability

$$\frac{\Gamma, A[x\backslash a] \implies \Delta}{\Gamma, \exists x\, A \implies \Delta}\ \exists\text{-left}$$

a is a constant that does not occur in the conculsion

- Assume that $\mathcal{I} = (D, \iota)$ falsifies the conclusion $\Gamma, \exists x\, A \implies \Delta$.
- $\mathcal{I}$ makes all formulae in $\Gamma \cup \{\exists x A\}$ true and alle formulae in Δ false.
- We have to find an interpretation that falsifies the premisse.
- But we can not simply assume that $\mathcal{I} \models A[x\backslash a]$.
- Since $\mathcal{I} \models \exists x\, A$ we know that $v_{\mathcal{I}}(\alpha\{x \leftarrow d\}, A) = T$ for some $d \in D$.
- Based on $\mathcal{I}$ and d, we define a new model $\mathcal{I}'$ as follows:
  - $\mathcal{I}' = (D, \iota')$ is identical to $\mathcal{I}$ except for the interpretation of the constant a.
  - a is interpreted as d, ie. $a^{\iota'} = d$.
- Then $\mathcal{I}'$ falsifies the premisse:
  - Since a does not occur in the conclusion, $\mathcal{I}'$ and $\mathcal{I}$ interpret the fml. in Γ og Δ equally. $\mathcal{I}'$ makes all fml. in Γ true and all fml. in Δ false.

Proof: ∃-left preserves falsifiability

$$\frac{\Gamma, A[x\backslash a] \implies \Delta}{\Gamma, \exists x\, A \implies \Delta}\ \exists\text{-left}$$

a is a constant that does not occur in the conculsion

- Assume that $\mathcal{I} = (D, \iota)$ falsifies the conclusion $\Gamma, \exists x\, A \implies \Delta$.
- $\mathcal{I}$ makes all formulae in $\Gamma \cup \{\exists x A\}$ true and alle formulae in Δ false.
- We have to find an interpretation that falsifies the premisse.
- But we can not simply assume that $\mathcal{I} \models A[x\backslash a]$.
- Since $\mathcal{I} \models \exists x\, A$ we know that $v_{\mathcal{I}}(\alpha\{x \leftarrow d\}, A) = T$ for some $d \in D$.
- Based on $\mathcal{I}$ and d, we define a new model $\mathcal{I}'$ as follows:
  - $\mathcal{I}' = (D, \iota')$ is identical to $\mathcal{I}$ except for the interpretation of the constant a.
  - a is interpreted as d, ie. $a^{\iota'} = d$.
- Then $\mathcal{I}'$ falsifies the premisse:
  - Since a does not occur in the conclusion, $\mathcal{I}'$ and $\mathcal{I}$ interpret the fml. in Γ og Δ equally. $\mathcal{I}'$ makes all fml. in Γ true and all fml. in Δ false.
  - $v_{\mathcal{I}'}(\alpha\{x \leftarrow d\}, A) = T$, and $d = v_{\mathcal{I}'}(\alpha, a)$, so $\mathcal{I}' \models A[x\backslash a]$.

An Example

An Example

- Assume that $\mathcal{I} = (D, \iota)$ is an interpretation with domain $D = \{1, 2\}$ and $p^{\iota} = \{2\}$.

An Example

- Assume that $\mathcal{I} = (D, \iota)$ is an interpretation with domain $D = \{1, 2\}$ and $p^\iota = \{2\}$.
- Assume that a og b are constants and $a^\iota = b^\iota = 1$.

An Example

- Assume that $\mathcal{I} = (D, \iota)$ is an interpretation with domain $D = \{1, 2\}$ and $p^\iota = \{2\}$.
- Assume that a og b are constants and $a^\iota = b^\iota = 1$.
- Then $\mathcal{I} \not\models p(a)$ og $\mathcal{I} \not\models p(b)$.

An Example

- Assume that $\mathcal{I} = (D, \iota)$ is an interpretation with domain $D = \{1, 2\}$ and $p^\iota = \{2\}$.
- Assume that a og b are constants and $a^\iota = b^\iota = 1$.
- Then $\mathcal{I} \not\models p(a)$ og $\mathcal{I} \not\models p(b)$.

$$\frac{p(b) \implies p(a)}{\exists x\, p(x) \implies p(a)}\ \exists\text{-left}$$

An Example

- Assume that $\mathcal{I} = (D, \iota)$ is an interpretation with domain $D = \{1, 2\}$ and $p^\iota = \{2\}$.
- Assume that a og b are constants and $a^\iota = b^\iota = 1$.
- Then $\mathcal{I} \not\models p(a)$ og $\mathcal{I} \not\models p(b)$.

$$\frac{p(b) \implies p(a)}{\exists x\, p(x) \implies p(a)}\ \exists\text{-left}$$

- $\mathcal{I}$ falsifies the conclusion:

An Example

- Assume that $\mathcal{I} = (D, \iota)$ is an interpretation with domain $D = \{1, 2\}$ and $p^\iota = \{2\}$.
- Assume that a og b are constants and $a^\iota = b^\iota = 1$.
- Then $\mathcal{I} \not\models p(a)$ og $\mathcal{I} \not\models p(b)$.

$$\frac{p(b) \implies p(a)}{\exists x\, p(x) \implies p(a)}\ \exists\text{-left}$$

- $\mathcal{I}$ falsifies the conclusion:

  $\mathcal{I} \models \exists x\, p(x)$, since $v_{\mathcal{I}}(\alpha\{x{\leftarrow}2\}, p(x)) = T$

An Example

- Assume that $\mathcal{I} = (D, \iota)$ is an interpretation with domain $D = \{1, 2\}$ and $p^\iota = \{2\}$.
- Assume that a og b are constants and $a^\iota = b^\iota = 1$.
- Then $\mathcal{I} \not\models p(a)$ og $\mathcal{I} \not\models p(b)$.

$$\frac{p(b) \implies p(a)}{\exists x\, p(x) \implies p(a)}\ \exists\text{-left}$$

- $\mathcal{I}$ falsifies the conclusion:

  $\mathcal{I} \models \exists x\, p(x)$, since $v_{\mathcal{I}}(\alpha\{x \leftarrow 2\}, p(x)) = T$

  $\mathcal{I} \not\models p(a)$.

An Example

- Assume that $\mathcal{I} = (D, \iota)$ is an interpretation with domain $D = \{1, 2\}$ and $p^\iota = \{2\}$.
- Assume that a og b are constants and $a^\iota = b^\iota = 1$.
- Then $\mathcal{I} \not\models p(a)$ og $\mathcal{I} \not\models p(b)$.

$$\frac{p(b) \implies p(a)}{\exists x\, p(x) \implies p(a)}\ \exists\text{-left}$$

- $\mathcal{I}$ falsifies the conclusion:

  $\mathcal{I} \models \exists x\, p(x)$, since $v_{\mathcal{I}}(\alpha\{x \leftarrow 2\}, p(x)) = T$

  $\mathcal{I} \not\models p(a)$.

An Example

- Assume that $\mathcal{I} = (D, \iota)$ is an interpretation with domain $D = \{1, 2\}$ and $p^\iota = \{2\}$.
- Assume that a og b are constants and $a^\iota = b^\iota = 1$.
- Then $\mathcal{I} \not\models p(a)$ og $\mathcal{I} \not\models p(b)$.

$$\dfrac{p(b) \implies p(a)}{\exists x\, p(x) \implies p(a)}\ \exists\text{-left}$$

- $\mathcal{I}$ falsifies the conclusion:

  $\mathcal{I} \models \exists x\, p(x)$, since $v_{\mathcal{I}}(\alpha\{x \leftarrow 2\}, p(x)) = T$

  $\mathcal{I} \not\models p(a)$.
- But $\mathcal{I}$ does not falsify the premisse because $\mathcal{I} \not\models p(b)$.

An Example

- Assume that $\mathcal{I} = (D, \iota)$ is an interpretation with domain $D = \{1, 2\}$ and $p^\iota = \{2\}$.
- Assume that a og b are constants and $a^\iota = b^\iota = 1$.
- Then $\mathcal{I} \not\models p(a)$ og $\mathcal{I} \not\models p(b)$.

$$\frac{p(b) \implies p(a)}{\exists x\, p(x) \implies p(a)}\ \exists\text{-left}$$

- $\mathcal{I}$ falsifies the conclusion:

  $\mathcal{I} \models \exists x\, p(x)$, since $v_{\mathcal{I}}(\alpha\{x \leftarrow 2\}, p(x)) = T$

  $\mathcal{I} \not\models p(a)$.
- But $\mathcal{I}$ does not falsify the premisse because $\mathcal{I} \not\models p(b)$.
- We define a new interpretation $\mathcal{I}' = (D, \iota')$ such that $b^{\iota'} = 2$.

An Example

- Assume that $\mathcal{I} = (D, \iota)$ is an interpretation with domain $D = \{1, 2\}$ and $p^\iota = \{2\}$.
- Assume that a og b are constants and $a^\iota = b^\iota = 1$.
- Then $\mathcal{I} \not\models p(a)$ og $\mathcal{I} \not\models p(b)$.

$$\frac{p(b) \implies p(a)}{\exists x\, p(x) \implies p(a)}\ \exists\text{-left}$$

- $\mathcal{I}$ falsifies the conclusion:
  $\mathcal{I} \models \exists x\, p(x)$, since $v_{\mathcal{I}}(\alpha\{x{\leftarrow}2\}, p(x)) = T$
  $\mathcal{I} \not\models p(a)$.
- But $\mathcal{I}$ does not falsify the premisse because $\mathcal{I} \not\models p(b)$.
- We define a new interpretation $\mathcal{I}' = (D, \iota')$ such that $b^{\iota'} = 2$.
- Then $\mathcal{I}'$ falsifies the premisse.

Proof: ∃-right and ∀-right preserve satisfiability

- The proof for ∀-right is dual to that for ∃-left
- The proof for ∃-right is dual to that for ∀-left

How to show the Soundness Theorem?

As for propositional logic, we show the following lemmas:

1. All LK-rules preserve falsifiability upwards.
2. An LK-derivation with a falsifiable root sequent has at least one falsifiable leaf sequent
3. All axioms are valid

Finally, we use these lemmas to show the soundness theorem.

Existence of a falsifiable leaf sequent

Lemma 2.2.

If the root sequent $\mathcal{I}$ of an an LK*-derivation is falsifiable, then at least one of the leaf sequents is satisfiable.*

- As for propositional logic, the proof is by structural induction on the LK-derivation.
- The base case (one sequent $\Gamma \implies \Delta$) is trivial since $\Gamma \implies \Delta$ is both root and leaf sequent.
- Two induction steps, for one-premisse and two-premisse rules
- Both use the lemma that falsifiability is preserved upwards.

Existence of a falsifiable leaf sequent

Lemma 2.2.

If the root sequent $\mathcal{I}$ of an an LK*-derivation is falsifiable, then at least one of the leaf sequents is satisfiable.*

- As for propositional logic, the proof is by structural induction on the LK-derivation.
- The base case (one sequent $\Gamma \implies \Delta$) is trivial since $\Gamma \implies \Delta$ is both root and leaf sequent.
- Two induction steps, for one-premisse and two-premisse rules
- Both use the lemma that falsifiability is preserved upwards.

Difference from propositional logic: not necessarily the same interpretation!

How to show the Soundness Theorem?

As for propositional logic, we show the following lemmas:

1. All LK-rules preserve falsifiability upwards.
2. An LK-derivation with a falsifiable root sequent has at least one falsifiable leaf sequent
3. All axioms are valid

Finally, we use these lemmas to show the soundness theorem.

All axioms are valid

Lemma 2.3.

All axioms are valid

All axioms are valid

Lemma 2.3.

All axioms are valid

- The proof is the same as for propositional logic

All axioms are valid

Lemma 2.3.

All axioms are valid

- The proof is the same as for propositional logic
- An axiom has the form

$$\Gamma, p(t_1, \ldots, t_n) \implies p(t_1, \ldots, t_n), \Delta$$

All axioms are valid

Lemma 2.3.

All axioms are valid

- The proof is the same as for propositional logic
- An axiom has the form

$$\Gamma, p(t_1, \ldots, t_n) \implies p(t_1, \ldots, t_n), \Delta$$

- Any interpretation that satisfies the antecedent satisfies $p(t_1, \ldots, t_n)$.

All axioms are valid

Lemma 2.3.

All axioms are valid

- The proof is the same as for propositional logic
- An axiom has the form

$$\Gamma, p(t_1, \ldots, t_n) \implies p(t_1, \ldots, t_n), \Delta$$

- Any interpretation that satisfies the antecedent satisfies $p(t_1, \ldots, t_n)$.
- Therefore, the same formula $p(t_1, \ldots, t_n)$ is satisfied in the succedent.

Proof of the Soundness Theorem for LK

Proof of soundness.

- Assume that $\mathcal{P}$ is an LK-proof for the sequent $\Gamma \implies \Delta$.

$\square$

Proof of the Soundness Theorem for LK

Proof of soundness.

- Assume that $\mathcal{P}$ is an LK-proof for the sequent $\Gamma \implies \Delta$.
  - $\mathcal{P}$ is an LK-derivation where every leaf is an axiom.

□

Proof of the Soundness Theorem for LK

Proof of soundness.

- Assume that $\mathcal{P}$ is an LK-proof for the sequent $\Gamma \implies \Delta$.
  - $\mathcal{P}$ is an LK-derivation where every leaf is an axiom.
- For the sake of contradiction, assume that $\Gamma \implies \Delta$ is not valid.

□

Proof of the Soundness Theorem for LK

Proof of soundness.

- Assume that $\mathcal{P}$ is an LK-proof for the sequent $\Gamma \implies \Delta$.
  - $\mathcal{P}$ is an LK-derivation where every leaf is an axiom.
- For the sake of contradiction, assume that $\Gamma \implies \Delta$ is not valid.
- Then there is a countermodel $\mathcal{I}$ that falsifies $\Gamma \implies \Delta$.

□

Proof of the Soundness Theorem for LK

Proof of soundness.

- Assume that $\mathcal{P}$ is an LK-proof for the sequent $\Gamma \implies \Delta$.
  - $\mathcal{P}$ is an LK-derivation where every leaf is an axiom.
- For the sake of contradiction, assume that $\Gamma \implies \Delta$ is not valid.
- Then there is a countermodel $\mathcal{I}$ that falsifies $\Gamma \implies \Delta$.
- We know from the previous Lemma that there is an $\mathcal{I}'$ that falsifies at least one leaf sequent of $\mathcal{P}$.

□

Proof of the Soundness Theorem for LK

Proof of soundness.

- Assume that $\mathcal{P}$ is an LK-proof for the sequent $\Gamma \implies \Delta$.
  - $\mathcal{P}$ is an LK-derivation where every leaf is an axiom.
- For the sake of contradiction, assume that $\Gamma \implies \Delta$ is not valid.
- Then there is a countermodel $\mathcal{I}$ that falsifies $\Gamma \implies \Delta$.
- We know from the previous Lemma that there is an $\mathcal{I}'$ that falsifies at least one leaf sequent of $\mathcal{P}$.
- Then $\mathcal{P}$ has a leaf sequent that is not an axiom, since axioms are not falsifiable.

□

Proof of the Soundness Theorem for LK

Proof of soundness.

- Assume that $\mathcal{P}$ is an LK-proof for the sequent $\Gamma \implies \Delta$.
  - $\mathcal{P}$ is an LK-derivation where every leaf is an axiom.
- For the sake of contradiction, assume that $\Gamma \implies \Delta$ is not valid.
- Then there is a countermodel $\mathcal{I}$ that falsifies $\Gamma \implies \Delta$.
- We know from the previous Lemma that there is an $\mathcal{I}'$ that falsifies at least one leaf sequent of $\mathcal{P}$.
- Then $\mathcal{P}$ has a leaf sequent that is not an axiom, since axioms are not falsifiable.
- So $\mathcal{P}$ cannot be an LK-proof.

□

Outline

- Preliminaries and Reminders
- Soundness Proof
- Completeness: Preliminaries
- Proof of Completeness
- Examples of Counter-model Construction

Herbrand Universe

Definition 3.1 (Herbrand universe).

Herbrand Universe

Definition 3.1 (Herbrand universe).

Let T be a set of terms.

Herbrand Universe

Definition 3.1 (Herbrand universe).

Let T be a set of terms. Then $\mathcal{H}(T)$, the Herbrand universe of T, is the smallest set such that

Herbrand Universe

Definition 3.1 (Herbrand universe).

Let T be a set of terms. Then $\mathcal{H}(T)$, the Herbrand universe of T, is the smallest set such that

- *$\mathcal{H}(T)$ contains all constant symbols from T.*

Herbrand Universe

Definition 3.1 (Herbrand universe).

Let T be a set of terms. Then $\mathcal{H}(T)$, the Herbrand universe of T, is the smallest set such that

- *$\mathcal{H}(T)$ contains all constant symbols from T. If there are no constants in T, we include some constant symbol o from $\mathcal{A}$ (called a dummy constant) in $\mathcal{H}(T)$.*

Herbrand Universe

Definition 3.1 (Herbrand universe).

Let T be a set of terms. Then $\mathcal{H}(T)$, the Herbrand universe of T, is the smallest set such that

- *$\mathcal{H}(T)$ contains all constant symbols from T. If there are no constants in T, we include some constant symbol o from $\mathcal{A}$ (called a dummy constant) in $\mathcal{H}(T)$.*
- *If f is a function symbol in T, with arity n and $t_1, \ldots, t_n$ are terms in $\mathcal{H}(T)$, then $f(t_1, \ldots, t_n) \in \mathcal{H}(T)$.*

Herbrand Universe

Definition 3.1 (Herbrand universe).

Let T be a set of terms. Then $\mathcal{H}(T)$, the Herbrand universe of T, is the smallest set such that

- *$\mathcal{H}(T)$ contains all constant symbols from T. If there are no constants in T, we include some constant symbol o from $\mathcal{A}$ (called a dummy constant) in $\mathcal{H}(T)$.*
- *If f is a function symbol in T, with arity n and $t_1, \ldots, t_n$ are terms in $\mathcal{H}(T)$, then $f(t_1, \ldots, t_n) \in \mathcal{H}(T)$.*

The Herbrand universe of a set of formulae is the Herbrand universe of the set of terms occuring in the formulae.

Herbrand Universe

Definition 3.1 (Herbrand universe).

Let T be a set of terms. Then $\mathcal{H}(T)$, the Herbrand universe of T, is the smallest set such that

- *$\mathcal{H}(T)$ contains all constant symbols from T. If there are no constants in T, we include some constant symbol o from $\mathcal{A}$ (called a dummy constant) in $\mathcal{H}(T)$.*
- *If f is a function symbol in T, with arity n and $t_1, \ldots, t_n$ are terms in $\mathcal{H}(T)$, then $f(t_1, \ldots, t_n) \in \mathcal{H}(T)$.*

The Herbrand universe of a set of formulae is the Herbrand universe of the set of terms occuring in the formulae. The Herbrand universe of a branch of a derivation is the Herbrand universe of the set of formulae occurring on that branch.

Herbrand Universe

Definition 3.1 (Herbrand universe).

Let T be a set of terms. Then $\mathcal{H}(T)$, the Herbrand universe of T, is the smallest set such that

- *$\mathcal{H}(T)$ contains all constant symbols from T. If there are no constants in T, we include some constant symbol o from $\mathcal{A}$ (called a dummy constant) in $\mathcal{H}(T)$.*
- *If f is a function symbol in T, with arity n and $t_1, \ldots, t_n$ are terms in $\mathcal{H}(T)$, then $f(t_1, \ldots, t_n) \in \mathcal{H}(T)$.*

The Herbrand universe of a set of formulae is the Herbrand universe of the set of terms occuring in the formulae. The Herbrand universe of a branch of a derivation is the Herbrand universe of the set of formulae occurring on that branch.

- Intuitively, the Herbrand universe of T is the set of all *closed* terms that can be constructed from the constant and function symbols in T.

Herbrand Universe: Examples

Example.

Let $T = \{f(x)\}$.

Herbrand Universe: Examples

Example.

Let $T = \{f(x)\}$. *Then the Herbrand universe of* T *is*

Herbrand Universe: Examples

Example.

Let $T = \{f(x)\}$*. Then the Herbrand universe of* T *is the set*

$$\{o, f(o), f(f(o)), f(f(f(o))), \ldots\}$$

Herbrand Universe: Examples

Example.

Let $T = \{f(x)\}$. *Then the Herbrand universe of* T *is the set*

$$\{o, f(o), f(f(o)), f(f(f(o))), \ldots\}$$

Example.

Let $T = \{a, f(x)\}$.

Herbrand Universe: Examples

Example.

Let $T = \{f(x)\}$. *Then the Herbrand universe of* T *is the set*

$$\{o, f(o), f(f(o)), f(f(f(o))), \ldots\}$$

Example.

Let $T = \{a, f(x)\}$. *Then the Herbrand universe of* T *is*

Herbrand Universe: Examples

Example.

Let $T = \{f(x)\}$. *Then the Herbrand universe of* T *is the set*

$$\{o, f(o), f(f(o)), f(f(f(o))), \ldots\}$$

Example.

Let $T = \{a, f(x)\}$. *Then the Herbrand universe of* T *is the set*

$$\{a, f(a), f(f(a)), f(f(f(a))), \ldots\}$$

Herbrand Universe: Examples

Example.

Let $T = \{f(x)\}$*. Then the Herbrand universe of* T *is the set*

$$\{o, f(o), f(f(o)), f(f(f(o))), \ldots\}$$

Example.

Let $T = \{a, f(x)\}$*. Then the Herbrand universe of* T *is the set*

$$\{a, f(a), f(f(a)), f(f(f(a))), \ldots\}$$

Example.

Let $F = \{\forall x\, p(f(g(x)))\}$

Herbrand Universe: Examples

Example.

Let $T = \{f(x)\}$. *Then the Herbrand universe of* T *is the set*

$$\{o, f(o), f(f(o)), f(f(f(o))), \ldots\}$$

Example.

Let $T = \{a, f(x)\}$. *Then the Herbrand universe of* T *is the set*

$$\{a, f(a), f(f(a)), f(f(f(a))), \ldots\}$$

Example.

Let $F = \{\forall x\, p(f(g(x)))\}$ *Then the Herbrand universe of* F *is*

Herbrand Universe: Examples

Example.

Let $T = \{f(x)\}$*. Then the Herbrand universe of* T *is the set*

$$\{o, f(o), f(f(o)), f(f(f(o))), \ldots\}$$

Example.

Let $T = \{a, f(x)\}$*. Then the Herbrand universe of* T *is the set*

$$\{a, f(a), f(f(a)), f(f(f(a))), \ldots\}$$

Example.

Let $F = \{\forall x\, p(f(g(x)))\}$ *Then the Herbrand universe of* F *is the set*

$$\{o, f(o), g(o), f(g(o)), g(f(o)), f(f(o)), g(g(o)), \ldots\}$$

Fairness

- To guarantee that a proof is found

Fairness

- To guarantee that a proof is found
  - all formulae have to be used in a rule eventually, and

Fairness

- To guarantee that a proof is found
  - all formulae have to be used in a rule eventually, and
  - all $\forall$-left and $\exists$-right rules are applied with *all terms* eventually.

Fairness

- To guarantee that a proof is found
  - all formulae have to be used in a rule eventually, and
  - all ∀-left and ∃-right rules are applied with *all terms* eventually.
- If we try to guarantee this,

Fairness

- To guarantee that a proof is found
  - all formulae have to be used in a rule eventually, and
  - all ∀-left and ∃-right rules are applied with *all terms* eventually.
- If we try to guarantee this,
  1. Either all branches can be closed, giving a proof,

Fairness

- To guarantee that a proof is found
  - all formulae have to be used in a rule eventually, and
  - all ∀-left and ∃-right rules are applied with *all terms* eventually.
- If we try to guarantee this,
  1. Either all branches can be closed, giving a proof,
  2. or there is an open branch that we can generate a counterexample from.

Fairness

- To guarantee that a proof is found
  - all formulae have to be used in a rule eventually, and
  - all $\forall$-left and $\exists$-right rules are applied with *all terms* eventually.
- If we try to guarantee this,
  1. Either all branches can be closed, giving a proof,
  2. or there is an open branch that we can generate a counterexample from.
- This only makes sense if we include infinite derivations,
  i.e. derivations with infinitely long branches.

Fairness

- To guarantee that a proof is found
  - all formulae have to be used in a rule eventually, and
  - all $\forall$-left and $\exists$-right rules are applied with *all terms* eventually.
- If we try to guarantee this,
  1. Either all branches can be closed, giving a proof,
  2. or there is an open branch that we can generate a counterexample from.
- This only makes sense if we include infinite derivations, i.e. derivations with infinitely long branches.
- We construct a *limit* by either continuing until no more rules can be applied, or continuing to apply rules indefinitely. We call the result of this process a limit derivation.

Fairness

- To guarantee that a proof is found
  - all formulae have to be used in a rule eventually, and
  - all $\forall$-left and $\exists$-right rules are applied with *all terms* eventually.
- If we try to guarantee this,
  1. Either all branches can be closed, giving a proof,
  2. or there is an open branch that we can generate a counterexample from.
- This only makes sense if we include infinite derivations,
  i.e. derivations with infinitely long branches.
- We construct a *limit* by either continuing until no more rules can be applied, or continuing to apply rules indefinitely. We call the result of this process a limit derivation.
- When we talk about limit derivations, we include infinite trees.

Fairness

- To guarantee that a proof is found
  - all formulae have to be used in a rule eventually, and
  - all ∀-left and ∃-right rules are applied with *all terms* eventually.
- If we try to guarantee this,
  1. Either all branches can be closed, giving a proof,
  2. or there is an open branch that we can generate a counterexample from.
- This only makes sense if we include infinite derivations, i.e. derivations with infinitely long branches.
- We construct a *limit* by either continuing until no more rules can be applied, or continuing to apply rules indefinitely. We call the result of this process a limit derivation.
- When we talk about limit derivations, we include infinite trees.
- We won't define these formally.

Fairness

- To guarantee that a proof is found
  - all formulae have to be used in a rule eventually, and
  - all $\forall$-left and $\exists$-right rules are applied with *all terms* eventually.
- If we try to guarantee this,
  1. Either all branches can be closed, giving a proof,
  2. or there is an open branch that we can generate a counterexample from.
- This only makes sense if we include infinite derivations, i.e. derivations with infinitely long branches.
- We construct a *limit* by either continuing until no more rules can be applied, or continuing to apply rules indefinitely. We call the result of this process a limit derivation.
- When we talk about limit derivations, we include infinite trees.
- We won't define these formally.
- If all branches in a derivation can be closed, then the derivation is finite. I.e. proofs are finite.

Fairness

Definition 3.2 (Fair derivations).

Fairness

Definition 3.2 (Fair derivations).

A limit derivation is fair if each open branch has the following properties:

Fairness

Definition 3.2 (Fair derivations).

A limit derivation is fair if each open branch has the following properties:

1. *There are no sequents* $\Gamma, A \implies A, \Delta$ *on the branch that could be closed using the axiom.*

Fairness

Definition 3.2 (Fair derivations).

A limit derivation is fair if each open branch has the following properties:

1. *There are no sequents* $\Gamma, A \implies A, \Delta$ *on the branch that could be closed using the axiom.*
2. *If a* $\wedge$, $\vee$, $\rightarrow$, *or* $\neg$ *formula occurs, then the corresponding* LK *rule is applied to the formula on that branch.*

Fairness

Definition 3.2 (Fair derivations).

A limit derivation is fair if each open branch has the following properties:

1. *There are no sequents* $\Gamma, A \implies A, \Delta$ *on the branch that could be closed using the axiom.*
2. *If a* $\wedge$, $\vee$, $\rightarrow$, *or* $\neg$ *formula occurs, then the corresponding* LK *rule is applied to the formula on that branch.*
3. *If a* $\exists$ *formula occurs in an antecedent, or a* $\forall$ *formula in a succedent, then the* $\exists$*-left, resp.* $\forall$*-right rules are applied to the formula on that branch.*

Fairness

Definition 3.2 (Fair derivations).

A limit derivation is fair if each open branch has the following properties:

1. *There are no sequents $\Gamma, A \implies A, \Delta$ on the branch that could be closed using the axiom.*
2. *If a $\wedge$, $\vee$, $\rightarrow$, or $\neg$ formula occurs, then the corresponding* LK *rule is applied to the formula on that branch.*
3. *If a $\exists$ formula occurs in an antecedent, or a $\forall$ formula in a succedent, then the $\exists$-left, resp. $\forall$-right rules are applied to the formula on that branch.*
4. *If a $\forall$ formula occurs in an antecedent, or a $\exists$ formula in a succedent, then the $\forall$-left, resp. $\exists$-right rules are applied to the formula on that branch for every term t in the Herbrand universe of that branch.*

Königs Lemma

Königs Lemma

Lemma 3.1 (Königs lemma).

Königs Lemma

Lemma 3.1 (Königs lemma).

If T is an infinite tree, but finitely branching (all nodes have finitely many descendants), then T has an infinitely long branch.

Königs Lemma

Lemma 3.1 (Königs lemma).

If T is an infinite tree, but finitely branching (all nodes have finitely many descendants), then T has an infinitely long branch.

Proof.

Königs Lemma

Lemma 3.1 (Königs lemma).

If T is an infinite tree, but finitely branching (all nodes have finitely many descendants), then T has an infinitely long branch.

Proof.

We inductively define an infinitely long branch.

Könings Lemma

Lemma 3.1 (Könings lemma).

If T is an infinite tree, but finitely branching (all nodes have finitely many descendants), then T has an infinitely long branch.

Proof.

We inductively define an infinitely long branch. Let u_0 be the root node of the tree T.

Königs Lemma

Lemma 3.1 (Königs lemma).

If T is an infinite tree, but finitely branching (all nodes have finitely many descendants), then T has an infinitely long branch.

Proof.

We inductively define an infinitely long branch. Let u_0 be the root node of the tree T. Since T is infinite and u_0 has finitely many descendants, one of u_0's descendents must be infinite.

Königs Lemma

Lemma 3.1 (Königs lemma).

If T is an infinite tree, but finitely branching (all nodes have finitely many descendants), then T has an infinitely long branch.

Proof.

We inductively define an infinitely long branch. Let u_0 be the root node of the tree T. Since T is infinite and u_0 has finitely many descendants, one of u_0's descendents must be infinite. (Otherwise T would be finite.)

Königs Lemma

Lemma 3.1 (Königs lemma).

If T is an infinite tree, but finitely branching (all nodes have finitely many descendants), then T has an infinitely long branch.

Proof.

We inductively define an infinitely long branch. Let u_0 be the root node of the tree T. Since T is infinite and u_0 has finitely many descendants, one of u_0's descendents must be infinite. (Otherwise T would be finite.) Let u_1 be the root of such a sub-tree.

Königs Lemma

Lemma 3.1 (Königs lemma).

If T is an infinite tree, but finitely branching (all nodes have finitely many descendants), then T has an infinitely long branch.

Proof.

We inductively define an infinitely long branch. Let u_0 be the root node of the tree T. Since T is infinite and u_0 has finitely many descendants, one of u_0's descendents must be infinite. (Otherwise T would be finite.) Let u_1 be the root of such a sub-tree. If the branch $u_0, u_1, \ldots, u_n$ is defined, we find the next node u_{n+1} by the same kind of reasoning.

Königs Lemma

Lemma 3.1 (Königs lemma).

If T is an infinite tree, but finitely branching (all nodes have finitely many descendants), then T has an infinitely long branch.

Proof.

We inductively define an infinitely long branch. Let u_0 be the root node of the tree T. Since T is infinite and u_0 has finitely many descendants, one of u_0's descendents must be infinite. (Otherwise T would be finite.) Let u_1 be the root of such a sub-tree. If the branch $u_0, u_1, \ldots, u_n$ is defined, we find the next node u_{n+1} by the same kind of reasoning. This process defines an infinitely long branch.

Königs Lemma

Lemma 3.1 (Königs lemma).

If T is an infinite tree, but finitely branching (all nodes have finitely many descendants), then T has an infinitely long branch.

Proof.

We inductively define an infinitely long branch. Let u_0 be the root node of the tree T. Since T is infinite and u_0 has finitely many descendants, one of u_0's descendents must be infinite. (Otherwise T would be finite.) Let u_1 be the root of such a sub-tree. If the branch $u_0, u_1, \ldots, u_n$ is defined, we find the next node u_{n+1} by the same kind of reasoning. This process defines an infinitely long branch. □

Corollary 3.1.

If T is a finitley branching tree, where all branches are finitely long, then T is finite.

Outline

- Preliminaries and Reminders
- Soundness Proof
- Completeness: Preliminaries
- Proof of Completeness
- Examples of Counter-model Construction

Proof of Completeness

Assume $\Gamma \implies \Delta$ is not provable.

Proof of Completeness

Assume $\Gamma \implies \Delta$ is not provable.

- Construct a fair (limit) derivation $\mathcal{D}$ from $\Gamma \implies \Delta$. Possibly infinite.

Proof of Completeness

Assume $\Gamma \implies \Delta$ is not provable.

- Construct a fair (limit) derivation $\mathcal{D}$ from $\Gamma \implies \Delta$. Possibly infinite.
- Then there is (at least) one branch $\mathcal{B}$ that does not end in an axiom.

Proof of Completeness

Assume $\Gamma \implies \Delta$ is not provable.

- Construct a fair (limit) derivation $\mathcal{D}$ from $\Gamma \implies \Delta$. Possibly infinite.
- Then there is (at least) one branch $\mathcal{B}$ that does not end in an axiom.
- We construct an interpretation that falsifies $\Gamma \implies \Delta$. Let

Proof of Completeness

Assume $\Gamma \implies \Delta$ is not provable.

- Construct a fair (limit) derivation $\mathcal{D}$ from $\Gamma \implies \Delta$. Possibly infinite.
- Then there is (at least) one branch $\mathcal{B}$ that does not end in an axiom.
- We construct an interpretation that falsifies $\Gamma \implies \Delta$. Let

$\mathcal{B}^\top$ be the set of formulae that occur in an antecedent on B, and

Proof of Completeness

Assume $\Gamma \implies \Delta$ is not provable.

- Construct a fair (limit) derivation $\mathcal{D}$ from $\Gamma \implies \Delta$. Possibly infinite.
- Then there is (at least) one branch $\mathcal{B}$ that does not end in an axiom.
- We construct an interpretation that falsifies $\Gamma \implies \Delta$. Let

  $\mathcal{B}^\top$ be the set of formulae that occur in an antecedent on B, and

  $\mathcal{B}^\bot$ be the set of formulae that occur in an succedent on $\mathcal{B}$, and

Proof of Completeness

Assume $\Gamma \implies \Delta$ is not provable.

- Construct a fair (limit) derivation $\mathcal{D}$ from $\Gamma \implies \Delta$. Possibly infinite.
- Then there is (at least) one branch $\mathcal{B}$ that does not end in an axiom.
- We construct an interpretation that falsifies $\Gamma \implies \Delta$. Let

  $\mathcal{B}^\top$ be the set of formulae that occur in an antecedent on B, and

  $\mathcal{B}^\perp$ be the set of formulae that occur in an succedent on $\mathcal{B}$, and

  At be the set of *atomic* formulae in $\mathcal{B}^\top$.

Proof of Completeness (Construction of counter-model)

- We construct a counter-model $\mathcal{I} = (D, \iota)$ for $\Gamma \implies \Delta$.

Proof of Completeness (Construction of counter-model)

- We construct a counter-model $\mathcal{I} = (D, \iota)$ for $\Gamma \implies \Delta$.
- Let the domain D be the Herbrand universe of the branch. (I.e. the set of all closed terms that can be generated from the terms on the branch).

Proof of Completeness (Construction of counter-model)

- We construct a counter-model $\mathcal{I} = (D, \iota)$ for $\Gamma \implies \Delta$.
- Let the domain D be the Herbrand universe of the branch. (I.e. the set of all closed terms that can be generated from the terms on the branch).
- Let $a^\iota = a$ for all constant symbols $a \in \mathcal{A}$.

Proof of Completeness (Construction of counter-model)

- We construct a counter-model $\mathcal{I} = (D, \iota)$ for $\Gamma \implies \Delta$.
- Let the domain D be the Herbrand universe of the branch. (I.e. the set of all closed terms that can be generated from the terms on the branch).
- Let $a^\iota = a$ for all constant symbols $a \in \mathcal{A}$.
- If $f \in \mathcal{F}$ is a function symbol with arity n, let $f^\iota(t_1, \ldots, t_n) = f(t_1, \ldots, t_n)$.

Proof of Completeness (Construction of counter-model)

- We construct a counter-model $\mathcal{I} = (D, \iota)$ for $\Gamma \implies \Delta$.
- Let the domain D be the Herbrand universe of the branch. (I.e. the set of all closed terms that can be generated from the terms on the branch).
- Let $a^\iota = a$ for all constant symbols $a \in \mathcal{A}$.
- If $f \in \mathcal{F}$ is a function symbol with arity n, let $f^\iota(t_1, \ldots, t_n) = f(t_1, \ldots, t_n)$.
  - Then $v_\mathcal{I}(t) = t$ for all closed terms t.

Proof of Completeness (Construction of counter-model)

- We construct a counter-model $\mathcal{I} = (D, \iota)$ for $\Gamma \implies \Delta$.
- Let the domain D be the Herbrand universe of the branch. (I.e. the set of all closed terms that can be generated from the terms on the branch).
- Let $a^\iota = a$ for all constant symbols $a \in \mathcal{A}$.
- If $f \in \mathcal{F}$ is a function symbol with arity n, let $f^\iota(t_1, \ldots, t_n) = f(t_1, \ldots, t_n)$.
  - Then $v_\mathcal{I}(t) = t$ for all closed terms t.
  - All terms are interpreted as themselves

Proof of Completeness (Construction of counter-model)

- We construct a counter-model $\mathcal{I} = (D, \iota)$ for $\Gamma \implies \Delta$.
- Let the domain D be the Herbrand universe of the branch. (I.e. the set of all closed terms that can be generated from the terms on the branch).
- Let $a^\iota = a$ for all constant symbols $a \in \mathcal{A}$.
- If $f \in \mathcal{F}$ is a function symbol with arity n, let $f^\iota(t_1, \ldots, t_n) = f(t_1, \ldots, t_n)$.
  - Then $v_\mathcal{I}(t) = t$ for all closed terms t.
  - All terms are interpreted as themselves
- If p is a predicate symbol with arity n, let $\langle t_1, \ldots, t_n \rangle \in p^\iota$ if and only if $p(t_1, \ldots, t_n) \in At$.

Proof of Completeness (Construction of counter-model)

- We construct a counter-model $\mathcal{I} = (D, \iota)$ for $\Gamma \implies \Delta$.
- Let the domain D be the Herbrand universe of the branch. (I.e. the set of all closed terms that can be generated from the terms on the branch).
- Let $a^\iota = a$ for all constant symbols $a \in \mathcal{A}$.
- If $f \in \mathcal{F}$ is a function symbol with arity n, let $f^\iota(t_1, \ldots, t_n) = f(t_1, \ldots, t_n)$.
  - Then $v_\mathcal{I}(t) = t$ for all closed terms t.
  - All terms are interpreted as themselves
- If p is a predicate symbol with arity n, let $\langle t_1, \ldots, t_n \rangle \in p^\iota$ if and only if $p(t_1, \ldots, t_n) \in At$.
- Such an intepretation is often called a Herbrand model or a term model.

Proof of Completeness (Properties of $\mathcal{I}$)

- We show by structural induction on first-order formlae that the interpretation $\mathcal{I}$ makes *all* formlae i $\mathcal{B}^\top$ true and all formulae in $\mathcal{B}^\perp$ false.

Proof of Completeness (Properties of $\mathcal{I}$)

- We show by structural induction on first-order formlae that the interpretation $\mathcal{I}$ makes *all* formlae i $\mathcal{B}^{\top}$ true and all formulae in $\mathcal{B}^{\perp}$ false.
- We show for all first-order formulae A that:

Proof of Completeness (Properties of $\mathcal{I}$)

- We show by structural induction on first-order formlae that the interpretation $\mathcal{I}$ makes *all* formlae i $\mathcal{B}^{\top}$ true and all formulae in $\mathcal{B}^{\perp}$ false.
- We show for all first-order formulae A that:

  If $A \in \mathcal{B}^{\top}$, then $\mathcal{I} \models A$, i.e. $v_{\mathcal{I}}(A) = T$

Proof of Completeness (Properties of $\mathcal{I}$)

- We show by structural induction on first-order formlae that the interpretation $\mathcal{I}$ makes *all* formlae i $\mathcal{B}^\top$ true and all formulae in $\mathcal{B}^\perp$ false.
- We show for all first-order formulae A that:

  If $A \in \mathcal{B}^\top$, then $\mathcal{I} \models A$, i.e. $v_\mathcal{I}(A) = T$
  If $A \in \mathcal{B}^\perp$, then $\mathcal{I} \not\models A$, i.e. $v_\mathcal{I}(A) = F$

Proof of Completeness (Properties of $\mathcal{I}$)

- We show by structural induction on first-order formlae that the interpretation $\mathcal{I}$ makes *all* formlae i $\mathcal{B}^\top$ true and all formulae in $\mathcal{B}^\perp$ false.
- We show for all first-order formulae A that:

  If $A \in \mathcal{B}^\top$, then $\mathcal{I} \models A$, i.e. $v_\mathcal{I}(A) = T$
  If $A \in \mathcal{B}^\perp$, then $\mathcal{I} \not\models A$, i.e. $v_\mathcal{I}(A) = F$

<u>Base case 1:</u> A is an atomic formula $p(t_1, \ldots, t_n)$ in $\mathcal{B}^\top$.

Proof of Completeness (Properties of $\mathcal{I}$)

- We show by structural induction on first-order formlae that the interpretation $\mathcal{I}$ makes *all* formlae i $\mathcal{B}^\top$ true and all formulae in $\mathcal{B}^\perp$ false.
- We show for all first-order formulae A that:

  If $A \in \mathcal{B}^\top$, then $\mathcal{I} \models A$, i.e. $v_\mathcal{I}(A) = T$
  If $A \in \mathcal{B}^\perp$, then $\mathcal{I} \not\models A$, i.e. $v_\mathcal{I}(A) = F$

<u>Base case 1:</u> A is an atomic formula $p(t_1, \ldots, t_n)$ in $\mathcal{B}^\top$.

- Then $p(t_1, \ldots, t_n) \in At$ og $\langle t_1, \ldots, t_n \rangle \in p^\iota$ by construction.

Proof of Completeness (Properties of $\mathcal{I}$)

- We show by structural induction on first-order formlae that the interpretation $\mathcal{I}$ makes *all* formlae i $\mathcal{B}^\top$ true and all formulae in $\mathcal{B}^\perp$ false.
- We show for all first-order formulae A that:

  If $A \in \mathcal{B}^\top$, then $\mathcal{I} \models A$, i.e. $v_\mathcal{I}(A) = T$
  If $A \in \mathcal{B}^\perp$, then $\mathcal{I} \not\models A$, i.e. $v_\mathcal{I}(A) = F$

<u>Base case 1:</u> A is an atomic formula $p(t_1, \ldots, t_n)$ in $\mathcal{B}^\top$.

- Then $p(t_1, \ldots, t_n) \in At$ og $\langle t_1, \ldots, t_n \rangle \in p^\iota$ by construction.
- Therefore $\mathcal{I} \models p(t_1, \ldots, t_n)$.

Proof of Completeness (Properties of $\mathcal{I}$)

- We show by structural induction on first-order formlae that the interpretation $\mathcal{I}$ makes *all* formlae i $\mathcal{B}^\top$ true and all formulae in $\mathcal{B}^\perp$ false.
- We show for all first-order formulae A that:

  If $A \in \mathcal{B}^\top$, then $\mathcal{I} \models A$, i.e. $v_\mathcal{I}(A) = T$
  If $A \in \mathcal{B}^\perp$, then $\mathcal{I} \not\models A$, i.e. $v_\mathcal{I}(A) = F$

Base case 1: A is an atomic formula $p(t_1, \ldots, t_n)$ in $\mathcal{B}^\top$.

- Then $p(t_1, \ldots, t_n) \in At$ og $\langle t_1, \ldots, t_n \rangle \in p^\iota$ by construction.
- Therefore $\mathcal{I} \models p(t_1, \ldots, t_n)$.

Base case 2: A is an atomic formula $p(t_1, \ldots, t_n)$ i $\mathcal{B}^\perp$.

Proof of Completeness (Properties of $\mathcal{I}$)

- We show by structural induction on first-order formlae that the interpretation $\mathcal{I}$ makes *all* formlae i $\mathcal{B}^\top$ true and all formulae in $\mathcal{B}^\perp$ false.
- We show for all first-order formulae A that:

  If $A \in \mathcal{B}^\top$, then $\mathcal{I} \models A$, i.e. $v_\mathcal{I}(A) = T$
  If $A \in \mathcal{B}^\perp$, then $\mathcal{I} \not\models A$, i.e. $v_\mathcal{I}(A) = F$

Base case 1: A is an atomic formula $p(t_1, \dots, t_n)$ in $\mathcal{B}^\top$.

- Then $p(t_1, \dots, t_n) \in At$ og $\langle t_1, \dots, t_n \rangle \in p^\iota$ by construction.
- Therefore $\mathcal{I} \models p(t_1, \dots, t_n)$.

Base case 2: A is an atomic formula $p(t_1, \dots, t_n)$ i $\mathcal{B}^\perp$.

- Since $\mathcal{B}$ does not end in an axiom, and the derivation is fair, $p(t_1, \dots, t_n) \notin At$ and $\langle t_1, \dots, t_n \rangle \notin p^\iota$.

Proof of Completeness (Properties of $\mathcal{I}$)

- We show by structural induction on first-order formlae that the interpretation $\mathcal{I}$ makes *all* formlae i $\mathcal{B}^\top$ true and all formulae in $\mathcal{B}^\bot$ false.
- We show for all first-order formulae A that:

  If $A \in \mathcal{B}^\top$, then $\mathcal{I} \models A$, i.e. $v_\mathcal{I}(A) = T$
  If $A \in \mathcal{B}^\bot$, then $\mathcal{I} \not\models A$, i.e. $v_\mathcal{I}(A) = F$

<u>Base case 1:</u> A is an atomic formula $p(t_1, \ldots, t_n)$ in $\mathcal{B}^\top$.

- Then $p(t_1, \ldots, t_n) \in At$ og $\langle t_1, \ldots, t_n \rangle \in p^\iota$ by construction.
- Therefore $\mathcal{I} \models p(t_1, \ldots, t_n)$.

<u>Base case 2:</u> A is an atomic formula $p(t_1, \ldots, t_n)$ i $\mathcal{B}^\bot$.

- Since $\mathcal{B}$ does not end in an axiom, and the derivation is fair, $p(t_1, \ldots, t_n) \notin At$ and $\langle t_1, \ldots, t_n \rangle \notin p^\iota$.
- Therefore $\mathcal{I} \not\models p(t_1, \ldots, t_n)$.

Proof of Completeness (Propositional connectives)

Induction step: From the assumption (induction hypothesis) that our statement holds for all smaller formulae, we have to show that it holds for $\neg A$, $(A \wedge B)$, $(A \vee B)$, $(A \to B)$, $\forall x\, A$, and $\exists x\, A$.

Proof of Completeness (Propositional connectives)

Induction step: From the assumption (induction hypothesis) that our statement holds for all smaller formulae, we have to show that it holds for $\neg A$, $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$, $\forall x\, A$, and $\exists x\, A$.
Most of this was done in the proof for propositional logic

Proof of Completeness (Propositional connectives)

Induction step: From the assumption (induction hypothesis) that our statement holds for all smaller formulae, we have to show that it holds for $\neg A$, $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$, $\forall x\, A$, and $\exists x\, A$.
Most of this was done in the proof for propositional logic
E.g. assume that $A \wedge B \in \mathcal{B}^{\top}$.

Proof of Completeness (Propositional connectives)

Induction step: From the assumption (induction hypothesis) that our statement holds for all smaller formulae, we have to show that it holds for $\neg A$, $(A \land B)$, $(A \lor B)$, $(A \to B)$, $\forall x\, A$, and $\exists x\, A$.
Most of this was done in the proof for propositional logic
E.g. assume that $A \land B \in \mathcal{B}^{\top}$.

- By fairness of the derivation, the $\land$-left rule has been applied to $A \land B$ on the branch $\mathcal{B}$.

Proof of Completeness (Propositional connectives)

Induction step: From the assumption (induction hypothesis) that our statement holds for all smaller formulae, we have to show that it holds for $\neg A$, $(A \wedge B)$, $(A \vee B)$, $(A \to B)$, $\forall x\, A$, and $\exists x\, A$.
Most of this was done in the proof for propositional logic
E.g. assume that $A \wedge B \in \mathcal{B}^\top$.

- By fairness of the derivation, the $\wedge$-left rule has been applied to $A \wedge B$ on the branch $\mathcal{B}$.
- Then $A \in \mathcal{B}^\top$ and $B \in \mathcal{B}^\top$.

Proof of Completeness (Propositional connectives)

Induction step: From the assumption (induction hypothesis) that our statement holds for all smaller formulae, we have to show that it holds for $\neg A$, $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$, $\forall x\, A$, and $\exists x\, A$.
Most of this was done in the proof for propositional logic
E.g. assume that $A \wedge B \in \mathcal{B}^\top$.

- By fairness of the derivation, the $\wedge$-left rule has been applied to $A \wedge B$ on the branch $\mathcal{B}$.
- Then $A \in \mathcal{B}^\top$ and $B \in \mathcal{B}^\top$.
- By the induction hypothesis, $\mathcal{I} \models A$ and $\mathcal{I} \models B$.

Proof of Completeness (Propositional connectives)

Induction step: From the assumption (induction hypothesis) that our statement holds for all smaller formulae, we have to show that it holds for $\neg A$, $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$, $\forall x\, A$, and $\exists x\, A$.

Most of this was done in the proof for propositional logic

E.g. assume that $A \wedge B \in \mathcal{B}^{\top}$.

- By fairness of the derivation, the $\wedge$-left rule has been applied to $A \wedge B$ on the branch $\mathcal{B}$.
- Then $A \in \mathcal{B}^{\top}$ and $B \in \mathcal{B}^{\top}$.
- By the induction hypothesis, $\mathcal{I} \models A$ and $\mathcal{I} \models B$.
- By model semantics, $\mathcal{I} \models A \wedge B$.

Proof of Completeness (Propositional connectives)

Induction step: From the assumption (induction hypothesis) that our statement holds for all smaller formulae, we have to show that it holds for $\neg A$, $(A \land B)$, $(A \lor B)$, $(A \to B)$, $\forall x\, A$, and $\exists x\, A$.
Most of this was done in the proof for propositional logic
E.g. assume that $A \land B \in \mathcal{B}^\top$.

- By fairness of the derivation, the $\land$-left rule has been applied to $A \land B$ on the branch $\mathcal{B}$.
- Then $A \in \mathcal{B}^\top$ and $B \in \mathcal{B}^\top$.
- By the induction hypothesis, $\mathcal{I} \models A$ and $\mathcal{I} \models B$.
- By model semantics, $\mathcal{I} \models A \land B$.

We only need to cover quantified formulae

Proof of Completeness (∃ in Antecedent)

Assume that $\exists x\, A \in \mathcal{B}^{\top}$.

Proof of Completeness ($\exists$ in Antecedent)

Assume that $\exists x\, A \in \mathcal{B}^{\top}$.

- By fairness of the derivation, $\exists$-left was applied to $\exists x\, A$ on the branch.

Proof of Completeness ($\exists$ in Antecedent)

Assume that $\exists x\, A \in \mathcal{B}^\top$.

- By fairness of the derivation, $\exists$-left was applied to $\exists x\, A$ on the branch.
- Then there is a constant a such that $A[x \backslash a] \in \mathcal{B}^\top$.

Proof of Completeness ($\exists$ in Antecedent)

Assume that $\exists x\, A \in \mathcal{B}^{\top}$.

- By fairness of the derivation, $\exists$-left was applied to $\exists x\, A$ on the branch.
- Then there is a constant a such that $A[x\backslash a] \in \mathcal{B}^{\top}$.
- By the ind. hyp., $\mathcal{I} \models A[x\backslash a]$.

Proof of Completeness ($\exists$ in Antecedent)

Assume that $\exists x\, A \in \mathcal{B}^\top$.

- By fairness of the derivation, $\exists$-left was applied to $\exists x\, A$ on the branch.
- Then there is a constant a such that $A[x\backslash a] \in \mathcal{B}^\top$.
- By the ind. hyp., $\mathcal{I} \models A[x\backslash a]$.
- I.e. $v_{\mathcal{I}}(\alpha, A[x\backslash a]) = T$ for any assignment α, since $A[x\backslash a]$ is closed

Proof of Completeness ($\exists$ in Antecedent)

Assume that $\exists x\, A \in \mathcal{B}^\top$.

- By fairness of the derivation, $\exists$-left was applied to $\exists x\, A$ on the branch.
- Then there is a constant a such that $A[x\backslash a] \in \mathcal{B}^\top$.
- By the ind. hyp., $\mathcal{I} \models A[x\backslash a]$.
- I.e. $v_{\mathcal{I}}(\alpha, A[x\backslash a]) = T$ for any assignment α, since $A[x\backslash a]$ is closed
- By the substitution lemma: $v_{\mathcal{I}}(\alpha\{x \leftarrow a^\iota\}, A) = T$.

Proof of Completeness ($\exists$ in Antecedent)

Assume that $\exists x\, A \in \mathcal{B}^{\top}$.

- By fairness of the derivation, $\exists$-left was applied to $\exists x\, A$ on the branch.
- Then there is a constant a such that $A[x \backslash a] \in \mathcal{B}^{\top}$.
- By the ind. hyp., $\mathcal{I} \models A[x \backslash a]$.
- I.e. $v_{\mathcal{I}}(\alpha, A[x \backslash a]) = T$ for any assignment α, since $A[x \backslash a]$ is closed
- By the substitution lemma: $v_{\mathcal{I}}(\alpha\{x \leftarrow a^{\iota}\}, A) = T$.
- By model semantics: $v_{\mathcal{I}}(\alpha, \exists x\, A) = T$

Proof of Completeness ($\exists$ in Antecedent)

Assume that $\exists x\, A \in \mathcal{B}^\top$.

- By fairness of the derivation, $\exists$-left was applied to $\exists x\, A$ on the branch.
- Then there is a constant a such that $A[x\backslash a] \in \mathcal{B}^\top$.
- By the ind. hyp., $\mathcal{I} \models A[x\backslash a]$.
- I.e. $v_\mathcal{I}(\alpha, A[x\backslash a]) = T$ for any assignment α, since $A[x\backslash a]$ is closed
- By the substitution lemma: $v_\mathcal{I}(\alpha\{x \leftarrow a^\iota\}, A) = T$.
- By model semantics: $v_\mathcal{I}(\alpha, \exists x\, A) = T$
- I.e. $\mathcal{I} \models \exists x\, A$.

Proof of Completeness ($\exists$ in Succedent)

Assume that $\exists x\, A \in \mathcal{B}^{\perp}$.

- We have to show that $\mathcal{I} \not\models \exists x\, A$. Assume that this does not hold.

Proof of Completeness ($\exists$ in Succedent)

Assume that $\exists x\, A \in \mathcal{B}^{\perp}$.

- We have to show that $\mathcal{I} \not\models \exists x\, A$. Assume that this does not hold.
- I.e. $\mathcal{I} \models \exists x\, A$

Proof of Completeness ($\exists$ in Succedent)

Assume that $\exists x\, A \in \mathcal{B}^{\perp}$.

- We have to show that $\mathcal{I} \not\models \exists x\, A$. Assume that this does not hold.
- I.e. $\mathcal{I} \models \exists x\, A$
- Remember that the domain D of $\mathcal{I} = (D, \iota)$ consists of terms

Proof of Completeness ($\exists$ in Succedent)

Assume that $\exists x\, A \in \mathcal{B}^{\perp}$.

- We have to show that $\mathcal{I} \not\models \exists x\, A$. Assume that this does not hold.
- I.e. $\mathcal{I} \models \exists x\, A$
- Remember that the domain D of $\mathcal{I} = (D, \iota)$ consists of terms
- Then $v_{\mathcal{I}}(\alpha\{x \leftarrow t\}, A) = T$ for some term $t \in D$.

Proof of Completeness ($\exists$ in Succedent)

Assume that $\exists x\, A \in \mathcal{B}^{\perp}$.

- We have to show that $\mathcal{I} \not\models \exists x\, A$. Assume that this does not hold.
- I.e. $\mathcal{I} \models \exists x\, A$
- Remember that the domain D of $\mathcal{I} = (D, \iota)$ consists of terms
- Then $v_{\mathcal{I}}(\alpha\{x \leftarrow t\}, A) = T$ for some term $t \in D$.
- By fairness of the derivation, the $\exists$-right rule was applied on $\exists x\, A$ with the term t.

Proof of Completeness (∃ in Succedent)

Assume that $\exists x\, A \in \mathcal{B}^{\perp}$.

- We have to show that $\mathcal{I} \not\models \exists x\, A$. Assume that this does not hold.
- I.e. $\mathcal{I} \models \exists x\, A$
- Remember that the domain D of $\mathcal{I} = (D, \iota)$ consists of terms
- Then $v_{\mathcal{I}}(\alpha\{x \leftarrow t\}, A) = T$ for some term $t \in D$.
- By fairness of the derivation, the ∃-right rule was applied on $\exists x\, A$ with the term t.
- It follows that:

Proof of Completeness ($\exists$ in Succedent)

Assume that $\exists x\, A \in \mathcal{B}^{\perp}$.

- We have to show that $\mathcal{I} \not\models \exists x\, A$. Assume that this does not hold.
- I.e. $\mathcal{I} \models \exists x\, A$
- Remember that the domain D of $\mathcal{I} = (D, \iota)$ consists of terms
- Then $v_{\mathcal{I}}(\alpha\{x{\leftarrow}t\}, A) = T$ for some term $t \in D$.
- By fairness of the derivation, the $\exists$-right rule was applied on $\exists x\, A$ with the term t.
- It follows that:
  - $A[x \backslash t] \in \mathcal{B}^{\perp}$

Proof of Completeness ($\exists$ in Succedent)

Assume that $\exists x\, A \in \mathcal{B}^{\perp}$.

- We have to show that $\mathcal{I} \not\models \exists x\, A$. Assume that this does not hold.
- I.e. $\mathcal{I} \models \exists x\, A$
- Remember that the domain D of $\mathcal{I} = (D, \iota)$ consists of terms
- Then $v_{\mathcal{I}}(\alpha\{x \leftarrow t\}, A) = T$ for some term $t \in D$.
- By fairness of the derivation, the $\exists$-right rule was applied on $\exists x\, A$ with the term t.
- It follows that:
  - $A[x \backslash t] \in \mathcal{B}^{\perp}$
  - $v_{\mathcal{I}}(A[x \backslash t]) = F$ (induction hypothesis)

Proof of Completeness ($\exists$ in Succedent)

Assume that $\exists x\, A \in \mathcal{B}^{\perp}$.

- We have to show that $\mathcal{I} \not\models \exists x\, A$. Assume that this does not hold.
- I.e. $\mathcal{I} \models \exists x\, A$
- Remember that the domain D of $\mathcal{I} = (D, \iota)$ consists of terms
- Then $v_{\mathcal{I}}(\alpha\{x \leftarrow t\}, A) = T$ for some term $t \in D$.
- By fairness of the derivation, the $\exists$-right rule was applied on $\exists x\, A$ with the term t.
- It follows that:
  - $A[x \backslash t] \in \mathcal{B}^{\perp}$
  - $v_{\mathcal{I}}(A[x \backslash t]) = F$ (induction hypothesis)
  - $v_{\mathcal{I}}(\alpha\{x \leftarrow v_{\mathcal{I}}(t)\}, A) = F$ for any α (substitution lemma)

Proof of Completeness ($\exists$ in Succedent)

Assume that $\exists x\, A \in \mathcal{B}^{\perp}$.

- We have to show that $\mathcal{I} \not\models \exists x\, A$. Assume that this does not hold.
- I.e. $\mathcal{I} \models \exists x\, A$
- Remember that the domain D of $\mathcal{I} = (D, \iota)$ consists of terms
- Then $v_{\mathcal{I}}(\alpha\{x{\leftarrow}t\}, A) = T$ for some term $t \in D$.
- By fairness of the derivation, the $\exists$-right rule was applied on $\exists x\, A$ with the term t.
- It follows that:
  - $A[x \backslash t] \in \mathcal{B}^{\perp}$
  - $v_{\mathcal{I}}(A[x \backslash t]) = F$ (induction hypothesis)
  - $v_{\mathcal{I}}(\alpha\{x{\leftarrow}v_{\mathcal{I}}(t)\}, A) = F$ for any α (substitution lemma)
  - $v_{\mathcal{I}}(\alpha\{x{\leftarrow}t\}, A) = F$ (since $v_{\mathcal{I}}(t) = t$)

Proof of Completeness ($\exists$ in Succedent)

Assume that $\exists x\, A \in \mathcal{B}^{\perp}$.

- We have to show that $\mathcal{I} \not\models \exists x\, A$. Assume that this does not hold.
- I.e. $\mathcal{I} \models \exists x\, A$
- Remember that the domain D of $\mathcal{I} = (D, \iota)$ consists of terms
- Then $v_{\mathcal{I}}(\alpha\{x \leftarrow t\}, A) = T$ for some term $t \in D$.
- By fairness of the derivation, the $\exists$-right rule was applied on $\exists x\, A$ with the term t.
- It follows that:
  - $A[x \backslash t] \in \mathcal{B}^{\perp}$
  - $v_{\mathcal{I}}(A[x \backslash t]) = F$ (induction hypothesis)
  - $v_{\mathcal{I}}(\alpha\{x \leftarrow v_{\mathcal{I}}(t)\}, A) = F$ for any α (substitution lemma)
  - $v_{\mathcal{I}}(\alpha\{x \leftarrow t\}, A) = F$ (since $v_{\mathcal{I}}(t) = t$)
- Contradiction!

Proof of Completeness ($\forall$ in Succedent)

Assume that $\forall x\, A \in \mathcal{B}^{\perp}$.

Proof of Completeness ($\forall$ in Succedent)

Assume that $\forall x\, A \in \mathcal{B}^{\perp}$.

- By fairness of the derivation, $\forall$-right was applied to $\exists x\, A$ on the branch.

Proof of Completeness ($\forall$ in Succedent)

Assume that $\forall x\, A \in \mathcal{B}^{\perp}$.

- By fairness of the derivation, $\forall$-right was applied to $\exists x\, A$ on the branch.
- Then there is a constant a such that $A[x \backslash a] \in \mathcal{B}^{\perp}$.

Proof of Completeness ($\forall$ in Succedent)

Assume that $\forall x\, A \in \mathcal{B}^{\perp}$.

- By fairness of the derivation, $\forall$-right was applied to $\exists x\, A$ on the branch.
- Then there is a constant a such that $A[x\backslash a] \in \mathcal{B}^{\perp}$.
- By the ind. hyp., $\mathcal{I} \not\models A[x\backslash a]$.

Proof of Completeness ($\forall$ in Succedent)

Assume that $\forall x\, A \in \mathcal{B}^\perp$.

- By fairness of the derivation, $\forall$-right was applied to $\exists x\, A$ on the branch.
- Then there is a constant a such that $A[x \backslash a] \in \mathcal{B}^\perp$.
- By the ind. hyp., $\mathcal{I} \not\models A[x \backslash a]$.
- I.e. $v_{\mathcal{I}}(\alpha, A[x \backslash a]) = F$ for any assignment α, since $A[x \backslash a]$ is closed

Proof of Completeness (∀ in Succedent)

Assume that $\forall x\, A \in \mathcal{B}^\perp$.

- By fairness of the derivation, ∀-right was applied to $\exists x\, A$ on the branch.
- Then there is a constant a such that $A[x\backslash a] \in \mathcal{B}^\perp$.
- By the ind. hyp., $\mathcal{I} \not\models A[x\backslash a]$.
- I.e. $v_\mathcal{I}(\alpha, A[x\backslash a]) = F$ for any assignment α, since $A[x\backslash a]$ is closed
- By the substitution lemma: $v_\mathcal{I}(\alpha\{x \leftarrow a^\iota\}, A) = F$.

Proof of Completeness (∀ in Succedent)

Assume that $\forall x\, A \in \mathcal{B}^{\perp}$.

- By fairness of the derivation, ∀-right was applied to $\exists x\, A$ on the branch.
- Then there is a constant a such that $A[x\backslash a] \in \mathcal{B}^{\perp}$.
- By the ind. hyp., $\mathcal{I} \not\models A[x\backslash a]$.
- I.e. $v_{\mathcal{I}}(\alpha, A[x\backslash a]) = F$ for any assignment α, since $A[x\backslash a]$ is closed
- By the substitution lemma: $v_{\mathcal{I}}(\alpha\{x \leftarrow a^{\iota}\}, A) = F$.
- By model semantics: $v_{\mathcal{I}}(\alpha, \forall x\, A) = F$

Proof of Completeness ($\forall$ in Succedent)

Assume that $\forall x\, A \in \mathcal{B}^{\perp}$.

- By fairness of the derivation, $\forall$-right was applied to $\exists x\, A$ on the branch.
- Then there is a constant a such that $A[x \backslash a] \in \mathcal{B}^{\perp}$.
- By the ind. hyp., $\mathcal{I} \not\models A[x \backslash a]$.
- I.e. $v_{\mathcal{I}}(\alpha, A[x \backslash a]) = F$ for any assignment α, since $A[x \backslash a]$ is closed
- By the substitution lemma: $v_{\mathcal{I}}(\alpha\{x \leftarrow a^{\iota}\}, A) = F$.
- By model semantics: $v_{\mathcal{I}}(\alpha, \forall x\, A) = F$
- I.e. $\mathcal{I} \not\models \forall x\, A$.

Proof of Completeness ($\forall$ in Antecedent)

Assume that $\forall x\, A \in \mathcal{B}^\top$.

- We have to show that $\mathcal{I} \models \forall x\, A$. Assume that this does not hold.

Proof of Completeness ($\forall$ in Antecedent)

Assume that $\forall x\, A \in \mathcal{B}^{\top}$.

- We have to show that $\mathcal{I} \models \forall x\, A$. Assume that this does not hold.
- I.e. $\mathcal{I} \not\models \forall x\, A$

Proof of Completeness ($\forall$ in Antecedent)

Assume that $\forall x\, A \in \mathcal{B}^\top$.

- We have to show that $\mathcal{I} \models \forall x\, A$. Assume that this does not hold.
- I.e. $\mathcal{I} \not\models \forall x\, A$
- Remember that the domain D of $\mathcal{I} = (D, \iota)$ consists of terms

Proof of Completeness ($\forall$ in Antecedent)

Assume that $\forall x\, A \in \mathcal{B}^\top$.

- We have to show that $\mathcal{I} \models \forall x\, A$. Assume that this does not hold.
- I.e. $\mathcal{I} \not\models \forall x\, A$
- Remember that the domain D of $\mathcal{I} = (D, \iota)$ consists of terms
- Then $v_{\mathcal{I}}(\alpha\{x \leftarrow t\}, A) = F$ for some term $t \in D$.

Proof of Completeness ($\forall$ in Antecedent)

Assume that $\forall x\, A \in \mathcal{B}^\top$.

- We have to show that $\mathcal{I} \models \forall x\, A$. Assume that this does not hold.
- I.e. $\mathcal{I} \not\models \forall x\, A$
- Remember that the domain D of $\mathcal{I} = (D, \iota)$ consists of terms
- Then $v_{\mathcal{I}}(\alpha\{x{\leftarrow}t\}, A) = F$ for some term $t \in D$.
- By fairness of the derivation, the $\forall$-left rule was applied on $\forall x\, A$ with the term t.

Proof of Completeness ($\forall$ in Antecedent)

Assume that $\forall x\, A \in \mathcal{B}^\top$.

- We have to show that $\mathcal{I} \models \forall x\, A$. Assume that this does not hold.
- I.e. $\mathcal{I} \not\models \forall x\, A$
- Remember that the domain D of $\mathcal{I} = (D, \iota)$ consists of terms
- Then $v_{\mathcal{I}}(\alpha\{x \leftarrow t\}, A) = F$ for some term $t \in D$.
- By fairness of the derivation, the $\forall$-left rule was applied on $\forall x\, A$ with the term t.
- It follows that:

Proof of Completeness ($\forall$ in Antecedent)

Assume that $\forall x\, A \in \mathcal{B}^\top$.

- We have to show that $\mathcal{I} \models \forall x\, A$. Assume that this does not hold.
- I.e. $\mathcal{I} \not\models \forall x\, A$
- Remember that the domain D of $\mathcal{I} = (D, \iota)$ consists of terms
- Then $v_{\mathcal{I}}(\alpha\{x \leftarrow t\}, A) = F$ for some term $t \in D$.
- By fairness of the derivation, the $\forall$-left rule was applied on $\forall x\, A$ with the term t.
- It follows that:
  - $A[x \backslash t] \in \mathcal{B}^\top$

Proof of Completeness ($\forall$ in Antecedent)

Assume that $\forall x\, A \in \mathcal{B}^\top$.

- We have to show that $\mathcal{I} \models \forall x\, A$. Assume that this does not hold.
- I.e. $\mathcal{I} \not\models \forall x\, A$
- Remember that the domain D of $\mathcal{I} = (D, \iota)$ consists of terms
- Then $v_\mathcal{I}(\alpha\{x{\leftarrow}t\}, A) = F$ for some term $t \in D$.
- By fairness of the derivation, the $\forall$-left rule was applied on $\forall x\, A$ with the term t.
- It follows that:
  - $A[x\backslash t] \in \mathcal{B}^\top$
  - $v_\mathcal{I}(A[x\backslash t]) = T$ (induction hypothesis)

Proof of Completeness ($\forall$ in Antecedent)

Assume that $\forall x\, A \in \mathcal{B}^\top$.

- We have to show that $\mathcal{I} \models \forall x\, A$. Assume that this does not hold.
- I.e. $\mathcal{I} \not\models \forall x\, A$
- Remember that the domain D of $\mathcal{I} = (D, \iota)$ consists of terms
- Then $v_{\mathcal{I}}(\alpha\{x \leftarrow t\}, A) = F$ for some term $t \in D$.
- By fairness of the derivation, the $\forall$-left rule was applied on $\forall x\, A$ with the term t.
- It follows that:
  - $A[x \backslash t] \in \mathcal{B}^\top$
  - $v_{\mathcal{I}}(A[x \backslash t]) = T$ (induction hypothesis)
  - $v_{\mathcal{I}}(\alpha\{x \leftarrow v_{\mathcal{I}}(t)\}, A) = T$ for any α (substitution lemma)

Proof of Completeness ($\forall$ in Antecedent)

Assume that $\forall x\, A \in \mathcal{B}^\top$.

- We have to show that $\mathcal{I} \models \forall x\, A$. Assume that this does not hold.
- I.e. $\mathcal{I} \not\models \forall x\, A$
- Remember that the domain D of $\mathcal{I} = (D, \iota)$ consists of terms
- Then $v_\mathcal{I}(\alpha\{x \leftarrow t\}, A) = F$ for some term $t \in D$.
- By fairness of the derivation, the $\forall$-left rule was applied on $\forall x\, A$ with the term t.
- It follows that:
  - $A[x\backslash t] \in \mathcal{B}^\top$
  - $v_\mathcal{I}(A[x\backslash t]) = T$ (induction hypothesis)
  - $v_\mathcal{I}(\alpha\{x \leftarrow v_\mathcal{I}(t)\}, A) = T$ for any α (substitution lemma)
  - $v_\mathcal{I}(\alpha\{x \leftarrow t\}, A) = T$ (since $v_\mathcal{I}(t) = t$)

Proof of Completeness ($\forall$ in Antecedent)

Assume that $\forall x\, A \in \mathcal{B}^\top$.

- We have to show that $\mathcal{I} \models \forall x\, A$. Assume that this does not hold.
- I.e. $\mathcal{I} \not\models \forall x\, A$
- Remember that the domain D of $\mathcal{I} = (D, \iota)$ consists of terms
- Then $v_\mathcal{I}(\alpha\{x \leftarrow t\}, A) = F$ for some term $t \in D$.
- By fairness of the derivation, the $\forall$-left rule was applied on $\forall x\, A$ with the term t.
- It follows that:
  - $A[x \backslash t] \in \mathcal{B}^\top$
  - $v_\mathcal{I}(A[x \backslash t]) = T$ (induction hypothesis)
  - $v_\mathcal{I}(\alpha\{x \leftarrow v_\mathcal{I}(t)\}, A) = T$ for any α (substitution lemma)
  - $v_\mathcal{I}(\alpha\{x \leftarrow t\}, A) = T$ (since $v_\mathcal{I}(t) = t$)
- Contradiction!

Some comments

Some comments

- We can see the construction of a limit derivation as approximating a counter-model for $\Gamma \implies \Delta$.

Some comments

- We can see the construction of a limit derivation as approximating a counter-model for $\Gamma \implies \Delta$.
- The more often we apply the $\forall$-left and $\exists$-right rules, the 'closer' we get to a possible counter-model

Some comments

- We can see the construction of a limit derivation as approximating a counter-model for $\Gamma \implies \Delta$.
- The more often we apply the $\forall$-left and $\exists$-right rules, the 'closer' we get to a possible counter-model
- But constructing a counter-model in this way may require using all rules infinitely often.

Some comments

- We can see the construction of a limit derivation as approximating a counter-model for $\Gamma \implies \Delta$.
- The more often we apply the $\forall$-left and $\exists$-right rules, the 'closer' we get to a possible counter-model
- But constructing a counter-model in this way may require using all rules infinitely often.
- So this is not an algorithm for finding counter-models!

Some comments

- We can see the construction of a limit derivation as approximating a counter-model for $\Gamma \implies \Delta$.
- The more often we apply the $\forall$-left and $\exists$-right rules, the 'closer' we get to a possible counter-model
- But constructing a counter-model in this way may require using all rules infinitely often.
- So this is not an algorithm for finding counter-models!
- It will find a proof if one exists, but may not terminate otherwise.

Some comments

- We can see the construction of a limit derivation as approximating a counter-model for $\Gamma \implies \Delta$.
- The more often we apply the $\forall$-left and $\exists$-right rules, the 'closer' we get to a possible counter-model
- But constructing a counter-model in this way may require using all rules infinitely often.
- So this is not an algorithm for finding counter-models!
- It will find a proof if one exists, but may not terminate otherwise.
- There may be finite counter-models even when this method does not terminate. Finding finite counter-models is a topic of active research.

Some comments

- We can see the construction of a limit derivation as approximating a counter-model for $\Gamma \implies \Delta$.
- The more often we apply the $\forall$-left and $\exists$-right rules, the 'closer' we get to a possible counter-model
- But constructing a counter-model in this way may require using all rules infinitely often.
- So this is not an algorithm for finding counter-models!
- It will find a proof if one exists, but may not terminate otherwise.
- There may be finite counter-models even when this method does not terminate. Finding finite counter-models is a topic of active research.
- The idea of the completeness proof is important: we construct an interpretation from something purely syntactic.

Outline

Counter-model Construction, Ex. 1

- Abbreviate px for $p(x)$, qb for $q(b)$, etc.

Counter-model Construction, Ex. 1

$$\underbrace{\forall x\,(px \rightarrow qx)}_{A},\, pa \Rightarrow \forall x\, qx$$

- Abbreviate px for $p(x)$, qb for $q(b)$, etc.

Counter-model Construction, Ex. 1

$$\underbrace{\forall x\,(px \to qx)}_{A}, pa \Rightarrow \forall x\, qx$$

- Abbreviate px for $p(x)$, qb for $q(b)$, etc.

Counter-model Construction, Ex. 1

$$\frac{A, pa \to qa, pa \Rightarrow \forall x\, qx}{\underbrace{\forall x\,(px \to qx)}_{A}, pa \Rightarrow \forall x\, qx}$$

- Abbreviate *px* for $p(x)$, *qb* for $q(b)$, etc.

Counter-model Construction, Ex. 1

$$\dfrac{A, {\color{red}pa \rightarrow qa}, pa \Rightarrow \forall x\, qx}{\underbrace{\forall x\,(px \rightarrow qx)}_{A}, pa \Rightarrow \forall x\, qx}$$

- Abbreviate px for $p(x)$, qb for $q(b)$, etc.

Counter-model Construction, Ex. 1

$$\dfrac{A, pa \Rightarrow \forall x\, qx, pa \qquad\qquad qa, A, pa \Rightarrow \forall x\, qx}{\dfrac{A, pa \rightarrow qa, pa \Rightarrow \forall x\, qx}{\underbrace{\forall x\,(px \rightarrow qx)}_{A}, pa \Rightarrow \forall x\, qx}}$$

- Abbreviate px for $p(x)$, qb for $q(b)$, etc.

Counter-model Construction, Ex. 1

$$\dfrac{\dfrac{A, \color{red}{pa} \Rightarrow \forall x\, qx, \color{red}{pa} \qquad qa, A, pa \Rightarrow \forall x\, qx}{A, pa \to qa, pa \Rightarrow \forall x\, qx}}{\underbrace{\forall x\,(px \to qx)}_{A}, pa \Rightarrow \forall x\, qx}$$

- Abbreviate px for $p(x)$, qb for $q(b)$, etc.

Counter-model Construction, Ex. 1

$$\dfrac{\dfrac{\overline{A, pa \Rightarrow \forall x\, qx, pa} \qquad qa, A, pa \Rightarrow \forall x\, qx}{A, pa \rightarrow qa, pa \Rightarrow \forall x\, qx}}{\underbrace{\forall x\,(px \rightarrow qx)}_{A}, pa \Rightarrow \forall x\, qx}$$

- Abbreviate px for $p(x)$, qb for $q(b)$, etc.

Counter-model Construction, Ex. 1

$$\dfrac{\dfrac{\dfrac{}{A, pa \Rightarrow \forall x\, qx, pa} \qquad qa, A, pa \Rightarrow {\color{red}\forall x\, qx}}{A, pa \rightarrow qa, pa \Rightarrow \forall x\, qx}}{\underbrace{\forall x\,(px \rightarrow qx)}_{A}, pa \Rightarrow \forall x\, qx}$$

- Abbreviate px for $p(x)$, qb for $q(b)$, etc.

Counter-model Construction, Ex. 1

$$
\dfrac{\dfrac{}{A, pa \Rightarrow \forall x\, qx, pa} \qquad \dfrac{qa, A, pa \Rightarrow qb}{qa, A, pa \Rightarrow \forall x\, qx}}{\dfrac{A, pa \to qa, pa \Rightarrow \forall x\, qx}{\underbrace{\forall x\, (px \to qx)}_{A}, pa \Rightarrow \forall x\, qx}}
$$

- Abbreviate px for $p(x)$, qb for $q(b)$, etc.

Counter-model Construction, Ex. 1

$$\dfrac{\dfrac{}{A, pa \Rightarrow \forall x\, qx, pa} \qquad \dfrac{qa, {\color{red}A}, pa \Rightarrow qb}{qa, A, pa \Rightarrow \forall x\, qx}}{\dfrac{A, pa \rightarrow qa, pa \Rightarrow \forall x\, qx}{\underbrace{\forall x\, (px \rightarrow qx)}_{A}, pa \Rightarrow \forall x\, qx}}$$

- Abbreviate px for $p(x)$, qb for $q(b)$, etc.

Counter-model Construction, Ex. 1

$$
\dfrac{\dfrac{\dfrac{}{A, pa \Rightarrow \forall x\, qx, pa} \qquad \dfrac{\dfrac{qa, A, pb \to qb, pa \Rightarrow qb}{qa, A, pa \Rightarrow qb}}{qa, A, pa \Rightarrow \forall x\, qx}}{A, pa \to qa, pa \Rightarrow \forall x\, qx}}{\underbrace{\forall x\,(px \to qx)}_{A}, pa \Rightarrow \forall x\, qx}
$$

- Abbreviate px for $p(x)$, qb for $q(b)$, etc.

Counter-model Construction, Ex. 1

$$\dfrac{\dfrac{}{A, pa \Rightarrow \forall x\, qx, pa} \quad \dfrac{\dfrac{\dfrac{qa, A, \color{red}{pb \to qb}, pa \Rightarrow qb}{qa, A, pa \Rightarrow qb}}{qa, A, pa \Rightarrow \forall x\, qx}}{A, pa \to qa, pa \Rightarrow \forall x\, qx}}{\underbrace{\forall x\,(px \to qx)}_{A}, pa \Rightarrow \forall x\, qx}$$

- Abbreviate px for $p(x)$, qb for $q(b)$, etc.

Counter-model Construction, Ex. 1

$$
\dfrac{\dfrac{}{A, pa \Rightarrow \forall x\, qx, pa} \qquad \dfrac{\dfrac{\dfrac{qa, A, pa \Rightarrow qb, pb \qquad qb, qa, A, pa \Rightarrow qb}{qa, A, pb \rightarrow qb, pa \Rightarrow qb}}{qa, A, pa \Rightarrow qb}}{qa, A, pa \Rightarrow \forall x\, qx}}{\dfrac{A, pa \rightarrow qa, pa \Rightarrow \forall x\, qx}{\underbrace{\forall x\, (px \rightarrow qx)}_{A}, pa \Rightarrow \forall x\, qx}}
$$

- Abbreviate px for $p(x)$, qb for $q(b)$, etc.

Counter-model Construction, Ex. 1

$$\dfrac{\dfrac{}{A, pa \Rightarrow \forall x\, qx, pa} \qquad \dfrac{\dfrac{\dfrac{qa, A, pa \Rightarrow qb, pb \qquad {\color{red}qb}, qa, A, pa \Rightarrow {\color{red}qb}}{qa, A, pb \to qb, pa \Rightarrow qb}}{qa, A, pa \Rightarrow qb}}{qa, A, pa \Rightarrow \forall x\, qx}}{\dfrac{A, pa \to qa, pa \Rightarrow \forall x\, qx}{\underbrace{\forall x\,(px \to qx)}_{A}, pa \Rightarrow \forall x\, qx}}$$

- Abbreviate px for $p(x)$, qb for $q(b)$, etc.

Counter-model Construction, Ex. 1

$$
\dfrac{\dfrac{}{A, pa \Rightarrow \forall x\, qx, pa} \qquad \dfrac{\dfrac{\dfrac{qa, A, pa \Rightarrow qb, pb \qquad \dfrac{}{\color{red}{qb}, qa, A, pa \Rightarrow \color{red}{qb}}}{qa, A, pb \to qb, pa \Rightarrow qb}}{qa, A, pa \Rightarrow qb}}{qa, A, pa \Rightarrow \forall x\, qx}}{\dfrac{A, pa \to qa, pa \Rightarrow \forall x\, qx}{\underbrace{\forall x\, (px \to qx)}_{A}, pa \Rightarrow \forall x\, qx}}
$$

- Abbreviate px for $p(x)$, qb for $q(b)$, etc.

Counter-model Construction, Ex. 1

$$\dfrac{\dfrac{}{A, pa \Rightarrow \forall x\, qx, pa} \qquad \dfrac{\dfrac{\dfrac{\overset{\mathcal{B}}{qa, A, pa \Rightarrow qb, pb} \qquad \dfrac{}{qb, qa, A, pa \Rightarrow qb}}{qa, A, pb \to qb, pa \Rightarrow qb}}{qa, A, pa \Rightarrow qb}}{qa, A, pa \Rightarrow \forall x\, qx}}{\dfrac{A, pa \to qa, pa \Rightarrow \forall x\, qx}{\underbrace{\forall x\,(px \to qx)}_{A}, pa \Rightarrow \forall x\, qx}}$$

- Abbreviate px for $p(x)$, qb for $q(b)$, etc.

Counter-model Construction, Ex. 1

$$
\dfrac{\dfrac{\dfrac{}{A, pa \Rightarrow \forall x\, qx, pa} \qquad \dfrac{\dfrac{\dfrac{\overset{\mathcal{B}}{qa, A, pa \Rightarrow qb, pb} \qquad \dfrac{}{qb, qa, A, pa \Rightarrow qb}}{qa, A, pb \rightarrow qb, pa \Rightarrow qb}}{qa, A, pa \Rightarrow qb}}{qa, A, pa \Rightarrow \forall x\, qx}}{A, pa \rightarrow qa, pa \Rightarrow \forall x\, qx}}{\underbrace{\forall x\,(px \rightarrow qx)}_{A}, pa \Rightarrow \forall x\, qx}
$$

- Abbreviate px for $p(x)$, qb for $q(b)$, etc.
- The Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.

Counter-model Construction, Ex. 1

$$\dfrac{\dfrac{}{A, pa \Rightarrow \forall x\, qx, pa} \qquad \dfrac{\dfrac{\dfrac{\overset{\mathcal{B}}{qa, A, pa \Rightarrow qb, pb} \qquad \dfrac{}{qb, qa, A, pa \Rightarrow qb}}{qa, A, pb \to qb, pa \Rightarrow qb}}{qa, A, pa \Rightarrow qb}}{qa, A, pa \Rightarrow \forall x\, qx}}{\dfrac{A, pa \to qa, pa \Rightarrow \forall x\, qx}{\underbrace{\forall x\,(px \to qx)}_{A}, pa \Rightarrow \forall x\, qx}}$$

- Abbreviate px for $p(x)$, qb for $q(b)$, etc.
- The Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- Since $pa \in \mathcal{B}^\top$, define $a \in p^\iota$, so $\mathcal{I} \models pa$.

Counter-model Construction, Ex. 1

$$
\dfrac{\dfrac{}{A, pa \Rightarrow \forall x\, qx, pa} \qquad \dfrac{\dfrac{\dfrac{\overset{\mathcal{B}}{qa, A, pa \Rightarrow qb, pb} \qquad \dfrac{}{qb, qa, A, pa \Rightarrow qb}}{qa, A, pb \rightarrow qb, pa \Rightarrow qb}}{qa, A, pa \Rightarrow qb}}{qa, A, pa \Rightarrow \forall x\, qx}}{\dfrac{A, pa \rightarrow qa, pa \Rightarrow \forall x\, qx}{\underbrace{\forall x\,(px \rightarrow qx)}_{A}, pa \Rightarrow \forall x\, qx}}
$$

- Abbreviate px for $p(x)$, qb for $q(b)$, etc.
- The Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- Since $pa \in \mathcal{B}^{\top}$, define $a \in p^{\iota}$, so $\mathcal{I} \models pa$.
- Since $qa \in \mathcal{B}^{\top}$, define $a \in q^{\iota}$, so $\mathcal{I} \models qa$ and thus $\mathcal{I} \models pa \rightarrow qa$.

Counter-model Construction, Ex. 1

$$
\dfrac{\dfrac{}{A, pa \Rightarrow \forall x\, qx, pa} \qquad \dfrac{\dfrac{\dfrac{\overset{\mathcal{B}}{qa, A, pa \Rightarrow qb, pb} \qquad \dfrac{}{qb, qa, A, pa \Rightarrow qb}}{qa, A, pb \rightarrow qb, pa \Rightarrow qb}}{qa, A, pa \Rightarrow qb}}{qa, A, pa \Rightarrow \forall x\, qx}}{\dfrac{A, pa \rightarrow qa, pa \Rightarrow \forall x\, qx}{\underbrace{\forall x\,(px \rightarrow qx)}_{A}, pa \Rightarrow \forall x\, qx}}
$$

- Abbreviate px for $p(x)$, qb for $q(b)$, etc.
- The Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- Since $pa \in \mathcal{B}^{\top}$, define $a \in p^{\iota}$, so $\mathcal{I} \models pa$.
- Since $qa \in \mathcal{B}^{\top}$, define $a \in q^{\iota}$, so $\mathcal{I} \models qa$ and thus $\mathcal{I} \models pa \rightarrow qa$.
- Since $qb \in \mathcal{B}^{\perp}$, define $b \notin q^{\iota}$, so $\mathcal{I} \not\models qb$ and thus $\mathcal{I} \not\models \forall x\, qx$.

Counter-model Construction, Ex. 1

$$
\dfrac{\dfrac{}{A, pa \Rightarrow \forall x\, qx, pa} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\overset{\mathcal{B}}{qa, A, pa \Rightarrow qb, pb} \qquad \dfrac{}{qb, qa, A, pa \Rightarrow qb}}{qa, A, pb \rightarrow qb, pa \Rightarrow qb}}{qa, A, pa \Rightarrow qb}}{qa, A, pa \Rightarrow \forall x\, qx}}{\dfrac{A, pa \rightarrow qa, pa \Rightarrow \forall x\, qx}{\underbrace{\forall x\,(px \rightarrow qx)}_{A}, pa \Rightarrow \forall x\, qx}}
$$

- Abbreviate px for $p(x)$, qb for $q(b)$, etc.
- The Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- Since $pa \in \mathcal{B}^{\top}$, define $a \in p^{\iota}$, so $\mathcal{I} \models pa$.
- Since $qa \in \mathcal{B}^{\top}$, define $a \in q^{\iota}$, so $\mathcal{I} \models qa$ and thus $\mathcal{I} \models pa \rightarrow qa$.
- Since $qb \in \mathcal{B}^{\perp}$, define $b \notin q^{\iota}$, so $\mathcal{I} \not\models qb$ and thus $\mathcal{I} \not\models \forall x\, qx$.
- Since $pb \in \mathcal{B}^{\perp}$, define $b \notin p^{\iota}$, so $\mathcal{I} \not\models pb$ and thus $\mathcal{I} \models pb \rightarrow qb$.

Counter-model Construction, Ex. 1

$$\mathcal{B}$$

$$\dfrac{\dfrac{A, pa \Rightarrow \forall x\, qx, pa}{} \qquad \dfrac{\dfrac{\dfrac{qa, A, pa \Rightarrow qb, pb \qquad \dfrac{}{qb, qa, A, pa \Rightarrow qb}}{qa, A, pb \to qb, pa \Rightarrow qb}}{qa, A, pa \Rightarrow qb}}{qa, A, pa \Rightarrow \forall x\, qx}}{\dfrac{A, pa \to qa, pa \Rightarrow \forall x\, qx}{\underbrace{\forall x\,(px \to qx)}_{A}, pa \Rightarrow \forall x\, qx}}$$

- Abbreviate px for $p(x)$, qb for $q(b)$, etc.
- The Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- Since $pa \in \mathcal{B}^{\top}$, define $a \in p^{\iota}$, so $\mathcal{I} \models pa$.
- Since $qa \in \mathcal{B}^{\top}$, define $a \in q^{\iota}$, so $\mathcal{I} \models qa$ and thus $\mathcal{I} \models pa \to qa$.
- Since $qb \in \mathcal{B}^{\perp}$, define $b \notin q^{\iota}$, so $\mathcal{I} \not\models qb$ and thus $\mathcal{I} \not\models \forall x\, qx$.
- Since $pb \in \mathcal{B}^{\perp}$, define $b \notin p^{\iota}$, so $\mathcal{I} \not\models pb$ and thus $\mathcal{I} \models pb \to qb$.
- Therefore also $\mathcal{I} \models \forall x\,(px \to qx)$.

Counter-model Construction, Ex. 1

$$\dfrac{\dfrac{\dfrac{}{A, pa \Rightarrow \forall x\, qx, pa} \qquad \dfrac{\dfrac{\dfrac{\overset{\mathcal{B}}{qa, A, pa \Rightarrow qb, pb} \qquad \dfrac{}{qb, qa, A, pa \Rightarrow qb}}{qa, A, pb \to qb, pa \Rightarrow qb}}{qa, A, pa \Rightarrow qb}}{qa, A, pa \Rightarrow \forall x\, qx}}{A, pa \to qa, pa \Rightarrow \forall x\, qx}}{\underbrace{\forall x\,(px \to qx)}_{A}, pa \Rightarrow \forall x\, qx}$$

- Abbreviate px for $p(x)$, qb for $q(b)$, etc.
- The Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- Since $pa \in \mathcal{B}^\top$, define $a \in p^\iota$, so $\mathcal{I} \models pa$.
- Since $qa \in \mathcal{B}^\top$, define $a \in q^\iota$, so $\mathcal{I} \models qa$ and thus $\mathcal{I} \models pa \to qa$.
- Since $qb \in \mathcal{B}^\perp$, define $b \notin q^\iota$, so $\mathcal{I} \not\models qb$ and thus $\mathcal{I} \not\models \forall x\, qx$.
- Since $pb \in \mathcal{B}^\perp$, define $b \notin p^\iota$, so $\mathcal{I} \not\models pb$ and thus $\mathcal{I} \models pb \to qb$.
- Therefore also $\mathcal{I} \models \forall x\,(px \to qx)$.
- $\mathcal{I}$ makes all of $\mathcal{B}^\top$ true and all of $\mathcal{B}^\perp$ false.

Counter-model Construction, Ex. 2

$$\underbrace{\forall x\,(pxa \rightarrow pxb)}_{A}, paa \vee pba \Rightarrow pab$$

Counter-model Construction, Ex. 2

$$\underbrace{\forall x\,(pxa \to pxb)}_{A}, paa \lor pba \Rightarrow pab$$

- Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.

Counter-model Construction, Ex. 2

$$\underbrace{\forall x\,(pxa \rightarrow pxb)}_{A}, paa \lor pba \Rightarrow \color{red}{pab}$$

- Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.

Counter-model Construction, Ex. 2

$$\underbrace{\forall x\,(pxa \rightarrow pxb)}_{A}, paa \vee pba \Rightarrow pab$$

- Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- Since $pab \in \mathcal{B}^{\perp}$, define $\langle a, b\rangle \notin p^{\iota}$, so $\mathcal{I} \not\models pab$.

Counter-model Construction, Ex. 2

$$\underbrace{\forall x\,(pxa \to pxb)}_{A},\ paa \lor pba \Rightarrow pab$$

- Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- Since $pab \in \mathcal{B}^{\perp}$, define $\langle a, b\rangle \notin p^{\iota}$, so $\mathcal{I} \not\models pab$.

Counter-model Construction, Ex. 2

$$\dfrac{A, paa \Rightarrow pab \qquad\qquad A, pba \Rightarrow pab}{\underbrace{\forall x\,(pxa \rightarrow pxb)}_{A}, paa \vee pba \Rightarrow pab}$$

- Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- Since $pab \in \mathcal{B}^{\perp}$, define $\langle a, b\rangle \notin p^{\iota}$, so $\mathcal{I} \not\models pab$.

Counter-model Construction, Ex. 2

$$\dfrac{A, paa \Rightarrow pab \qquad\qquad A, pba \Rightarrow pab}{\underbrace{\forall x\,(pxa \rightarrow pxb)}_{A}, paa \lor pba \Rightarrow pab}$$

- Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- Since $pab \in \mathcal{B}^{\perp}$, define $\langle a, b\rangle \notin p^{\iota}$, so $\mathcal{I} \not\models pab$.

Counter-model Construction, Ex. 2

$$\cfrac{\cfrac{A, paa \to pab, paa \Rightarrow pab}{A, paa \Rightarrow pab} \qquad A, pba \Rightarrow pab}{\underbrace{\forall x\,(pxa \to pxb)}_{A}, paa \lor pba \Rightarrow pab}$$

- Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- Since $pab \in \mathcal{B}^{\perp}$, define $\langle a, b\rangle \notin p^{\iota}$, so $\mathcal{I} \not\models pab$.

Counter-model Construction, Ex. 2

$$\dfrac{\dfrac{A, paa \to pab, paa \Rightarrow pab}{A, paa \Rightarrow pab} \qquad A, pba \Rightarrow pab}{\underbrace{\forall x\,(pxa \to pxb)}_{A}, paa \lor pba \Rightarrow pab}$$

- Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- Since $pab \in \mathcal{B}^{\perp}$, define $\langle a, b \rangle \notin p^{\iota}$, so $\mathcal{I} \not\models pab$.

Counter-model Construction, Ex. 2

$$\dfrac{\dfrac{\dfrac{\text{(Both branches closeable)}}{A, paa \to pab, paa \Rightarrow pab}}{A, paa \Rightarrow pab} \qquad A, pba \Rightarrow pab}{\underbrace{\forall x\,(pxa \to pxb)}_{A}, paa \lor pba \Rightarrow pab}$$

- Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- Since $pab \in \mathcal{B}^{\perp}$, define $\langle a, b \rangle \notin p^{\iota}$, so $\mathcal{I} \not\models pab$.

Counter-model Construction, Ex. 2

$$\dfrac{\dfrac{\text{(Both branches closeable)}}{\dfrac{A, paa \rightarrow pab, paa \Rightarrow pab}{A, paa \Rightarrow pab}} \qquad A, \textcolor{red}{pba} \Rightarrow pab}{\underbrace{\forall x\,(pxa \rightarrow pxb)}_{A}, paa \vee pba \Rightarrow pab}$$

- Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- Since $pab \in \mathcal{B}^{\perp}$, define $\langle a, b \rangle \notin p^{\iota}$, so $\mathcal{I} \not\models pab$.

Counter-model Construction, Ex. 2

$$\dfrac{\dfrac{\dfrac{\text{(Both branches closeable)}}{A, paa \rightarrow pab, paa \Rightarrow pab}}{A, paa \Rightarrow pab} \qquad A, {\color{red}pba} \Rightarrow pab}{\underbrace{\forall x\,(pxa \rightarrow pxb)}_{A}, paa \lor pba \Rightarrow pab}$$

- Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- Since $pab \in \mathcal{B}^{\perp}$, define $\langle a, b\rangle \notin p^{\iota}$, so $\mathcal{I} \not\models pab$.
- Since $pba \in \mathcal{B}^{\top}$ vil $\langle b, a\rangle \in p^{\iota}$, so $\mathcal{I} \models pba$ and $\mathcal{I} \models paa \lor pba$.

Counter-model Construction, Ex. 2

(Both branches closeable)

$$\dfrac{\dfrac{A, paa \rightarrow pab, paa \Rightarrow pab}{A, paa \Rightarrow pab} \qquad {\color{red}A}, pba \Rightarrow pab}{\underbrace{\forall x\,(pxa \rightarrow pxb)}_{A}, paa \vee pba \Rightarrow pab}$$

- Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- Since $pab \in \mathcal{B}^{\perp}$, define $\langle a, b \rangle \notin p^{\iota}$, so $\mathcal{I} \not\models pab$.
- Since $pba \in \mathcal{B}^{\top}$ vil $\langle b, a \rangle \in p^{\iota}$, so $\mathcal{I} \models pba$ and $\mathcal{I} \models paa \vee pba$.

Counter-model Construction, Ex. 2

(Both branches closeable)

$$\dfrac{\dfrac{A, paa \to pab, paa \Rightarrow pab}{A, paa \Rightarrow pab} \qquad \dfrac{A, paa \to pab, pba \Rightarrow pab}{A, pba \Rightarrow pab}}{\underbrace{\forall x\,(pxa \to pxb)}_{A}, paa \lor pba \Rightarrow pab}$$

- Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- Since $pab \in \mathcal{B}^{\perp}$, define $\langle a, b\rangle \notin p^{\iota}$, so $\mathcal{I} \not\models pab$.
- Since $pba \in \mathcal{B}^{\top}$ vil $\langle b, a\rangle \in p^{\iota}$, so $\mathcal{I} \models pba$ and $\mathcal{I} \models paa \lor pba$.

Counter-model Construction, Ex. 2

$$\dfrac{\dfrac{\overset{\text{(Both branches closeable)}}{A, paa \to pab, paa \Rightarrow pab}}{A, paa \Rightarrow pab} \qquad \dfrac{A, \color{red}{paa \to pab}, pba \Rightarrow pab}{A, pba \Rightarrow pab}}{\underbrace{\forall x\,(pxa \to pxb)}_{A}, paa \lor pba \Rightarrow pab}$$

- Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- Since $pab \in \mathcal{B}^{\perp}$, define $\langle a, b\rangle \notin p^{\iota}$, so $\mathcal{I} \not\models pab$.
- Since $pba \in \mathcal{B}^{\top}$ vil $\langle b, a\rangle \in p^{\iota}$, so $\mathcal{I} \models pba$ and $\mathcal{I} \models paa \lor pba$.

Counter-model Construction, Ex. 2

$$\dfrac{\dfrac{\begin{array}{c}\text{(Both branches closeable)}\\ A, paa \to pab, paa \Rightarrow pab\end{array}}{A, paa \Rightarrow pab} \qquad \dfrac{\dfrac{A, pba \Rightarrow pab, paa \qquad pab, A, pba \Rightarrow pab}{A, paa \to pab, pba \Rightarrow pab}}{A, pba \Rightarrow pab}}{\underbrace{\forall x\,(pxa \to pxb)}_{A}, paa \lor pba \Rightarrow pab}$$

- Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- Since $pab \in \mathcal{B}^{\perp}$, define $\langle a, b\rangle \notin p^{\iota}$, so $\mathcal{I} \not\models pab$.
- Since $pba \in \mathcal{B}^{\top}$ vil $\langle b, a\rangle \in p^{\iota}$, so $\mathcal{I} \models pba$ and $\mathcal{I} \models paa \lor pba$.

Counter-model Construction, Ex. 2

$$
\dfrac{\dfrac{\dfrac{\text{(Both branches closeable)}}{A, paa \rightarrow pab, paa \Rightarrow pab}}{A, paa \Rightarrow pab} \qquad \dfrac{\dfrac{A, pba \Rightarrow pab, paa \qquad {\color{red}pab}, A, pba \Rightarrow {\color{red}pab}}{A, paa \rightarrow pab, pba \Rightarrow pab}}{A, pba \Rightarrow pab}}{\underbrace{\forall x\,(pxa \rightarrow pxb)}_{A}, paa \vee pba \Rightarrow pab}
$$

- Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- Since $pab \in \mathcal{B}^{\perp}$, define $\langle a, b\rangle \notin p^{\iota}$, so $\mathcal{I} \not\models pab$.
- Since $pba \in \mathcal{B}^{\top}$ vil $\langle b, a\rangle \in p^{\iota}$, so $\mathcal{I} \models pba$ and $\mathcal{I} \models paa \vee pba$.

Counter-model Construction, Ex. 2

$$\dfrac{\dfrac{\dfrac{\text{(Both branches closeable)}}{A, paa \to pab, paa \Rightarrow pab}}{A, paa \Rightarrow pab} \qquad \dfrac{\dfrac{A, pba \Rightarrow pab, paa \qquad \dfrac{}{\color{red}{pab}, A, pba \Rightarrow \color{red}{pab}}}{A, paa \to pab, pba \Rightarrow pab}}{A, pba \Rightarrow pab}}{\underbrace{\forall x\,(pxa \to pxb)}_{A}, paa \vee pba \Rightarrow pab}$$

- Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- Since $pab \in \mathcal{B}^{\perp}$, define $\langle a, b\rangle \notin p^{\iota}$, so $\mathcal{I} \not\models pab$.
- Since $pba \in \mathcal{B}^{\top}$ vil $\langle b, a\rangle \in p^{\iota}$, so $\mathcal{I} \models pba$ and $\mathcal{I} \models paa \vee pba$.

Counter-model Construction, Ex. 2

$$
\dfrac{\dfrac{\dfrac{\text{(Both branches closeable)}}{A, paa \to pab, paa \Rightarrow pab}}{A, paa \Rightarrow pab}
\qquad
\dfrac{\dfrac{A, pba \Rightarrow pab, {\color{red}paa} \qquad \dfrac{}{pab, A, pba \Rightarrow pab}}{A, paa \to pab, pba \Rightarrow pab}}{A, pba \Rightarrow pab}}
{\underbrace{\forall x\,(pxa \to pxb)}_{A}, paa \lor pba \Rightarrow pab}
$$

- Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- Since $pab \in \mathcal{B}^{\bot}$, define $\langle a, b\rangle \notin p^{\iota}$, so $\mathcal{I} \not\models pab$.
- Since $pba \in \mathcal{B}^{\top}$ vil $\langle b, a\rangle \in p^{\iota}$, so $\mathcal{I} \models pba$ and $\mathcal{I} \models paa \lor pba$.

Counter-model Construction, Ex. 2

$$\dfrac{\dfrac{\begin{array}{c}\text{(Both branches closeable)}\\ A, paa \to pab, paa \Rightarrow pab\end{array}}{A, paa \Rightarrow pab} \qquad \dfrac{\dfrac{A, pba \Rightarrow pab, \color{red}{paa} \qquad \dfrac{}{pab, A, pba \Rightarrow pab}}{A, paa \to pab, pba \Rightarrow pab}}{A, pba \Rightarrow pab}}{\underbrace{\forall x\,(pxa \to pxb)}_{A}, paa \lor pba \Rightarrow pab}$$

- Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- Since $pab \in \mathcal{B}^{\bot}$, define $\langle a, b\rangle \notin p^{\iota}$, so $\mathcal{I} \not\models pab$.
- Since $pba \in \mathcal{B}^{\top}$ vil $\langle b, a\rangle \in p^{\iota}$, so $\mathcal{I} \models pba$ and $\mathcal{I} \models paa \lor pba$.
- Since $paa \in \mathcal{B}^{\bot}$ vil $\langle a, a\rangle \notin p^{\iota}$, so $\mathcal{I} \not\models paa$ and $\mathcal{I} \models paa \to pab$.

Counter-model Construction, Ex. 2

$$\dfrac{\dfrac{\begin{array}{c}\text{(Both branches closeable)}\\ A, paa \to pab, paa \Rightarrow pab\end{array}}{A, paa \Rightarrow pab} \qquad \dfrac{\dfrac{{\color{red}A}, pba \Rightarrow pab, paa \qquad \dfrac{}{pab, A, pba \Rightarrow pab}}{A, paa \to pab, pba \Rightarrow pab}}{A, pba \Rightarrow pab}}{\underbrace{\forall x\,(pxa \to pxb)}_{A}, paa \lor pba \Rightarrow pab}$$

- Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- Since $pab \in \mathcal{B}^{\perp}$, define $\langle a, b\rangle \notin p^{\iota}$, so $\mathcal{I} \not\models pab$.
- Since $pba \in \mathcal{B}^{\top}$ vil $\langle b, a\rangle \in p^{\iota}$, so $\mathcal{I} \models pba$ and $\mathcal{I} \models paa \lor pba$.
- Since $paa \in \mathcal{B}^{\perp}$ vil $\langle a, a\rangle \notin p^{\iota}$, so $\mathcal{I} \not\models paa$ and $\mathcal{I} \models paa \to pab$.

Counter-model Construction, Ex. 2

$$
\dfrac{
\dfrac{\dfrac{\text{(Both branches closeable)}}{A, paa \rightarrow pab, paa \Rightarrow pab}}{A, paa \Rightarrow pab}
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{A, pba \rightarrow pbb, pba \Rightarrow pab, paa}{A, pba \Rightarrow pab, paa} \qquad \dfrac{}{pab, A, pba \Rightarrow pab}}{A, paa \rightarrow pab, pba \Rightarrow pab}}{A, pba \Rightarrow pab}}{}
}{\underbrace{\forall x\,(pxa \rightarrow pxb)}_{A}, paa \lor pba \Rightarrow pab}
$$

- Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- Since $pab \in \mathcal{B}^{\bot}$, define $\langle a, b\rangle \notin p^{\iota}$, so $\mathcal{I} \not\models pab$.
- Since $pba \in \mathcal{B}^{\top}$ vil $\langle b, a\rangle \in p^{\iota}$, so $\mathcal{I} \models pba$ and $\mathcal{I} \models paa \lor pba$.
- Since $paa \in \mathcal{B}^{\bot}$ vil $\langle a, a\rangle \notin p^{\iota}$, so $\mathcal{I} \not\models paa$ and $\mathcal{I} \models paa \rightarrow pab$.

Counter-model Construction, Ex. 2

$$
\dfrac{
\dfrac{\dfrac{\text{(Both branches closeable)}}{A, paa \rightarrow pab, paa \Rightarrow pab}}{A, paa \Rightarrow pab}
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{A, {\color{red} pba \rightarrow pbb}, pba \Rightarrow pab, paa}{A, pba \Rightarrow pab, paa} \qquad \dfrac{}{pab, A, pba \Rightarrow pab}}{A, paa \rightarrow pab, pba \Rightarrow pab}}{A, pba \Rightarrow pab}}{}
}{
\underbrace{\forall x\,(pxa \rightarrow pxb)}_{A}, paa \vee pba \Rightarrow pab
}
$$

- Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- Since $pab \in \mathcal{B}^{\perp}$, define $\langle a, b\rangle \notin p^{\iota}$, so $\mathcal{I} \not\models pab$.
- Since $pba \in \mathcal{B}^{\top}$ vil $\langle b, a\rangle \in p^{\iota}$, so $\mathcal{I} \models pba$ and $\mathcal{I} \models paa \vee pba$.
- Since $paa \in \mathcal{B}^{\perp}$ vil $\langle a, a\rangle \notin p^{\iota}$, so $\mathcal{I} \not\models paa$ and $\mathcal{I} \models paa \rightarrow pab$.

Counter-model Construction, Ex. 2

$$\dfrac{\dfrac{\dfrac{\text{(Both branches closeable)}}{A, paa \to pab, paa \Rightarrow pab}}{A, paa \Rightarrow pab} \qquad \dfrac{\dfrac{\dfrac{\dfrac{A, pba \Rightarrow pab, paa, pba \qquad pbb, A, pba \Rightarrow pab, paa}{A, pba \to pbb, pba \Rightarrow pab, paa}}{A, pba \Rightarrow pab, paa} \qquad \dfrac{}{pab, A, pba \Rightarrow pab}}{A, paa \to pab, pba \Rightarrow pab}}{A, pba \Rightarrow pab}}{\underbrace{\forall x\,(pxa \to pxb)}_{A}, paa \lor pba \Rightarrow pab}$$

- Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- Since $pab \in \mathcal{B}^{\perp}$, define $\langle a, b\rangle \notin p^{\iota}$, so $\mathcal{I} \not\models pab$.
- Since $pba \in \mathcal{B}^{\top}$ vil $\langle b, a\rangle \in p^{\iota}$, so $\mathcal{I} \models pba$ and $\mathcal{I} \models paa \lor pba$.
- Since $paa \in \mathcal{B}^{\perp}$ vil $\langle a, a\rangle \notin p^{\iota}$, so $\mathcal{I} \not\models paa$ and $\mathcal{I} \models paa \to pab$.

Counter-model Construction, Ex. 2

$$
\dfrac{
  \dfrac{\dfrac{\text{(Both branches closeable)}}{A, paa \to pab, paa \Rightarrow pab}}{A, paa \Rightarrow pab}
  \qquad
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{A, {\color{red}pba} \Rightarrow pab, paa, {\color{red}pba} \qquad pbb, A, pba \Rightarrow pab, paa}{A, pba \to pbb, pba \Rightarrow pab, paa}
      }{A, pba \Rightarrow pab, paa}
      \qquad
      \dfrac{}{pab, A, pba \Rightarrow pab}
    }{A, paa \to pab, pba \Rightarrow pab}
  }{A, pba \Rightarrow pab}
}{\underbrace{\forall x\,(pxa \to pxb)}_{A}, paa \lor pba \Rightarrow pab}
$$

- Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- Since $pab \in \mathcal{B}^{\perp}$, define $\langle a, b\rangle \notin p^{\iota}$, so $\mathcal{I} \not\models pab$.
- Since $pba \in \mathcal{B}^{\top}$ vil $\langle b, a\rangle \in p^{\iota}$, so $\mathcal{I} \models pba$ and $\mathcal{I} \models paa \lor pba$.
- Since $paa \in \mathcal{B}^{\perp}$ vil $\langle a, a\rangle \notin p^{\iota}$, so $\mathcal{I} \not\models paa$ and $\mathcal{I} \models paa \to pab$.

Counter-model Construction, Ex. 2

$$
\dfrac{
\dfrac{\dfrac{\text{(Both branches closeable)}}{A, paa \to pab, paa \Rightarrow pab}}{A, paa \Rightarrow pab}
\qquad
\dfrac{\dfrac{
\dfrac{\dfrac{\dfrac{}{A, pba \Rightarrow pab, paa, pba} \quad pbb, A, pba \Rightarrow pab, paa}{A, pba \to pbb, pba \Rightarrow pab, paa}}{A, pba \Rightarrow pab, paa}
\quad
\dfrac{}{pab, A, pba \Rightarrow pab}
}{A, paa \to pab, pba \Rightarrow pab}}{A, pba \Rightarrow pab}
}{\underbrace{\forall x\,(pxa \to pxb)}_{A}, paa \lor pba \Rightarrow pab}
$$

- Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- Since $pab \in \mathcal{B}^{\bot}$, define $\langle a, b\rangle \notin p^{\iota}$, so $\mathcal{I} \not\models pab$.
- Since $pba \in \mathcal{B}^{\top}$ vil $\langle b, a\rangle \in p^{\iota}$, so $\mathcal{I} \models pba$ and $\mathcal{I} \models paa \lor pba$.
- Since $paa \in \mathcal{B}^{\bot}$ vil $\langle a, a\rangle \notin p^{\iota}$, so $\mathcal{I} \not\models paa$ and $\mathcal{I} \models paa \to pab$.

Counter-model Construction, Ex. 2

$$\overline{A, pba \Rightarrow pab, paa, pba} \qquad {\color{red}pbb}, A, pba \Rightarrow pab, paa$$

$$A, pba \to pbb, pba \Rightarrow pab, paa$$

(Both branches closeable)

$$A, pba \Rightarrow pab, paa \qquad \overline{pab, A, pba \Rightarrow pab}$$

$$A, paa \to pab, paa \Rightarrow pab \qquad A, paa \to pab, pba \Rightarrow pab$$

$$A, paa \Rightarrow pab \qquad A, pba \Rightarrow pab$$

$$\underbrace{\forall x\,(pxa \to pxb)}_{A}, paa \lor pba \Rightarrow pab$$

- Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- Since $pab \in \mathcal{B}^{\perp}$, define $\langle a, b\rangle \notin p^{\iota}$, so $\mathcal{I} \not\models pab$.
- Since $pba \in \mathcal{B}^{\top}$ vil $\langle b, a\rangle \in p^{\iota}$, so $\mathcal{I} \models pba$ and $\mathcal{I} \models paa \lor pba$.
- Since $paa \in \mathcal{B}^{\perp}$ vil $\langle a, a\rangle \notin p^{\iota}$, so $\mathcal{I} \not\models paa$ and $\mathcal{I} \models paa \to pab$.

Counter-model Construction, Ex. 2

(Both branches closeable)

$$
\dfrac{
\dfrac{A, paa \to pab, paa \Rightarrow pab}{A, paa \Rightarrow pab}
\qquad
\dfrac{
\dfrac{
\dfrac{
\dfrac{\dfrac{}{A, pba \Rightarrow pab, paa, pba} \qquad {\color{red}pbb}, A, pba \Rightarrow pab, paa}{A, pba \to pbb, pba \Rightarrow pab, paa}
}{A, pba \Rightarrow pab, paa}
\qquad
\dfrac{}{pab, A, pba \Rightarrow pab}
}{A, paa \to pab, pba \Rightarrow pab}
}{A, pba \Rightarrow pab}
}{\underbrace{\forall x\,(pxa \to pxb)}_{A}, paa \lor pba \Rightarrow pab}
$$

- Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- Since $pab \in \mathcal{B}^{\perp}$, define $\langle a, b\rangle \notin p^{\iota}$, so $\mathcal{I} \not\models pab$.
- Since $pba \in \mathcal{B}^{\top}$ vil $\langle b, a\rangle \in p^{\iota}$, so $\mathcal{I} \models pba$ and $\mathcal{I} \models paa \lor pba$.
- Since $paa \in \mathcal{B}^{\perp}$ vil $\langle a, a\rangle \notin p^{\iota}$, so $\mathcal{I} \not\models paa$ and $\mathcal{I} \models paa \to pab$.
- Since $pbb \in \mathcal{B}^{\top}$ vil $\langle b, b\rangle \in p^{\iota}$, so $\mathcal{I} \models pbb$ and $\mathcal{I} \models pba \to pbb$.

Counter-model Construction, Ex. 2

$$
\dfrac{
  \dfrac{\dfrac{\text{(Both branches closeable)}}{A, paa \to pab, paa \Rightarrow pab}}{A, paa \Rightarrow pab}
  \qquad
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{
          \dfrac{}{A, pba \Rightarrow pab, paa, pba}
          \quad
          pbb, A, pba \Rightarrow pab, paa
        }{A, pba \to pbb, pba \Rightarrow pab, paa}
      }{A, pba \Rightarrow pab, paa}
      \quad
      \dfrac{}{pab, A, pba \Rightarrow pab}
    }{A, paa \to pab, pba \Rightarrow pab}
  }{A, pba \Rightarrow pab}
}{\underbrace{\forall x\,(pxa \to pxb)}_{A}, paa \lor pba \Rightarrow pab}
$$

- Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- Since $pab \in \mathcal{B}^{\bot}$, define $\langle a, b\rangle \notin p^{\iota}$, so $\mathcal{I} \not\models pab$.
- Since $pba \in \mathcal{B}^{\top}$ vil $\langle b, a\rangle \in p^{\iota}$, so $\mathcal{I} \models pba$ and $\mathcal{I} \models paa \lor pba$.
- Since $paa \in \mathcal{B}^{\bot}$ vil $\langle a, a\rangle \notin p^{\iota}$, so $\mathcal{I} \not\models paa$ and $\mathcal{I} \models paa \to pab$.
- Since $pbb \in \mathcal{B}^{\top}$ vil $\langle b, b\rangle \in p^{\iota}$, so $\mathcal{I} \models pbb$ and $\mathcal{I} \models pba \to pbb$.
- We thus have $\mathcal{I} \models \forall x\,(pxa \to pxb)$.

Counter-model Construction, Ex. 2

$$
\dfrac{
\dfrac{\text{(Both branches closeable)}}{\dfrac{A, paa \to pab, paa \Rightarrow pab}{A, paa \Rightarrow pab}}
\qquad
\dfrac{
\dfrac{
\dfrac{
\dfrac{\dfrac{}{A, pba \Rightarrow pab, paa, pba} \qquad \overset{\mathcal{B}}{pbb, A, pba \Rightarrow pab, paa}}{A, pba \to pbb, pba \Rightarrow pab, paa}
}{A, pba \Rightarrow pab, paa}
\qquad
\dfrac{}{pab, A, pba \Rightarrow pab}
}{A, paa \to pab, pba \Rightarrow pab}
}{A, pba \Rightarrow pab}
}{\underbrace{\forall x\,(pxa \to pxb)}_{A}, paa \lor pba \Rightarrow pab}
$$

- Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- Since $pab \in \mathcal{B}^{\perp}$, define $\langle a, b\rangle \notin p^{\iota}$, so $\mathcal{I} \not\models pab$.
- Since $pba \in \mathcal{B}^{\top}$ vil $\langle b, a\rangle \in p^{\iota}$, so $\mathcal{I} \models pba$ and $\mathcal{I} \models paa \lor pba$.
- Since $paa \in \mathcal{B}^{\perp}$ vil $\langle a, a\rangle \notin p^{\iota}$, so $\mathcal{I} \not\models paa$ and $\mathcal{I} \models paa \to pab$.
- Since $pbb \in \mathcal{B}^{\top}$ vil $\langle b, b\rangle \in p^{\iota}$, so $\mathcal{I} \models pbb$ and $\mathcal{I} \models pba \to pbb$.
- We thus have $\mathcal{I} \models \forall x\,(pxa \to pxb)$.

Counter-model Construction, Ex. 2

(Both branches closeable)

$$\dfrac{\dfrac{A, paa \to pab, paa \Rightarrow pab}{A, paa \Rightarrow pab} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A, pba \Rightarrow pab, paa, pba} \quad \begin{matrix}\mathcal{B}\\ pbb, A, pba \Rightarrow pab, paa\end{matrix}}{A, pba \to pbb, pba \Rightarrow pab, paa}}{A, pba \Rightarrow pab, paa} \qquad \dfrac{}{pab, A, pba \Rightarrow pab}}{A, paa \to pab, pba \Rightarrow pab}}{A, pba \Rightarrow pab}}{\underbrace{\forall x\,(pxa \to pxb)}_{A}, paa \lor pba \Rightarrow pab}$$

- Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- Since $pab \in \mathcal{B}^{\perp}$, define $\langle a, b\rangle \notin p^{\iota}$, so $\mathcal{I} \not\models pab$.
- Since $pba \in \mathcal{B}^{\top}$ vil $\langle b, a\rangle \in p^{\iota}$, so $\mathcal{I} \models pba$ and $\mathcal{I} \models paa \lor pba$.
- Since $paa \in \mathcal{B}^{\perp}$ vil $\langle a, a\rangle \notin p^{\iota}$, so $\mathcal{I} \not\models paa$ and $\mathcal{I} \models paa \to pab$.
- Since $pbb \in \mathcal{B}^{\top}$ vil $\langle b, b\rangle \in p^{\iota}$, so $\mathcal{I} \models pbb$ and $\mathcal{I} \models pba \to pbb$.
- We thus have $\mathcal{I} \models \forall x\,(pxa \to pxb)$.
- $\mathcal{I}$ makes all of $\mathcal{B}^{\top}$ true and all of $\mathcal{B}^{\perp}$ false.

Summary and Outlook

- We can show things for ∞ many interpretations using finite proofs!

Summary and Outlook

- We can show things for ∞ many interpretations using finite proofs!
- OMG! Amazing!

Summary and Outlook

- We can show things for ∞ many interpretations using finite proofs!
- OMG! Amazing!
- Uncloseable branches give counter-models

Summary and Outlook

- We can show things for ∞ many interpretations using finite proofs!
- OMG! Amazing!
- Uncloseable branches give counter-models
- Might be infinite: if there is no proof, we might search for ever

Summary and Outlook

- We can show things for ∞ many interpretations using finite proofs!
- OMG! Amazing!
- Uncloseable branches give counter-models
- Might be infinite: if there is no proof, we might search for ever
- First-order validity is undecidable

Summary and Outlook

- We can show things for ∞ many interpretations using finite proofs!
- OMG! Amazing!
- Uncloseable branches give counter-models
- Might be infinite: if there is no proof, we might search for ever
- First-order validity is undecidable

- Can this be automated?

Summary and Outlook

- We can show things for ∞ many interpretations using finite proofs!
- OMG! Amazing!
- Uncloseable branches give counter-models
- Might be infinite: if there is no proof, we might search for ever
- First-order validity is undecidable

- Can this be automated?
- Sure! But...

Summary and Outlook

- We can show things for ∞ many interpretations using finite proofs!
- OMG! Amazing!
- Uncloseable branches give counter-models
- Might be infinite: if there is no proof, we might search for ever
- First-order validity is undecidable

- Can this be automated?
- Sure! But. . .
- Instantiating quantifiers with *every* possible term is wasteful

Summary and Outlook

- We can show things for ∞ many interpretations using finite proofs!
- OMG! Amazing!
- Uncloseable branches give counter-models
- Might be infinite: if there is no proof, we might search for ever
- First-order validity is undecidable

- Can this be automated?
- Sure! But. . .
- Instantiating quantifiers with *every* possible term is wasteful
- More goal-oriented ways of doing this?

Summary and Outlook

- We can show things for ∞ many interpretations using finite proofs!
- OMG! Amazing!
- Uncloseable branches give counter-models
- Might be infinite: if there is no proof, we might search for ever
- First-order validity is undecidable

- Can this be automated?
- Sure! But...
- Instantiating quantifiers with *every* possible term is wasteful
- More goal-oriented ways of doing this?
- Coming up...