

IN3070/4070 – Logic – Autumn 2019

Lecture 6: Unification, Normal Forms

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25th September 2019



DEPARTMENT OF
INFORMATICS



UNIVERSITY OF
OSLO

Today's Plan

- ▶ Unification
- ▶ Normal Forms
- ▶ Negation Normal Form
- ▶ Conjunctive Normal Form
- ▶ Clausal Form
- ▶ Prenex Normal Forms
- ▶ Skolemization

Outline

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- ▶ Two terms are **unifiable** if they have a unifier.

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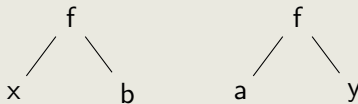
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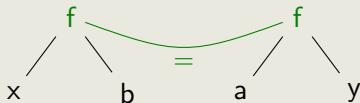
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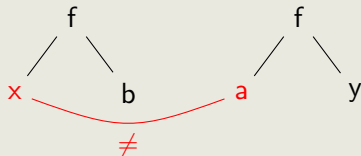
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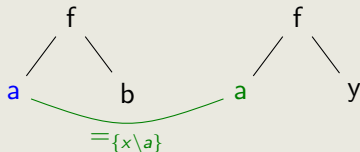
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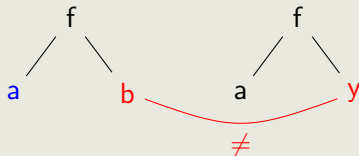
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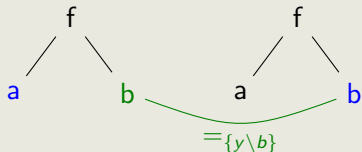
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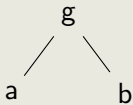
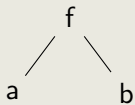
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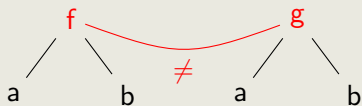
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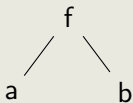
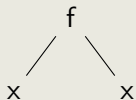
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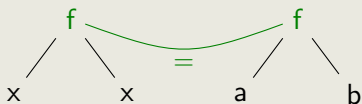
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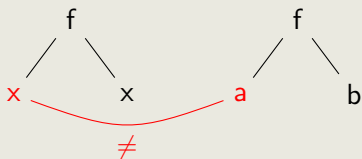


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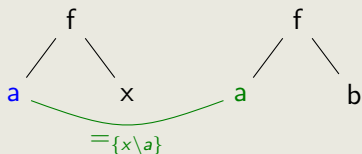
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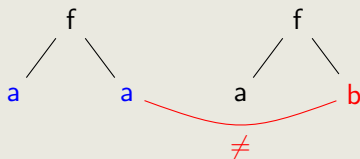
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- ▶ The right children are now different, and can *not* be unified!

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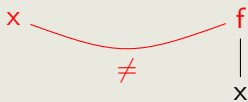
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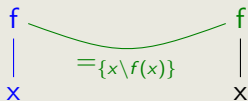
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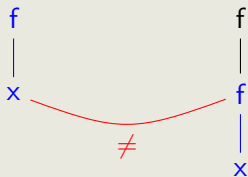
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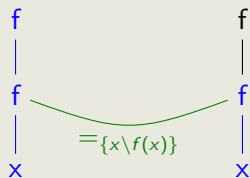
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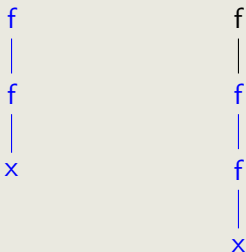
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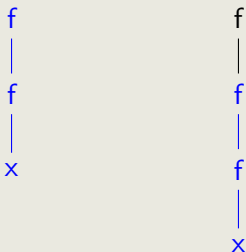
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- ▶ This continues indefinitely

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- ▶ Such a unifier is known as a **most general unifier**.

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- ▶ Exercise: show that $(\tau\sigma)(A) = \tau(\sigma(A))$ for all formulae A and all substitutions σ and τ .

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Yes, since $\{x \setminus f(y)\} = \{\}\{x \setminus f(y)\}$, where $\{\}$ is the identity substitution.

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Uniqueness “up to variable renaming”

Proposition 1.2.

If σ_1 and σ_2 are most general unifiers for two terms s and t , then there is a variable renaming η such that $\eta\sigma_1 = \sigma_2$.

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- ▶ So every term is a subterm of itself, but not a strict subterm.

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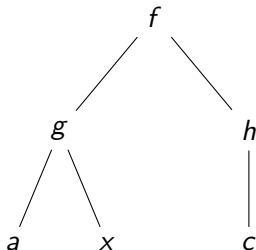
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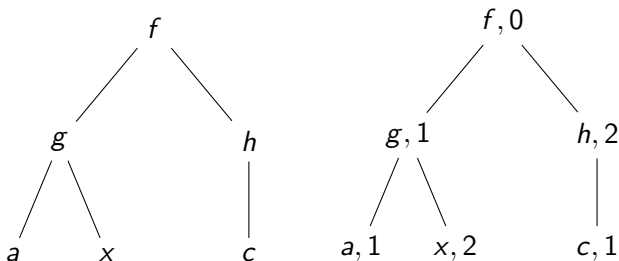
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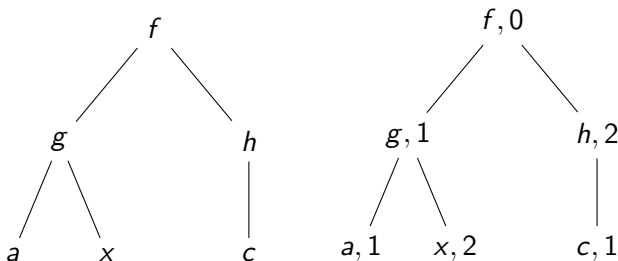
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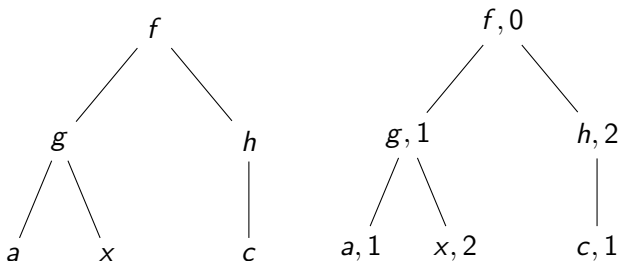
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- ▶ We write the root of the numbered term tree of t as $\text{root}(t)$.

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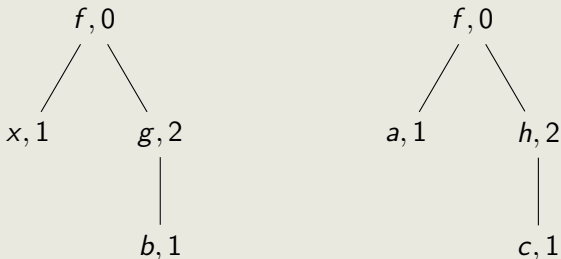
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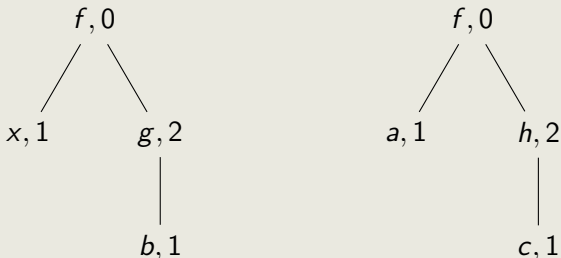
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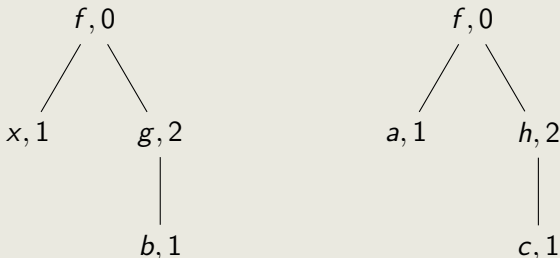
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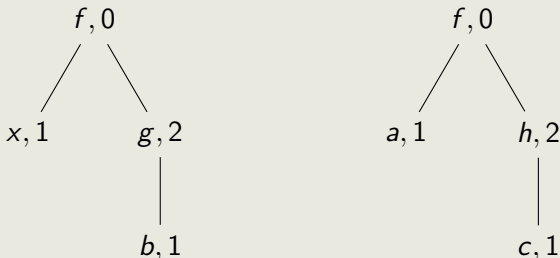
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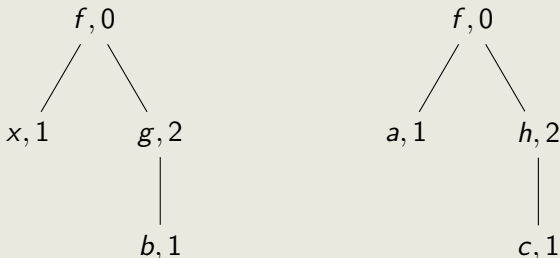
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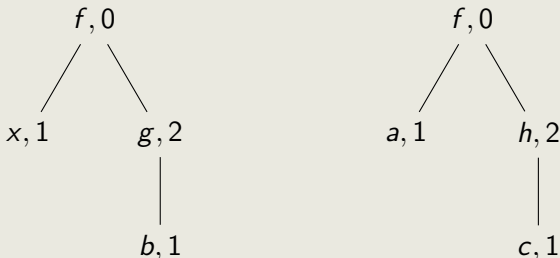
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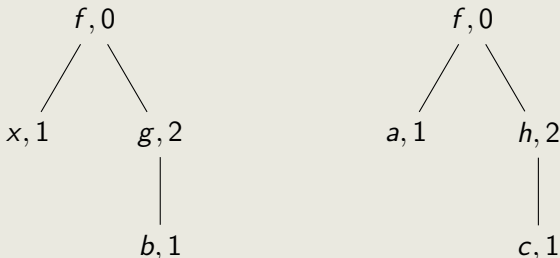
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- ▶ Unification
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Example: Normal Form for Rational Numbers

- ▶ Let Q be the set of pairs $\langle m, n \rangle$, where we think of $\frac{m}{n}$
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- ▶ So $\langle m/k, n/k \rangle$ is the “reduced normal form” of $\langle m, n \rangle$

Outline

- ▶ Unification
- ▶ Normal Forms
- ▶ **Negation Normal Form**
- ▶ Conjunctive Normal Form
- ▶ Clausal Form
- ▶ Prenex Normal Forms
- ▶ Skolemization

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Theorem 3.1.

Every formula in first-order logic can be transformed into an equivalent formula in NNF.

Proof.

To convert an arbitrary formula to a formula in NNF, remove implications, and push negations inwards, preserving equivalence, using the following:

$$A \rightarrow B \equiv \neg A \vee B$$

$$\neg(A \wedge B) \equiv \neg A \vee \neg B$$

$$\neg(A \vee B) \equiv \neg A \wedge \neg B$$

$$\neg(\forall x A) \equiv \exists x \neg A$$

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$$\neg(\neg A) \equiv A$$



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- ▶ Soundness and completeness proofs also have fewer cases.

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- (1) Convert to negation normal form.
- (2) Use the distributive laws to move conjunctions inside disjunctions to the outside

$$A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$$



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$(p \vee \neg q) \wedge (\neg p \vee q)$ in clausal form: $\{\{p, \neg q\}, \{\neg p, q\}\}$

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This follows from the previous theorem, where we transformed a formula to CNF. Each disjunction is then transformed to a clause (of literals), and the clausal form is the set of these clauses. □

Empty Clause and Empty Set of Clauses

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Note: The universal quantifiers are implicit.

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“Satisfiability is more interesting than validity. Always true or always false are extremes.”

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- ▶ The formula is now in PCNF.

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- ▶ If there are no universal quantifiers preceding $\exists x$, replace x by a new constant (0-ary function).
- ▶ These new function symbols are **Skolem functions** and the process of replacing existential quantifiers by functions is **Skolemization**.

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- ▶ Look at the formulas $\forall x \exists y p(x, y)$ and $\forall x p(x, f(x))$.
- ▶ Are they *equivalent*? No!
- ▶ Are they *equisatisfiable*? Yes!
- ▶ The Skolemization of $\forall x \exists y p(x, y)$ is $\forall x p(x, f(x))$, and if one of them has a model, so does the other.

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- ▶ For any choice of elements d_1, \dots, d_n from D , there is an element d_{n+1} in D such that

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- ▶ Let $f^{\iota'}(d_1, \dots, d_n) = d_{n+1}$. This ensures that the claim holds.

Example

- ▶ Clause form of $\neg\exists x(p(x) \rightarrow \forall y p(y))$

Outlook

- ▶ We have seen the LK calculus for propositional and first-order logic
- ▶ Sound and complete, but not machine-oriented
- ▶ Machine-oriented calculi use:
 - ▶ Unification to find the right instantiations
 - ▶ Normal forms to simplify reasoning steps
- ▶ Free variable calculi
 - ▶ Similar to LK, but with unification
 - ▶ Often used with NNF or clause form
- ▶ Resolution
 - ▶ Basis of many theorem provers, uses unification
 - ▶ Almost always on clause form