IN3070/4070 – Logic – Autumn 2019

Lecture 7: Free-variable calculi

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Today's Plan

- Introduction
- The Free-variable Sequent Calculus
- Soundness
- Completeness

Outline

- Introduction
- The Free-variable Sequent Calculus
- Soundness
- Completeness

Smullyan's categories: $\alpha/\beta/\gamma/\delta$

- Many similar cases in proofs and implementations
- Categorise rules, rule applications, and formulae

α-rules

Propositional, one branch, e.g.

$$\frac{\Gamma, A, B \implies \Delta}{\Gamma, A \land B \implies \Delta}\ \land\text{-left}$$

β-rules

Propositional, splitting, e.g.

$$\frac{\Gamma \implies A, \Delta \qquad \Gamma, B \implies \Delta}{\Gamma, A \to B \implies \Delta}$$

γ-rules

Apply for all terms t, e.g.

$$\frac{\Gamma, A[x\backslash t], \forall x\, A \implies \Delta}{\Gamma, \forall x\, A \implies \Delta}\ \forall\text{-left}$$

δ-rules

Introduce new constant c, e.g.

$$\frac{\Gamma, A[x\backslash c], \implies \Delta}{\Gamma, \exists x\, A \implies \Delta}\ \exists\text{-left}$$

γ-rules – The Biggest Problem in 1st-Order Proof Search

- Let us look at the γ-rules of LK:

$$\frac{\Gamma, \forall x\, A, A[x\backslash t] \implies \Delta}{\Gamma, \forall x\, A \implies \Delta}\ \forall\text{-left} \qquad \frac{\Gamma \implies \Delta, \exists x\, A, A[x\backslash t]}{\Gamma \implies \Delta, \exists x\, A}\ \exists\text{-right}$$

- We can substitute an arbitrary closed term t for x.
- For a complete proof search we must instantiate every γ-formula with every term in the Herbrand universe.
- We can enumerate terms in the Herbrand universe and instantiate the γ-formulae in that order.
- But which order should we use to find a proof as early as possible?

$$\frac{\dfrac{\dfrac{\forall x p(x), p(a), \ldots, p(fffa) \implies p(fffa), q(ga)}{\vdots}}{\forall x p(x), p(a) \implies p(fffa), q(ga)}}{\forall x p(x) \implies p(fffa), q(ga)}$$

a,	fa,	ga,	ffa,	fga,	$\ldots$,	$fffa$,	$\ldots$
1	2	3	4	5		i	

Postponing the choice of γ-terms

- A better idea: postpone the choice of instantiation until later.
- Let γ-rules introduce free variables as placeholders.

$$\dfrac{\dfrac{\overset{U \backslash a}{\forall x\, p(x), p(U) \implies p(a)}}{\forall x\, p(x) \implies p(a)} \qquad \dfrac{\overset{V \backslash b}{\forall x\, p(x), p(V) \implies p(b)}}{\forall x\, p(x) \implies p(b)}}{\forall x\, p(x) \implies p(a) \land p(b)}$$

- Substitute terms for free variables that make leaves into axioms
- Which substitutions can we use to make leaves into axioms?
- That's a unification problem!

δ-rules

- If we use free variables in γ-rules we run into a problem with the δ-rules
- How can we ensure that the constant they introduce is *new* if we don't know what we will substitute for the free variables?

$$\cfrac{\cfrac{\begin{array}{c} U \backslash b,\ V \backslash a \\ I(U,a) \implies I(b,V) \end{array}}{\exists y\, I(U,y) \implies \forall x\, I(x,V)}}{\forall x \exists y\, I(x,y) \implies \exists y \forall x\, I(x,y)}$$

can't be closed

$$\cfrac{\cfrac{I(U,f(U)) \implies I(g(V),V)}{\exists y\, I(U,y) \implies \forall x\, I(x,V)}}{\forall x \exists y\, I(x,y) \implies \exists y \forall x\, I(x,y)}$$

- We let δ rules introduce a Skolem term:

$$f(U_1,\ldots,U_n),$$

  where f is a new function symbol, called a Skolem function, and $U_1,\ldots,U_n$ are all variables that occur free in the δ-formula.
- This will ensure soundness of the δ-rule no matter how the free variables are instantiated.

Outline

Free-variable Sequent Calculus

- The choice of a term in γ-rules is postponed by instantiating with a free variable.
- Introducing free variables in γ-rules means that δ-rules must introduce Skolem terms.
- Unification allows us to find a substitution that will replace free variables by terms that close the derivation.

$$U \backslash a, V \backslash f(a)$$

$$\dfrac{\dfrac{p(U, f(U)) \implies p(a, V)}{\exists y\, p(U, y) \implies p(a, V)}}{\forall x \exists y\, p(x, y) \implies \exists x\, p(a, x)}$$

γ-rules

Definition 2.1 (γ-rules in free-variable LK).

The γ-rules in free variable LK *are:*

$$\frac{\Gamma, \forall x\, A, A[x \backslash U] \implies \Delta}{\Gamma, \forall x\, A \implies \Delta}\ \forall\text{-}left \qquad \frac{\Gamma \implies \Delta, \exists x\, A, A[x \backslash U]}{\Gamma \implies \Delta, \exists x\, A}\ \exists\text{-}right$$

U is a new free variable

- By *new*, we mean that it does not previously occur in the derivation.

δ-rules

Definition 2.2 (δ-rules in free-variable LK).

The δ-rules in free-variable LK *are:*

$$\frac{\Gamma, A[x \backslash f(\vec{U})] \implies \Delta}{\Gamma, \exists x\, A \implies \Delta}\ \exists\text{-}left \qquad \frac{\Gamma \implies \Delta, A[x \backslash f(\vec{U})]}{\Gamma \implies \Delta, \forall x\, A}\ \forall\text{-}right$$

f is a new Skolem function
$\vec{U} = U_1, \ldots, U_n$ are the free variables occurring in $\exists x\, A$.

- By *new*, we mean that f does not previously occur in the derivation.

Rules and Derivations

Definition 2.3 (Rules of free-variable LK).

The rules of free-variable LK *are*

- *the γ- and δ-rules for free variables, and*
- *the α- and β-rules for propositional* LK*.*

- Free-variable derivations are defined inductively:
  - The induction base consists of *closed* sequents.
  - The derivations are closed under applications of the free-variable LK rules.
- So we still allow *only closed formulae* in the root sequent.
- Formulae later on are not necessarily closed.

Examples of free-variable derivations

$$\dfrac{\dfrac{\forall x\, p(x), p(U) \implies \exists x\, p(x), p(V)}{\forall x\, p(x), p(U) \implies \exists x\, p(x)}}{\forall x\, p(x) \implies \exists x\, p(x)}$$

$\exists$-right can not introduce U, since it already occurs in the derivation.

$$\dfrac{\dfrac{\forall x\, p(x), p(U) \implies p(a)}{\forall x\, p(x) \implies p(a)} \qquad \dfrac{\forall x\, p(x), p(V) \implies p(b)}{\forall x\, p(x) \implies p(b)}}{\forall x\, p(x) \implies p(a) \land p(b)}$$

$\forall$-left on the two branches can not introduce the same free variable, for the same reason as above.

Examples of free-variable derivations

$$
\dfrac{\dfrac{\dfrac{\forall x \ldots, p(U) \implies p(a), \forall x\, q(x)}{\forall x \ldots, p(U) \implies \forall x\, p(x), \forall x\, q(x)} \qquad \dfrac{\forall x \ldots, q(U) \implies \forall x\, p(x), q(b)}{\forall x \ldots, q(U) \implies \forall x\, p(x), \forall x\, q(x)}}{\forall x\,(p(x) \lor q(x)), p(U) \lor q(U) \implies \forall x\, p(x), \forall x\, q(x)}}{\forall x\,(p(x) \lor q(x)) \implies \forall x\, p(x), \forall x\, q(x)}
$$

- Every δ-rule application has to introduce a new Skolem symbol, i.e. one that does not yet occur in the derivation.
- Therefore, $\forall$-right in the right branch cannot introduce the same Skolem constant as $\forall$-right in the left branch.
- This is a *stronger* requirement than for δ-rules in LK without free variables, where the introduced constant must not occur in the *conclusion*.
- In this derivation, we don't know which symbols occur in the conclusion until we have instantiated U!

Examples of Objects that are *not* Derivations

$$\dfrac{\dfrac{p(x) \implies p(a)}{p(x) \implies \forall y\, p(y)}\ \forall\text{-right}}{\implies p(x) \rightarrow \forall y\, p(y)}\ \rightarrow\text{-right}$$

Root sequent not closed.

$$\dfrac{\dfrac{\dfrac{\dfrac{\forall x \exists y\, p(x,y), p(U, f(U)) \implies p(f(V), V), \exists x \forall y\, p(y,x)}{\forall x \exists y\, p(x,y), p(U, f(U)) \implies \forall y\, p(y, V), \exists x \forall y\, p(y,x)}\ \forall\text{-right}}{\forall x \exists y\, p(x,y), p(U, f(U)) \implies \exists x \forall y\, p(y,x)}\ \exists\text{-right}}{\forall x \exists y\, p(x,y), \exists y\, p(U, y) \implies \exists x \forall y\, p(y,x)}\ \exists\text{-left}}{\forall x \exists y\, p(x,y) \implies \exists x \forall y\, p(y,x)}\ \forall\text{-left}$$

The two δ-applications introduce the same Skolem symbol.

Closed Derivations

- For an LK-derivation to be a proof, we have to instantiate the free variables in the derivation such that all leaf sequents become axioms.
- We call this closing a derivation.

Definition 2.4 (Closing).

Let π be a free-variable derivation, and let σ be a substitution.

- σ *closes a leaf sequent* $\Gamma \implies \Delta$ *of* π *if there are atomic formulae* $A \in \Gamma$ *and* $B \in \Delta$ *such that* $\sigma(A) = \sigma(B)$.
- σ *closes* π *if* σ *closes all leaf sequents of* π.

Proofs

Definition 2.5 (Proofs).

A free-variable LK proof for a sequent $\Gamma \implies \Delta$ is a pair $\langle \pi, \sigma \rangle$ where

- *π is a derivation with root sequent $\Gamma \implies \Delta$, and*
- *σ is a ground substitution that closes π.*

- A ground substitution is a substitution σ such that $\sigma(x)$ is closed (contains no variables) for all $x \in \mathcal{V}$.
- We restrict closing substitutions to be ground, since this makes the soundness proof *slightly* simpler.
- We will later see that this requirement can be dropped to allow arbitrary closing substitutions.

Examples

Example (1).

Let π *be the derivation*

$$\cfrac{\cfrac{\forall x\, p(x), p(U) \implies \exists x\, p(x), p(V)}{\forall x\, p(x), p(U) \implies \exists x\, p(x)}\ \exists\text{-}right}{\forall x\, p(x) \implies \exists x\, p(x)}\ \forall\text{-}left$$

and let $\sigma = \{U \backslash a, V \backslash a\}$.

- σ *closes the leaf sequent:* $\sigma(p(U)) = p(a) = \sigma(p(V))$.
- σ *closes* π, *since it closes the only leaf sequent.*
- *Thus,* $\langle \pi, \sigma \rangle$ *is a proof for the sequent* $\forall x\, p(x) \implies \exists x\, p(x)$.

Note:

With our current definition of free-variable LK, e.g. $\langle \pi, \sigma' \rangle$ where $\sigma' = \{U \backslash V\}$ is not a proof, since σ' is not ground.

Example (2).

Let π *be the derivation*

$$\dfrac{\dfrac{\forall x\, p(x), p(U) \implies p(a)}{\forall x\, p(x) \implies p(a)}\ \forall\text{-}left \qquad \dfrac{\forall x\, p(x), p(V) \implies p(b)}{\forall x\, p(x) \implies p(b)}\ \forall\text{-}left}{\forall x\, p(x) \implies p(a) \land p(b)}\ \land\text{-}right$$

and let $\sigma = \{U \backslash a, V \backslash b\}$.

- σ *closes the left leaf:* $\sigma(p(U)) = p(a)$.
- σ *closes the right leaf:* $\sigma(p(V)) = p(b)$.
- σ *closes* π, *since it closes both leaf sequents.*
- *Thus,* $\langle \pi, \sigma \rangle$ *is a proof for the sequent* $\forall x\, p(x) \implies p(a) \land p(b)$.

Example (3).

Let π be the derivation

$$\dfrac{\dfrac{\forall x, p(U) \implies p(a), \forall x\, q(x)}{\forall x, p(U) \implies \forall x\, p(x), \forall x\, q(x)}\ \forall\text{-}r \qquad \dfrac{\forall x, q(U) \implies \forall x\, p(x), q(b)}{\forall x, q(U) \implies \forall x\, p(x), \forall x\, q(x)}\ \forall\text{-}r}{\dfrac{\forall x \ldots, p(U) \vee q(U) \implies \forall x\, p(x), \forall x\, q(x)}{\forall x\, (p(x) \vee q(x)) \implies \forall x\, p(x), \forall x\, q(x)}\ \forall\text{-}left}\ \vee\text{-}l$$

- *There is no substitution that closes both leaf sequents, since U can not be instantiated with a and b simultaneously.*
- *There is therefore no proof of $\forall x\, (p(x) \vee q(x)) \implies \forall x\, p(x), \forall x\, q(x)$ based on the derivation π.*
- *Is the root sequent valid. . . ?*

Outline

Soundess Proof: Basic Idea & Difficulty

- LK with free variables is mostly like LK without free variables
- Can we just show that proofs are sound after applying a closing substitution?
- But what if $\sigma(U) = f(\cdots)$ for a U introduced before f?
- After applying σ, the eigenvariable condition of the 'closed' δ rule is violated.
- Example: $\sigma = \{U \backslash a\}$

$$\dfrac{\dfrac{\forall x\, p(x), p(a) \implies p(a)}{\forall x\, p(x), p(a) \implies \forall y\, p(y)}\ \exists\text{-right}}{\forall x\, p(x) \implies \forall y\, p(y)}\ \forall\text{-left}$$

- Remember: new constant was important for soundness of δ rule.
- Defined a new interpretation $\mathcal{I}'$ that is the same as $\mathcal{I}$ for old formulae.
- Now we have $f(\vec{U})$…but how will that help?

Extending the Interpretation

- Let's define all the new interpretations up-front.
- Given a free-variable LK proof π, let

$$f_1, f_2, \dots, f_n$$

  be the Skolem symbols introduced by δ-rules in π, for formulae $\exists/\forall x_1\, A_1, \dots, \exists/\forall x_n\, A_n$, ordered such that $i < j$ whenever f_i is inroduced below f_j on a branch.
- Then A_{i+1} contains only symbols $f_1, \dots, f_i$
- Given an interpretation $\mathcal{I} = (D, \iota)$, we define a series of interpretations $\mathcal{I}_0 = (D, \iota_0), \dots, \mathcal{I}_n = (D, \iota_n)$ such that
  - $\mathcal{I}_0 = \mathcal{I}$
  - Each $\mathcal{I}_{i+1}$ is identical to $\mathcal{I}_i$ on $f_1, \dots f_i$ but defines ι_{i+1} for f_{i+1}.
- We then call $\mathcal{I}_n$ the Skolem extension of $\mathcal{I}$ with respect to π.

Extending the Interpretation (cont.)

- We now define $\mathcal{I}_{i+1}$ based on $\mathcal{I}_i$
- Let $U_1, \ldots U_m$ be the free variables in $\exists/\forall x_{i+1}\, A_{i+1}$.
- Have to define f in $\mathcal{I}_{n+1}$ for all m-tuples of domain elements.
- For any tuple of domain elements $d_1, \ldots d_m \in D$,
- define a variable assignment α with $\alpha(U_j) = d_j$ for $j = 1, \ldots, m$.
- Case 1: $\exists x_{i+1}\, A_{i+1}$ occurs in an antecedent
  - If $v_{\mathcal{I}_i}(\alpha, \exists x_{i+1}\, A_{i+1}) = T$, pick one $d \in D$ with $v_{\mathcal{I}_i}(\alpha\{x \leftarrow d\}, A_{i+1}) = T$, and define $f_{i+1}^{\iota_{i+1}}(d_1, \ldots, d_m) := d$
  - If $v_{\mathcal{I}_i}(\alpha, \exists x_{i+1}\, A_{i+1}) = F$, define $f_{i+1}^{\iota_{i+1}}(d_1, \ldots, d_m)$ to an arbitrary domain element.
- Case 2: $\forall x_{i+1}\, A_{i+1}$ occurs in a succedent
  - If $v_{\mathcal{I}_i}(\alpha, \forall x_{i+1}\, A_{i+1}) = F$, pick one $d \in D$ with $v_{\mathcal{I}_i}(\alpha\{x \leftarrow d\}, A_{i+1}) = F$, and define $f_{i+1}^{\iota_{i+1}}(d_1, \ldots, d_m) := d$
  - If $v_{\mathcal{I}_i}(\alpha, \exists x_{i+1}\, A_{i+1}) = T$, define $f_{i+1}^{\iota_{i+1}}(d_1, \ldots, d_m)$ to an arbitrary domain element.

Properties of the Skolem Extension

- Let $\mathcal{I}_n$ be the Skolem extension of $\mathcal{I}$ with respect to a derivation π.
- Then $\mathcal{I}_n$ satisfies the root sequent of π iff $\mathcal{I}$ does.
  - Because none of the Skolem symbols f_i occur in the root
- For any δ rule that introduces $f(\vec{U})$ for a formula $\exists/\forall x\, A$ in π, and for any variable assignemnt α:

$$v_{\mathcal{I}_n}(\alpha, \exists/\forall x\, A) = v_{\mathcal{I}_n}(\alpha, A[x \backslash f(\vec{U})])$$

Example of Skolem Extension

- Intuition: think of $p(x, y, z)$ as $x + y = z$ on $\mathbb{Z}$
- There is a neutral element (0) and an inverse $(-x)$ for every x

$$\dfrac{\dfrac{\dfrac{\dfrac{p(a, U, U), p(U, f(U), a) \implies}{p(a, U, U), \exists z\, p(U, z, a) \implies}}{p(a, U, U) \land \exists z\, p(U, z, a) \implies}}{\forall y (p(a, y, y) \land \exists z\, p(y, z, a)) \implies}}{\exists x \forall y (p(x, y, y) \land \exists z\, p(y, z, x)) \implies}$$

- Start with $\mathcal{I}_0 = (\mathbb{Z}, \iota)$ with $p^\iota = \{\langle x, y, z\rangle \mid x + y = z\}$.
- $\mathcal{I}_1$ has to interpret a.
- $v_{\mathcal{I}_0}(\{x \leftarrow 0\}, \forall y (p(x, y, y) \land \exists z\, p(y, z, x))) = T$
- So we set $a^{\iota_1} = 0$
- $\mathcal{I}_2$ has to interpret f on all domain elements.
- For any $k \in \mathbb{Z}$, $v_{\mathcal{I}_1}(\{U \leftarrow k\}\{z \leftarrow (-k)\}, p(U, z, a)) = T$
- So we set $f^{\iota_2}(k) = -k$

Soundness for free-variable LK

Similarly to closed LK, we show the following lemmas:

1. All LK-rules preserve σ-falsifiability upwards for all Skolem extensions
2. An LK-derivation with a σ-falsifiable root sequent has at least one σ-falsifiable leaf sequent
3. If σ closes a leaf $\Gamma \implies \Delta$, then $\sigma(\Gamma \implies \Delta)$ is valid

Where the notion of σ-falsifiability will be defined.

Finally, we use these lemmas to show the soundness theorem.

Falsifiability with Free Variables

Definition 3.1.

Given a ground substitution σ *such that* $\sigma(\Gamma \implies \Delta)$ *is closed, an interpretation* $\mathcal{I}$ *σ-falsifies* $\Gamma \implies \Delta$ *if it falsifies* $\sigma(\Gamma \implies \Delta)$*.*

$\Gamma \implies \Delta$ *is called σ-falsifiable if there is an interpretation that σ-falsifies it.*

Preservation of σ-Falsifiability

Definition 3.2.

An LK*-rule θ preserves σ-falsifiability (upwards) by an inprerretation $\mathcal{I}$ if whenever the conclusion w of an instance $\frac{w_1 \cdots w_n}{w}$ of θ is σ-falsified by $\mathcal{I}$, then also at least one of the premises w_i is σ-falsified by $\mathcal{I}$.*

In our proof, the falsifying interpretation will remain the same, since we already did all necessary modification in the Skolem extension.

Lemma 3.1.

All LK*-rules preserve σ-falsifiability by any interpretation $\mathcal{I}_n$ that is the Skolem extension of an interpretation $\mathcal{I}$.*

- The proofs for the propositional connectives are almost identical. We show $\rightarrow$-left.
- It remains to show that also the $\forall$ and $\exists$ rules preserve σ-falsifiability.

Proof for $\rightarrow$-left

Proof for $\rightarrow$-left.

$$\frac{\Gamma \implies A, \Delta \qquad \Gamma, B \implies \Delta}{\Gamma, A \rightarrow B \implies \Delta} \rightarrow\text{-left}$$

- Assume that $\mathcal{I}_n$ σ-falsifies the conclusion.
- Then $\mathcal{I}_n$ satisfies $\sigma(\Gamma) \cup \{\sigma(A \rightarrow B)\}$ and falsifies all formlae in $\sigma(\Delta)$.
- Since $\mathcal{I}_n$ satisfies $\sigma(A) \rightarrow \sigma(B)$, by definition of model semantics,
  (1) $\mathcal{I}_n \not\models \sigma(A)$, or
  (2) $\mathcal{I}_n \models \sigma(B)$.
- In case (1), $\mathcal{I}_n$ σ-falsifies the left premisse.
- In case (2), $\mathcal{I}_n$ σ-falsifies the right premisse.

□

Proof: $\forall$-left preserves σ-falsifiability

$$\frac{\Gamma, \forall x\, A, A[x \backslash U] \implies \Delta}{\Gamma, \forall x\, A \implies \Delta}\ \forall\text{-left} \qquad U \text{ is a new free variable}$$

- Assume that $\mathcal{I}_n = (D, \iota)$ σ-falsifies the conclusion $\Gamma, \forall x\, A \implies \Delta$.
- $\mathcal{I}_n$ makes all formulae in $\sigma(\Gamma) \cup \{\sigma(\forall x A)\}$ true and all formulae in $\sigma(\Delta)$ false.
- It suffices to show that $\mathcal{I}_n \models \sigma(A[x \backslash U])$. Then, the premiss is falsified by $\mathcal{I}_n$.
- Since $\mathcal{I}_n \models \sigma(\forall x\, A)$, and $\sigma(\forall x\, A) = \forall x\, \sigma_x(A)$, we know that $v_{\mathcal{I}_n}(\alpha\{x \leftarrow d\}, \sigma_x(A)) = T$ for all $d \in D$ and any α.
- In particular, $v_{\mathcal{I}_n}(\alpha\{x \leftarrow v_{\mathcal{I}_n}(\alpha, \sigma(U))\}, \sigma_x(A)) = T$
- By the substitution lemma: $v_{\mathcal{I}_n}(\alpha, \sigma_x(A)[x \backslash \sigma(U)]) = T$
- Which is the same as $v_{\mathcal{I}_n}(\alpha, \sigma(A[x \backslash U])) = T$
- And therefore: $\mathcal{I}_n \models \sigma(A[x \backslash U])$.

Proof: $\exists$-left preserves σ-falsifiability

$$\frac{\Gamma, A[x\backslash f(\vec{U})] \implies \Delta}{\Gamma, \exists x\, A \implies \Delta}\ \exists\text{-left}$$

f is a function that does not occur in the conculsion and $\vec{U}$ are the free variables in A.

- Assume that $\mathcal{I}_n = (D, \iota)$ σ-falsifies the conclusion $\Gamma, \exists x\, A \implies \Delta$.
- $\mathcal{I}_n$ makes all formulae in $\sigma(\Gamma) \cup \{\sigma(\exists x A)\}$ true and alle formulae in $\sigma(\Delta)$ false.
- Define a variable assignment α with $\alpha(U) = v_{\mathcal{I}_n}(\sigma(U))$ for all U in $\vec{U}$.
- Since free variables are always new, x is not in $\vec{U}$.
- By the subst. lemma, $v_{\mathcal{I}_n}(\alpha, \exists x\, A) = T$
- By construction of the Skolem extension, $v_{\mathcal{I}_n}(\alpha, A[x\backslash f(\vec{U})]) = T$
- And therefore $\mathcal{I}_n \models \sigma(A[x\backslash f(\vec{U})])$
- So $\mathcal{I}_n$ σ-falsifies the premisse.

Substitution Lemma for Multiple Variables

A variant of the Substitution Lemma, for multiple variables, but only ground substitutions

Lemma 3.2.

Let $\mathcal{I}$ be an interpretation, and let σ be a ground substitution. Let α be a variable assignment such that $\alpha(x) = v_{\mathcal{I}}(\sigma(x))$ for all variables x with $\sigma(x) \neq x$. If A is a formula such that $\sigma(A)$ is closed, then $v_{\mathcal{I}}(\alpha, A) = v_{\mathcal{I}}(\sigma(A))$

Proof.

Exercise. □

Soundness for free-variable LK

Similarly to closed LK, we show the following lemmas:

1. All LK-rules preserve σ-falsifiability upwards for all Skolem extensions
2. An LK-derivation with a σ-falsifiable root sequent has at least one σ-falsifiable leaf sequent
3. If σ closes a leaf $\Gamma \implies \Delta$, then $\sigma(\Gamma \implies \Delta)$ is valid

Where the notion of σ-falsifiability will be defined.

Finally, we use these lemmas to show the soundness theorem.

Existence of a σ-falsifiable leaf sequent

Lemma 3.3.

If the root sequent $\mathcal{I}$ of an an LK*-derivation is σ-falsifiable, then at least one of the leaf sequents is σ-falsifiable.*

- If $\sigma(\Gamma \implies \Delta)$ is falsified by $\mathcal{I}$, it is also falsified by the Skolem extension $\mathcal{I}_n$.
- We can inductively use that σ-falsifiability by Skolem extensions is preserved.

Soundness for free-variable LK

Similarly to closed LK, we show the following lemmas:

1. All LK-rules preserve σ-falsifiability upwards for all Skolem extensions
2. An LK-derivation with a σ-falsifiable root sequent has at least one σ-falsifiable leaf sequent
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Where the notion of σ-falsifiability will be defined.

Finally, we use these lemmas to show the soundness theorem.

Validity of closed leaves

Lemma 3.4.

If σ closes a leaf $\Gamma \implies \Delta$, then $\sigma(\Gamma \implies \Delta)$ is valid.

- If σ closes a leaf $\Gamma \implies \Delta$, then there are atomic formulae $A \in \Gamma$ and $B \in \Delta$ such that $\sigma(A) = \sigma(B)$.
- Any interpretation that satisfies the antecedent satisfies $\sigma(A)$.
- Therefore, the same formula $\sigma(B)$ is satisfied in the succedent.
- Thus $\sigma(\Gamma \implies \Delta)$ is valid.

Theorem 3.1 (Soundness).

The free-variable sequent calculus LK *is sound.*

Proof.

- Let $\langle \pi, \sigma \rangle$ be a proof for $\Gamma \implies \Delta$.
- Assume for the sake of contradiction that $\Gamma \implies \Delta$ is not valid, but falsifiable by some $\mathcal{I}$.
- $\Gamma \implies \Delta$ is closed so $\mathcal{I}$ falsifies $\sigma(\Gamma \implies \Delta)$.
- $\Gamma \implies \Delta$ is σ-falsifiable.
- One of the leaf sequents $\Gamma' \implies \Delta'$ is σ-falsifiable (Lemma)
- But if σ closes π then $\sigma(\Gamma' \implies \Delta')$ must be valid.
- Contradiction!

□

Outline

Completeness

Theorem 4.1 (Completeness).

If $\Gamma \implies \Delta$ *is valid, then it is provable in free-variable* LK.

To prove *completeness*, we show the equivalent statement:

Lemma 4.1 (Modelleksistens).

If $\Gamma \implies \Delta$ *is not provable in LK, then it is falsifiable.*

Remember:

- All formulae in $\Gamma \implies \Delta$ are closed
- No need to worry about a closing substitution at this point.

Reminder: model existence for ground LK

The proof of model existence for free-variable LK is based on that for ground LK. Main points of the proof for ground LK:

- We assume that a sequent $\Gamma \implies \Delta$ is not provable in ground LK.
  - I.e. derivations from root $\Gamma \implies \Delta$ have an open branch.
- We construct a fair limit derivation for $\Gamma \implies \Delta$ where every open branch $\mathcal{B}$ has the following properties:
  - all α-, β- and δ-formulae are treated by a respective rule on $\mathcal{B}$, and
  - if $\mathcal{B}$ contains a γ-formula $\forall/\exists x\, A$, then $A[x \backslash t]$ is introduced on $\mathcal{B}$ by a γ rule for every term t in the Herbrand universe of $\mathcal{B}$.
  - *"All possible rule applications have been tried on all open branches."*
- Since $\Gamma \implies \Delta$ is not provable, the limit derivation must contain an open branch $\mathcal{B}$ (by König's lemma).

Reminder: model existence for ground LK, cont.

- We use the information in $\mathcal{B}$ to construct a Herbrand model $\mathcal{I}$ as follows:
  - The domain of $\mathcal{I}$ is the Herbrand universe of $\mathcal{B}$.
  - For closed terms t on $\mathcal{B}$: $v_{\mathcal{I}}(t) = t$
  - For n-ary relation symbols p on $\mathcal{B}$:
    $\langle t_1, \ldots, t_n \rangle \in p^{\iota} \Leftrightarrow p(t_1, \ldots, t_n) \in \mathcal{B}^{\top}$
- *Remember: the Herbrand universe of a branch is the set of all closed terms that can be constructed from the constant and function symbols on the branch.*
- We then show by structural induction on formulae in $\mathcal{B}$ that $\mathcal{I}$ satisfies all formulae in $\mathcal{B}^{\top}$ and falsifies all formulae in $\mathcal{B}^{\perp}$.
- It follows that $\mathcal{I}$ falsifies $\Gamma \implies \Delta$, since $\Gamma \subseteq \mathcal{B}^{\top}$ og $\Delta \subseteq \mathcal{B}^{\perp}$.

Model Existence for Free-variable LK

- To prove model existence for free-variable LK, we will
  - construct a fair limit derivation
  - use a specific ground substitution on all formulae such that all free variables are replaced by closed terms.
- We can then construct a Herbrand-modell from a grounded open branch as for ground LK, and use the same induction argument.
- The γ-rules of free-variable LK introduce free variables instead of terms, so we need a new definition of fair limit derivations.
- We have to choose the substitution in such a way that a grounded open branch will have the same properteis w.r.t. γ-formlae as in ground LK.

Fairness for free-variable LK

Definition 4.1 (Fair derivations).

A limit derivation is fair if each branch has the following properties:

1. *If an α, β, or δ formula occurs, then the corresponding* LK *rule is applied to the formula on that branch.*
2. *If a γ formula occurs, then the $\forall$-left, resp. $\exists$-right rules are applied to the formula on that branch infinitely often on that branch, introducing infinitely many free variables.*

Fair Substitutions

- This is a fair limit derivation
- $\forall x\, p(x)$ introduces infinitely many free variables U_i.
- Let σ be such that $\sigma(U_i) = a$ for all U_i.

$$\cfrac{\cfrac{\cfrac{\cfrac{\vdots}{\forall x\, p(x), p(U_1), p(U_2), p(U_3) \implies q(fa)}}{\forall x\, p(x), p(U_1), p(U_2) \implies q(fa)}}{\forall x\, p(x), p(U_1) \implies q(fa)}}{\forall x\, p(x) \implies q(fa)}$$

- The derivation has only one branch $\mathcal{B}$. If we apply σ to all formulae on $\mathcal{B}$, then all $p(U_i)$ become $p(a)$.
- The Herbrand universe of $\sigma(\mathcal{B})$ is $a, fa, ffa, fffa, \ldots$ There are terms t in the Herbrand universe such that $p()t)$ is not on $\sigma(\mathcal{B})$, e.g. $t = fa$.
- The Herbrand model generated from $\sigma(\mathcal{B})$ will therefore not make $\forall x\, p(x)$ true.

Fair Substitutions (cont.)

- Recursively define τ such that
  - $\tau(U_1) = a$, and
  - $\tau(U_{i+1}) = f\tau(U_i)$.
- We apply τ to $\mathcal{B}$.

$$\dfrac{\dfrac{\dfrac{\vdots}{\forall x\, p(x), p(U_1), p(U_2), p(U_3) \implies q(fa)}}{\forall x\, p(x), p(U_1), p(U_2) \implies q(fa)}}{\dfrac{\forall x\, p(x), p(U_1) \implies q(fa)}{\forall x\, p(x) \implies q(fa)}}$$

- We now have $p(t) \in \tau(\mathcal{B})$ for all terms t in the Herbrand universe of $\tau(\mathcal{B})$.
- Therefore, the Herbrand model generated from $\tau(\mathcal{B})$ will satisfy $\forall x\, p(x)$.
- We call τ a fair substitution.

Fair Substitutions (cont.)

Definition 4.2 (Fair substitution).

Let π be a fair limit derivation, and let $\mathcal{B}$ be a branch of π. let σ be a substitution.

- *σ is fair w.r.t. a γ-formulae $\forall/\exists x\, A$ on $\mathcal{B}$ if for all terms t in the Herbrand-universe of $\mathcal{B}$, there is a γ application on $\mathcal{B}$ that introduces $A[x\backslash U]$ such that $\sigma(U) = t$.*
- *σ is fair w.r.t. the branch $\mathcal{B}$ if σ is fair w.r.t. all γ-formlae in $\mathcal{B}$.*
- *σ is fair w.r.t. the derivation π if σ is fair w.r.t. every branch of π.*

Model existence for free variable LK – Proof

- Assume that $\Gamma \implies \Delta$ is not provable in free-variable LK.
  - For all derivations from the root sequent $\Gamma \implies \Delta$, there is no substitution that closes all branches.
- We construct a fair limit derivation π with root sequent $\Gamma \implies \Delta$.
- We construct a substitution σ that is fair w.r.t. π and that replaces all free variables in π by ground terms.
- Since σ does not close π, there must be a branch $\mathcal{B}$ in π that is not closed by σ (by König's Lemma).
- We apply σ to all formulae on $\mathcal{B}$. Note that $\sigma(\mathcal{B})$ only contains closed formlae. We can now apply the model existence theorem for ground LK.
- Note that $\Gamma \implies \Delta$ is closed. The Herbrand model generated from $\sigma(\mathcal{B})$ will therefore falsify $\Gamma \implies \Delta$ independently of σ.