

# IN3070/4070 – Logic – Autumn 2019

## Lecture 7: Free-variable calculi

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## Today's Plan

- ▶ Introduction
- ▶ The Free-variable Sequent Calculus
- ▶ Soundness
- ▶ Completeness

## Outline

- ▶ Introduction
- ▶ The Free-variable Sequent Calculus
- ▶ Soundness
- ▶ Completeness

## Smullyan's categories: $\alpha/\beta/\gamma/\delta$

- ▶ Many similar cases in proofs and implementations
- ▶ Categorise rules, rule applications, and formulae

### $\alpha$ -rules

Propositional, one branch, e.g.

$$\frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta} \wedge\text{-left}$$

### $\beta$ -rules

Propositional, splitting, e.g.

$$\frac{\Gamma \Rightarrow A, \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \rightarrow B \Rightarrow \Delta}$$

### $\gamma$ -rules

Apply for all terms  $t$ , e.g.

$$\frac{\Gamma, A[x \setminus t], \forall x A \Rightarrow \Delta}{\Gamma, \forall x A \Rightarrow \Delta} \forall\text{-left}$$

### $\delta$ -rules

Introduce new constant  $c$ , e.g.

$$\frac{\Gamma, A[x \setminus c], \Rightarrow \Delta}{\Gamma, \exists x A \Rightarrow \Delta} \exists\text{-left}$$

## $\gamma$ -rules – The Biggest Problem in 1st-Order Proof Search

- ▶ Let us look at the  $\gamma$ -rules of LK:

$$\frac{\Gamma, \forall x A, A[x \setminus t] \Rightarrow \Delta}{\Gamma, \forall x A \Rightarrow \Delta} \forall\text{-left} \quad \frac{\Gamma \Rightarrow \Delta, \exists x A, A[x \setminus t]}{\Gamma \Rightarrow \Delta, \exists x A} \exists\text{-right}$$

- ▶ We can substitute an arbitrary closed term  $t$  for  $x$ .
- ▶ For a complete proof search we **must** instantiate every  $\gamma$ -formula with **every** term in the Herbrand universe.
- ▶ We can enumerate terms in the Herbrand universe and instantiate the  $\gamma$ -formulae in that order.
- ▶ But which order should we use to **find a proof as early as possible**?

$$\frac{\forall x p(x), p(a), \dots, p(\text{ffa}) \Rightarrow p(\text{ffa}), q(ga)}{\vdots} \quad \frac{\forall x p(x), p(a) \Rightarrow p(\text{ffa}), q(ga)}{\forall x p(x) \Rightarrow p(\text{ffa}), q(ga)} \quad \begin{matrix} a, fa, ga, ffa, fga, \dots, fffa, \dots \\ 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad i \end{matrix}$$

## Postponing the choice of $\gamma$ -terms

- ▶ A better idea: postpone the choice of instantiation until later.
- ▶ Let  $\gamma$ -rules introduce **free variables** as placeholders.

$$\frac{\frac{U \setminus a}{\forall x p(x), p(U) \Rightarrow p(a)} \quad \frac{V \setminus b}{\forall x p(x), p(V) \Rightarrow p(b)}}{\forall x p(x) \Rightarrow p(a)} \quad \frac{\forall x p(x), p(V) \Rightarrow p(b)}{\forall x p(x) \Rightarrow p(b)} \quad \frac{\forall x p(x) \Rightarrow p(a) \quad \forall x p(x) \Rightarrow p(b)}{\forall x p(x) \Rightarrow p(a) \wedge p(b)}$$

- ▶ Substitute terms for free variables that make leaves into axioms
- ▶ Which substitutions can we use to make leaves into axioms?
- ▶ That's a **unification problem**!

## $\delta$ -rules

- ▶ If we use free variables in  $\gamma$ -rules we run into a problem with the  $\delta$ -rules
- ▶ How can we ensure that the constant they introduce is *new* if we don't know what we will substitute for the free variables?

~~$$\frac{\frac{U \setminus b, V \setminus a}{I(U, a) \Rightarrow I(b, V)} \quad \frac{\exists y I(U, y) \Rightarrow \forall x I(x, V)}{\forall x \exists y I(x, y) \Rightarrow \exists y \forall x I(x, y)}}{\forall x \exists y I(x, y) \Rightarrow \exists y \forall x I(x, y)} \quad \frac{\text{can't be closed} \quad \frac{I(U, f(U)) \Rightarrow I(g(V), V)}{\exists y I(U, y) \Rightarrow \forall x I(x, V)}}{\forall x \exists y I(x, y) \Rightarrow \exists y \forall x I(x, y)}$$~~

- ▶ We let  $\delta$  rules introduce a **Skolem term**:  
 $f(U_1, \dots, U_n)$ ,  
 where  $f$  is a new function symbol, called a **Skolem function**, and  $U_1, \dots, U_n$  are all variables that occur free in the  $\delta$ -formula.
- ▶ This will ensure soundness of the  $\delta$ -rule no matter how the free variables are instantiated.

## Outline

- ▶ Introduction
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## Free-variable Sequent Calculus

- ▶ The choice of a term in  $\gamma$ -rules is postponed by instantiating with a **free** variable.
- ▶ Introducing free variables in  $\gamma$ -rules means that  $\delta$ -rules must introduce **Skolem terms**.
- ▶ Unification allows us to find a substitution that will replace free variables by terms that close the derivation.

$$\frac{U \setminus a, V \setminus f(a)}{\frac{p(U, f(U)) \implies p(a, V)}{\exists y p(U, y) \implies p(a, V)}} \quad \frac{\exists y p(U, y) \implies p(a, V)}{\forall x \exists y p(x, y) \implies \exists x p(a, x)}$$

 $\gamma$ -rules**Definition 2.1 ( $\gamma$ -rules in free-variable LK).**

The  $\gamma$ -rules in free variable LK are:

$$\frac{\Gamma, \forall x A, A[x \setminus U] \implies \Delta}{\Gamma, \forall x A \implies \Delta} \forall\text{-left} \quad \frac{\Gamma \implies \Delta, \exists x A, A[x \setminus U]}{\Gamma \implies \Delta, \exists x A} \exists\text{-right}$$

$U$  is a **new** free variable

- ▶ By *new*, we mean that it does not previously occur in the derivation.

 $\delta$ -rules**Definition 2.2 ( $\delta$ -rules in free-variable LK).**

The  $\delta$ -rules in free-variable LK are:

$$\frac{\Gamma, A[x \setminus f(\vec{U})] \implies \Delta}{\Gamma, \exists x A \implies \Delta} \exists\text{-left} \quad \frac{\Gamma \implies \Delta, A[x \setminus f(\vec{U})]}{\Gamma \implies \Delta, \forall x A} \forall\text{-right}$$

$f$  is a **new** Skolem function

$\vec{U} = U_1, \dots, U_n$  are the free variables occurring in  $\exists x A$ .

- ▶ By *new*, we mean that  $f$  does not previously occur in the derivation.

## Rules and Derivations

**Definition 2.3 (Rules of free-variable LK).**

The rules of free-variable LK are

- ▶ the  $\gamma$ - and  $\delta$ -rules for free variables, and
- ▶ the  $\alpha$ - and  $\beta$ -rules for propositional LK.
- ▶ **Free-variable derivations** are defined inductively:
  - ▶ The **induction base** consists of **closed** sequents.
  - ▶ The derivations are closed under applications of the free-variable LK rules.
- ▶ So we still allow *only closed formulae* in the root sequent.
- ▶ Formulae later on are not necessarily closed.

### Examples of free-variable derivations

$$\frac{\frac{\forall x p(x), p(U) \implies \exists x p(x), p(V)}{\forall x p(x), p(U) \implies \exists x p(x)}}{\forall x p(x) \implies \exists x p(x)} \quad \exists\text{-right can not introduce } U, \text{ since it already occurs in the derivation.}$$

$$\frac{\frac{\forall x p(x), p(U) \implies p(a)}{\forall x p(x) \implies p(a)} \quad \frac{\forall x p(x), p(V) \implies p(b)}{\forall x p(x) \implies p(b)}}{\forall x p(x) \implies p(a) \wedge p(b)}$$

$\forall$ -left on the two branches can not introduce the same free variable, for the same reason as above.

### Examples of free-variable derivations

$$\frac{\frac{\forall x \dots, p(U) \implies p(a), \forall x q(x)}{\forall x \dots, p(U) \implies \forall x p(x), \forall x q(x)} \quad \frac{\forall x \dots, q(U) \implies \forall x p(x), q(b)}{\forall x \dots, q(U) \implies \forall x p(x), \forall x q(x)}}{\frac{\forall x (p(x) \vee q(x)), p(U) \vee q(U) \implies \forall x p(x), \forall x q(x)}{\forall x (p(x) \vee q(x)) \implies \forall x p(x), \forall x q(x)}}$$

- ▶ Every  $\delta$ -rule application has to introduce a **new** Skolem symbol, i.e. one that does **not** yet occur in the derivation.
- ▶ Therefore,  $\forall$ -right in the right branch **cannot** introduce the same Skolem constant as  $\forall$ -right in the left branch.
- ▶ This is a *stronger* requirement than for  $\delta$ -rules in LK without free variables, where the introduced constant must not occur in the *conclusion*.
- ▶ In this derivation, we don't know which symbols occur in the conclusion until we have instantiated  $U$ !

### Examples of Objects that are *not* Derivations

~~$$\frac{\frac{p(x) \implies p(a)}{p(x) \implies \forall y p(y)} \quad \forall\text{-right}}{\implies p(x) \rightarrow \forall y p(y)} \rightarrow\text{-right}$$

Root sequent not closed.~~

~~$$\frac{\frac{\forall x \exists y p(x, y), p(U, f(U)) \implies p(f(V), V), \exists x \forall y p(y, x)}{\forall x \exists y p(x, y), p(U, f(U)) \implies \forall y p(y, V), \exists x \forall y p(y, x)} \quad \forall\text{-right}}{\forall x \exists y p(x, y), p(U, f(U)) \implies \exists x \forall y p(y, x)} \quad \exists\text{-right}}{\frac{\forall x \exists y p(x, y), \exists y p(U, y) \implies \exists x \forall y p(y, x)}{\forall x \exists y p(x, y) \implies \exists x \forall y p(y, x)} \quad \forall\text{-left}} \quad \exists\text{-left}$$~~

The two  $\delta$ -applications introduce the same Skolem symbol.

### Closed Derivations

- ▶ For an LK-derivation to be a proof, we have to instantiate the free variables in the derivation such that all leaf sequents become axioms.
- ▶ We call this **closing** a derivation.

#### Definition 2.4 (Closing).

Let  $\pi$  be a free-variable derivation, and let  $\sigma$  be a substitution.

- ▶  $\sigma$  **closes** a leaf sequent  $\Gamma \implies \Delta$  of  $\pi$  if there are atomic formulae  $A \in \Gamma$  and  $B \in \Delta$  such that  $\sigma(A) = \sigma(B)$ .
- ▶  $\sigma$  **closes**  $\pi$  if  $\sigma$  closes all leaf sequents of  $\pi$ .

## Proofs

## Definition 2.5 (Proofs).

A *free-variable LK proof* for a sequent  $\Gamma \Rightarrow \Delta$  is a pair  $\langle \pi, \sigma \rangle$  where

- ▶  $\pi$  is a derivation with root sequent  $\Gamma \Rightarrow \Delta$ , and
  - ▶  $\sigma$  is a *ground* substitution that closes  $\pi$ .
- ▶ A *ground substitution* is a substitution  $\sigma$  such that  $\sigma(x)$  is closed (contains no variables) for all  $x \in \mathcal{V}$ .
  - ▶ We restrict closing substitutions to be *ground*, since this makes the soundness proof *slightly* simpler.
  - ▶ We will later see that this requirement can be dropped to allow arbitrary closing substitutions.

## Examples

## Example (1).

Let  $\pi$  be the derivation

$$\frac{\frac{\forall x p(x), p(U) \Rightarrow \exists x p(x), p(V)}{\forall x p(x), p(U) \Rightarrow \exists x p(x)} \exists\text{-right}}{\forall x p(x) \Rightarrow \exists x p(x)} \forall\text{-left}$$

and let  $\sigma = \{U \setminus a, V \setminus a\}$ .

- ▶  $\sigma$  closes the leaf sequent:  $\sigma(p(U)) = p(a) = \sigma(p(V))$ .
- ▶  $\sigma$  closes  $\pi$ , since it closes the only leaf sequent.
- ▶ Thus,  $\langle \pi, \sigma \rangle$  is a *proof* for the sequent  $\forall x p(x) \Rightarrow \exists x p(x)$ .

## Note:

With our current definition of free-variable LK, e.g.  $\langle \pi, \sigma' \rangle$  where  $\sigma' = \{U \setminus V\}$  is *not* a proof, since  $\sigma'$  is not ground.

## Example (2).

Let  $\pi$  be the derivation

$$\frac{\frac{\forall x p(x), p(U) \Rightarrow p(a)}{\forall x p(x) \Rightarrow p(a)} \forall\text{-left} \quad \frac{\forall x p(x), p(V) \Rightarrow p(b)}{\forall x p(x) \Rightarrow p(b)} \forall\text{-left}}{\forall x p(x) \Rightarrow p(a) \wedge p(b)} \wedge\text{-right}$$

and let  $\sigma = \{U \setminus a, V \setminus b\}$ .

- ▶  $\sigma$  closes the left leaf:  $\sigma(p(U)) = p(a)$ .
- ▶  $\sigma$  closes the right leaf:  $\sigma(p(V)) = p(b)$ .
- ▶  $\sigma$  closes  $\pi$ , since it closes both leaf sequents.
- ▶ Thus,  $\langle \pi, \sigma \rangle$  is a *proof* for the sequent  $\forall x p(x) \Rightarrow p(a) \wedge p(b)$ .

## Example (3).

Let  $\pi$  be the derivation

$$\frac{\frac{\forall x, p(U) \Rightarrow p(a), \forall x q(x)}{\forall x, p(U) \Rightarrow \forall x p(x), \forall x q(x)} \forall\text{-r} \quad \frac{\forall x, q(U) \Rightarrow \forall x p(x), q(b)}{\forall x, q(U) \Rightarrow \forall x p(x), \forall x q(x)} \forall\text{-l}}{\forall x \dots, p(U) \vee q(U) \Rightarrow \forall x p(x), \forall x q(x)} \forall\text{-left}}{\forall x (p(x) \vee q(x)) \Rightarrow \forall x p(x), \forall x q(x)} \forall\text{-left}$$

- ▶ There is no substitution that closes both leaf sequents, since  $U$  *can not* be instantiated with  $a$  and  $b$  simultaneously.
- ▶ There is therefore no proof of  $\forall x (p(x) \vee q(x)) \Rightarrow \forall x p(x), \forall x q(x)$  based on the derivation  $\pi$ .
- ▶ Is the root sequent valid...?

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## Soundness Proof: Basic Idea &amp; Difficulty

- ▶ LK with free variables is mostly like LK without free variables
- ▶ Can we just show that proofs are sound after applying a closing substitution?
- ▶ But what if  $\sigma(U) = f(\dots)$  for a  $U$  introduced before  $f$ ?
- ▶ After applying  $\sigma$ , the eigenvariable condition of the 'closed'  $\delta$  rule is violated.
- ▶ Example:  $\sigma = \{U \setminus a\}$

$$\frac{\forall x p(x), p(a) \implies p(a)}{\forall x p(x), p(a) \implies \forall y p(y)} \exists\text{-right}$$

$$\frac{\forall x p(x), p(a) \implies \forall y p(y)}{\forall x p(x) \implies \forall y p(y)} \forall\text{-left}$$

- ▶ Remember: new constant was important for soundness of  $\delta$  rule.
- ▶ Defined a new interpretation  $\mathcal{I}'$  that is the same as  $\mathcal{I}$  for old formulae.
- ▶ Now we have  $f(\vec{U})$ ... but how will that help?

## Extending the Interpretation

- ▶ Let's define all the new interpretations up-front.
- ▶ Given a free-variable LK proof  $\pi$ , let

$$f_1, f_2, \dots, f_n$$

be the Skolem symbols introduced by  $\delta$ -rules in  $\pi$ , for formulae  $\exists/\forall x_1 A_1, \dots, \exists/\forall x_n A_n$ , ordered such that  $i < j$  whenever  $f_i$  is introduced below  $f_j$  on a branch.

- ▶ Then  $A_{i+1}$  contains only symbols  $f_1, \dots, f_i$
- ▶ Given an interpretation  $\mathcal{I} = (D, \iota)$ , we define a series of interpretations  $\mathcal{I}_0 = (D, \iota_0), \dots, \mathcal{I}_n = (D, \iota_n)$  such that
  - ▶  $\mathcal{I}_0 = \mathcal{I}$
  - ▶ Each  $\mathcal{I}_{i+1}$  is identical to  $\mathcal{I}_i$  on  $f_1, \dots, f_i$  but defines  $\iota_{i+1}$  for  $f_{i+1}$ .
- ▶ We then call  $\mathcal{I}_n$  the **Skolem extension** of  $\mathcal{I}$  with respect to  $\pi$ .

## Extending the Interpretation (cont.)

- ▶ We now define  $\mathcal{I}_{i+1}$  based on  $\mathcal{I}_i$
- ▶ Let  $U_1, \dots, U_m$  be the free variables in  $\exists/\forall x_{i+1} A_{i+1}$ .
- ▶ Have to define  $f$  in  $\mathcal{I}_{n+1}$  for all  $m$ -tuples of domain elements.
- ▶ For any tuple of domain elements  $d_1, \dots, d_m \in D$ ,
- ▶ define a variable assignment  $\alpha$  with  $\alpha(U_j) = d_j$  for  $j = 1, \dots, m$ .
- ▶ Case 1:  $\exists x_{i+1} A_{i+1}$  occurs in an antecedent
  - ▶ If  $v_{\mathcal{I}_i}(\alpha, \exists x_{i+1} A_{i+1}) = T$ , pick one  $d \in D$  with  $v_{\mathcal{I}_i}(\alpha \{x \leftarrow d\}, A_{i+1}) = T$ , and define  $f_{i+1}^{i+1}(d_1, \dots, d_m) := d$
  - ▶ If  $v_{\mathcal{I}_i}(\alpha, \exists x_{i+1} A_{i+1}) = F$ , define  $f_{i+1}^{i+1}(d_1, \dots, d_m)$  to an arbitrary domain element.
- ▶ Case 2:  $\forall x_{i+1} A_{i+1}$  occurs in a succedent
  - ▶ If  $v_{\mathcal{I}_i}(\alpha, \forall x_{i+1} A_{i+1}) = F$ , pick one  $d \in D$  with  $v_{\mathcal{I}_i}(\alpha \{x \leftarrow d\}, A_{i+1}) = F$ , and define  $f_{i+1}^{i+1}(d_1, \dots, d_m) := d$
  - ▶ If  $v_{\mathcal{I}_i}(\alpha, \forall x_{i+1} A_{i+1}) = T$ , define  $f_{i+1}^{i+1}(d_1, \dots, d_m)$  to an arbitrary domain element.

## Properties of the Skolem Extension

- ▶ Let  $\mathcal{I}_n$  be the Skolem extension of  $\mathcal{I}$  with respect to a derivation  $\pi$ .
- ▶ Then  $\mathcal{I}_n$  satisfies the root sequent of  $\pi$  iff  $\mathcal{I}$  does.
  - ▶ Because none of the Skolem symbols  $f_i$  occur in the root
- ▶ For any  $\delta$  rule that introduces  $f(\vec{U})$  for a formula  $\exists/\forall x A$  in  $\pi$ , and for any variable assignment  $\alpha$ :

$$v_{\mathcal{I}_n}(\alpha, \exists/\forall x A) = v_{\mathcal{I}_n}(\alpha, A[x \setminus f(\vec{U})])$$

## Example of Skolem Extension

- ▶ Intuition: think of  $p(x, y, z)$  as  $x + y = z$  on  $\mathbb{Z}$
- ▶ There is a neutral element (0) and an inverse ( $-x$ ) for every  $x$

$$\frac{p(a, U, U), p(U, f(U), a) \implies}{p(a, U, U), \exists z p(U, z, a) \implies} \implies$$

$$\frac{p(a, U, U) \wedge \exists z p(U, z, a) \implies}{\forall y (p(a, y, y) \wedge \exists z p(y, z, a)) \implies} \implies$$

$$\frac{\forall y (p(a, y, y) \wedge \exists z p(y, z, a)) \implies}{\exists x \forall y (p(x, y, y) \wedge \exists z p(y, z, x)) \implies} \implies$$

- ▶ Start with  $\mathcal{I}_0 = (\mathbb{Z}, \iota)$  with  $p^t = \{\langle x, y, z \rangle \mid x + y = z\}$ .
- ▶  $\mathcal{I}_1$  has to interpret  $a$ .
- ▶  $v_{\mathcal{I}_0}(\{x \leftarrow 0\}, \forall y (p(x, y, y) \wedge \exists z p(y, z, x))) = T$
- ▶ So we set  $a^{t_1} = 0$
- ▶  $\mathcal{I}_2$  has to interpret  $f$  on all domain elements.
- ▶ For any  $k \in \mathbb{Z}$ ,  $v_{\mathcal{I}_1}(\{U \leftarrow k\} \{z \leftarrow (-k)\}, p(U, z, a)) = T$
- ▶ So we set  $f^{t_2}(k) = -k$

## Soundness for free-variable LK

Similarly to closed LK, we show the following lemmas:

1. All LK-rules preserve  $\sigma$ -falsifiability upwards for all Skolem extensions
2. An LK-derivation with a  $\sigma$ -falsifiable root sequent has at least one  $\sigma$ -falsifiable leaf sequent
3. If  $\sigma$  closes a leaf  $\Gamma \implies \Delta$ , then  $\sigma(\Gamma \implies \Delta)$  is valid

Where the notion of  $\sigma$ -falsifiability will be defined.

Finally, we use these lemmas to show the soundness theorem.

## Falsifiability with Free Variables

### Definition 3.1.

Given a ground substitution  $\sigma$  such that  $\sigma(\Gamma \implies \Delta)$  is closed, an interpretation  $\mathcal{I}$   $\sigma$ -falsifies  $\Gamma \implies \Delta$  if it falsifies  $\sigma(\Gamma \implies \Delta)$ .

$\Gamma \implies \Delta$  is called  $\sigma$ -falsifiable if there is an interpretation that  $\sigma$ -falsifies it.

## Preservation of $\sigma$ -Falsifiability

### Definition 3.2.

An LK-rule  $\theta$  **preserves  $\sigma$ -falsifiability (upwards)** by an interpretation  $\mathcal{I}$  if whenever the conclusion  $w$  of an instance  $\frac{w_1 \cdots w_n}{w}$  of  $\theta$  is  $\sigma$ -falsified by  $\mathcal{I}$ , then also at least one of the premises  $w_i$  is  $\sigma$ -falsified by  $\mathcal{I}$ .

In our proof, the falsifying interpretation will remain the same, since we already did all necessary modification in the Skolem extension.

### Lemma 3.1.

All LK-rules preserve  $\sigma$ -falsifiability by any interpretation  $\mathcal{I}_n$  that is the Skolem extension of an interpretation  $\mathcal{I}$ .

- ▶ The proofs for the propositional connectives are almost identical. We show  $\rightarrow$ -left.
- ▶ It remains to show that also the  $\forall$  and  $\exists$  rules preserve  $\sigma$ -falsifiability.

## Proof for $\rightarrow$ -left

### Proof for $\rightarrow$ -left.

$$\frac{\Gamma \Rightarrow A, \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \rightarrow B \Rightarrow \Delta} \rightarrow\text{-left}$$

- ▶ Assume that  $\mathcal{I}_n$   $\sigma$ -falsifies the conclusion.
- ▶ Then  $\mathcal{I}_n$  satisfies  $\sigma(\Gamma) \cup \{\sigma(A \rightarrow B)\}$  and falsifies all formulae in  $\sigma(\Delta)$ .
- ▶ Since  $\mathcal{I}_n$  satisfies  $\sigma(A) \rightarrow \sigma(B)$ , by definition of model semantics,
  - (1)  $\mathcal{I}_n \not\models \sigma(A)$ , or
  - (2)  $\mathcal{I}_n \models \sigma(B)$ .
- ▶ In case (1),  $\mathcal{I}_n$   $\sigma$ -falsifies the left premise.
- ▶ In case (2),  $\mathcal{I}_n$   $\sigma$ -falsifies the right premise. □

## Proof: $\forall$ -left preserves $\sigma$ -falsifiability

$$\frac{\Gamma, \forall x A, A[x \setminus U] \Rightarrow \Delta}{\Gamma, \forall x A \Rightarrow \Delta} \forall\text{-left} \quad U \text{ is a new free variable}$$

- ▶ Assume that  $\mathcal{I}_n = (D, \iota)$   $\sigma$ -falsifies the conclusion  $\Gamma, \forall x A \Rightarrow \Delta$ .
- ▶  $\mathcal{I}_n$  makes all formulae in  $\sigma(\Gamma) \cup \{\sigma(\forall x A)\}$  true and all formulae in  $\sigma(\Delta)$  false.
- ▶ It suffices to show that  $\mathcal{I}_n \models \sigma(A[x \setminus U])$ . Then, the premiss is falsified by  $\mathcal{I}_n$ .
- ▶ Since  $\mathcal{I}_n \models \sigma(\forall x A)$ , and  $\sigma(\forall x A) = \forall x \sigma_x(A)$ , we know that  $v_{\mathcal{I}_n}(\alpha \{x \leftarrow d\}, \sigma_x(A)) = T$  for all  $d \in D$  and any  $\alpha$ .
- ▶ In particular,  $v_{\mathcal{I}_n}(\alpha \{x \leftarrow v_{\mathcal{I}_n}(\alpha, \sigma(U))\}, \sigma_x(A)) = T$
- ▶ By the substitution lemma:  $v_{\mathcal{I}_n}(\alpha, \sigma_x(A)[x \setminus \sigma(U)]) = T$
- ▶ Which is the same as  $v_{\mathcal{I}_n}(\alpha, \sigma(A[x \setminus U])) = T$
- ▶ And therefore:  $\mathcal{I}_n \models \sigma(A[x \setminus U])$ .

## Proof: $\exists$ -left preserves $\sigma$ -falsifiability

$$\frac{\Gamma, A[x \setminus f(\vec{U})] \Rightarrow \Delta}{\Gamma, \exists x A \Rightarrow \Delta} \exists\text{-left} \quad \begin{array}{l} f \text{ is a function that does not} \\ \text{occur in the conclusion and} \\ \vec{U} \text{ are the free variables in } A. \end{array}$$

- ▶ Assume that  $\mathcal{I}_n = (D, \iota)$   $\sigma$ -falsifies the conclusion  $\Gamma, \exists x A \Rightarrow \Delta$ .
- ▶  $\mathcal{I}_n$  makes all formulae in  $\sigma(\Gamma) \cup \{\sigma(\exists x A)\}$  true and all formulae in  $\sigma(\Delta)$  false.
- ▶ Define a variable assignment  $\alpha$  with  $\alpha(U) = v_{\mathcal{I}_n}(\sigma(U))$  for all  $U$  in  $\vec{U}$ .
- ▶ Since free variables are always new,  $x$  is not in  $\vec{U}$ .
- ▶ By the subst. lemma,  $v_{\mathcal{I}_n}(\alpha, \exists x A) = T$
- ▶ By construction of the Skolem extension,  $v_{\mathcal{I}_n}(\alpha, A[x \setminus f(\vec{U})]) = T$
- ▶ And therefore  $\mathcal{I}_n \models \sigma(A[x \setminus f(\vec{U})])$
- ▶ So  $\mathcal{I}_n$   $\sigma$ -falsifies the premiss.



## Substitution Lemma for Multiple Variables

A variant of the Substitution Lemma, for multiple variables, but only ground substitutions

### Lemma 3.2.

Let  $\mathcal{I}$  be an interpretation, and let  $\sigma$  be a ground substitution. Let  $\alpha$  be a variable assignment such that  $\alpha(x) = v_{\mathcal{I}}(\sigma(x))$  for all variables  $x$  with  $\sigma(x) \neq x$ . If  $A$  is a formula such that  $\sigma(A)$  is closed, then  $v_{\mathcal{I}}(\alpha, A) = v_{\mathcal{I}}(\sigma(A))$

Proof.

Exercise. □

## Soundness for free-variable LK

Similarly to closed LK, we show the following lemmas:

1. All LK-rules preserve  $\sigma$ -falsifiability upwards for all Skolem extensions
2. An LK-derivation with a  $\sigma$ -falsifiable root sequent has at least one  $\sigma$ -falsifiable leaf sequent
3. If  $\sigma$  closes a leaf  $\Gamma \Longrightarrow \Delta$ , then  $\sigma(\Gamma \Longrightarrow \Delta)$  is valid

Where the notion of  $\sigma$ -falsifiability will be defined.

Finally, we use these lemmas to show the soundness theorem.

## Existence of a $\sigma$ -falsifiable leaf sequent

### Lemma 3.3.

If the root sequent  $\mathcal{I}$  of an LK-derivation is  $\sigma$ -falsifiable, then at least one of the leaf sequents is  $\sigma$ -falsifiable.

- ▶ If  $\sigma(\Gamma \Longrightarrow \Delta)$  is falsified by  $\mathcal{I}$ , it is also falsified by the Skolem extension  $\mathcal{I}_n$ .
- ▶ We can inductively use that  $\sigma$ -falsifiability by Skolem extensions is preserved.

## Soundness for free-variable LK

Similarly to closed LK, we show the following lemmas:

1. All LK-rules preserve  $\sigma$ -falsifiability upwards for all Skolem extensions
2. An LK-derivation with a  $\sigma$ -falsifiable root sequent has at least one  $\sigma$ -falsifiable leaf sequent
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Where the notion of  $\sigma$ -falsifiability will be defined.

Finally, we use these lemmas to show the soundness theorem.

## Validity of closed leaves

**Lemma 3.4.**

If  $\sigma$  closes a leaf  $\Gamma \Rightarrow \Delta$ , then  $\sigma(\Gamma \Rightarrow \Delta)$  is valid.

- ▶ If  $\sigma$  closes a leaf  $\Gamma \Rightarrow \Delta$ , then there are atomic formulae  $A \in \Gamma$  and  $B \in \Delta$  such that  $\sigma(A) = \sigma(B)$ .
- ▶ Any interpretation that satisfies the antecedent satisfies  $\sigma(A)$ .
- ▶ Therefore, the same formula  $\sigma(B)$  is satisfied in the succedent.
- ▶ Thus  $\sigma(\Gamma \Rightarrow \Delta)$  is valid.

**Theorem 3.1 (Soundness).**

The free-variable sequent calculus LK is sound.

**Proof.**

- ▶ Let  $\langle \pi, \sigma \rangle$  be a proof for  $\Gamma \Rightarrow \Delta$ .
- ▶ Assume for the sake of contradiction that  $\Gamma \Rightarrow \Delta$  is **not** valid, but falsifiable by some  $\mathcal{I}$ .
- ▶  $\Gamma \Rightarrow \Delta$  is closed so  $\mathcal{I}$  falsifies  $\sigma(\Gamma \Rightarrow \Delta)$ .
- ▶  $\Gamma \Rightarrow \Delta$  is  $\sigma$ -falsifiable.
- ▶ One of the leaf sequents  $\Gamma' \Rightarrow \Delta'$  is  $\sigma$ -falsifiable (Lemma)
- ▶ But if  $\sigma$  closes  $\pi$  then  $\sigma(\Gamma' \Rightarrow \Delta')$  must be valid.
- ▶ Contradiction!

□

## Outline

- ▶ Introduction
- ▶ The Free-variable Sequent Calculus
- ▶ Soundness
- ▶ Completeness

## Completeness

**Theorem 4.1 (Completeness).**

If  $\Gamma \Rightarrow \Delta$  is valid, then it is provable in free-variable LK.

To prove *completeness*, we show the equivalent statement:

**Lemma 4.1 (Modelleksistens).**

If  $\Gamma \Rightarrow \Delta$  is not provable in LK, then it is falsifiable.

Remember:

- ▶ All formulae in  $\Gamma \Rightarrow \Delta$  are closed
- ▶ No need to worry about a closing substitution at this point.

## Reminder: model existence for ground LK

The proof of model existence for free-variable LK is based on that for ground LK. Main points of the proof for ground LK:

- ▶ We assume that a sequent  $\Gamma \Rightarrow \Delta$  is **not** provable in ground LK.
  - ▶ I.e. derivations from root  $\Gamma \Rightarrow \Delta$  have an open branch.
- ▶ We construct a **fair limit derivation** for  $\Gamma \Rightarrow \Delta$  where every open branch  $\mathcal{B}$  has the following properties:
  - ▶ all  $\alpha$ -,  $\beta$ - and  $\delta$ -formulae are treated by a respective rule on  $\mathcal{B}$ , and
  - ▶ if  $\mathcal{B}$  contains a  $\gamma$ -formula  $\forall/\exists x A$ , then  $A[x \setminus t]$  is introduced on  $\mathcal{B}$  by a  $\gamma$  rule for every term  $t$  in the Herbrand universe of  $\mathcal{B}$ .
  - ▶ “All possible rule applications have been tried on all open branches.”
- ▶ Since  $\Gamma \Rightarrow \Delta$  is not provable, the limit derivation must contain an open branch  $\mathcal{B}$  (by König’s lemma).

## Reminder: model existence for ground LK, cont.

- ▶ We use the information in  $\mathcal{B}$  to construct a Herbrand model  $\mathcal{I}$  as follows:
  - ▶ The domain of  $\mathcal{I}$  is the Herbrand universe of  $\mathcal{B}$ .
  - ▶ For closed terms  $t$  on  $\mathcal{B}$ :  $v_{\mathcal{I}}(t) = t$
  - ▶ For  $n$ -ary relation symbols  $p$  on  $\mathcal{B}$ :  
 $\langle t_1, \dots, t_n \rangle \in p^{\mathcal{I}} \Leftrightarrow p(t_1, \dots, t_n) \in \mathcal{B}^{\top}$
- ▶ *Remember: the Herbrand universe of a branch is the set of all **closed terms that can be constructed from the constant and function symbols on the branch.***
- ▶ We then show by structural induction on formulae in  $\mathcal{B}$  that  $\mathcal{I}$  satisfies all formulae in  $\mathcal{B}^{\top}$  and falsifies all formulae in  $\mathcal{B}^{\perp}$ .
- ▶ It follows that  $\mathcal{I}$  falsifies  $\Gamma \Rightarrow \Delta$ , since  $\Gamma \subseteq \mathcal{B}^{\top}$  og  $\Delta \subseteq \mathcal{B}^{\perp}$ .

## Model Existence for Free-variable LK

- ▶ To prove model existence for free-variable LK, we will
  - ▶ construct a **fair limit derivation**
  - ▶ use a specific **ground** substitution on all formulae such that all free variables are replaced by closed terms.
- ▶ We can then construct a Herbrand-modell from a **grounded** open branch as for ground LK, and use the same induction argument.
- ▶ The  $\gamma$ -rules of free-variable LK introduce **free variables** instead of terms, so we need a new definition of **fair limit derivations**.
- ▶ We have to choose the substitution in such a way that a grounded open branch will have the same properteis w.r.t.  $\gamma$ -formlae as in ground LK.

## Fairness for free-variable LK

### Definition 4.1 (Fair derivations).

A limit derivation is **fair** if each branch has the following properties:

1. If an  $\alpha$ ,  $\beta$ , or  $\delta$  formula occurs, then the corresponding LK rule is applied to the formula on that branch.
2. If a  $\gamma$  formula occurs, then the  $\forall$ -left, resp.  $\exists$ -right rules are applied to the formula on that branch **infinitely often** on that branch, introducing infinitely many free variables.

## Fair Substitutions

- ▶ This is a fair limit derivation 
$$\frac{\forall x p(x), p(U_1), p(U_2), p(U_3) \implies q(fa)}{\forall x p(x), p(U_1), p(U_2) \implies q(fa)} \quad \vdots$$
- ▶  $\forall x p(x)$  introduces infinitely many free variables  $U_i$ .
- ▶ Let  $\sigma$  be such that  $\sigma(U_i) = a$  for all  $U_i$ .
- ▶ The derivation has only one branch  $\mathcal{B}$ . If we apply  $\sigma$  to all formulae on  $\mathcal{B}$ , then all  $p(U_i)$  become  $p(a)$ .
- ▶ The Herbrand universe of  $\sigma(\mathcal{B})$  is  $a, fa, ffa, fffa, \dots$ . There are terms  $t$  in the Herbrand universe such that  $p(t)$  is **not** on  $\sigma(\mathcal{B})$ , e.g.  $t = fa$ .
- ▶ The Herbrand model generated from  $\sigma(\mathcal{B})$  will therefore **not** make  $\forall x p(x)$  true.

## Fair Substitutions (cont.)

- ▶ Recursively define  $\tau$  such that 
$$\frac{\forall x p(x), p(U_1), p(U_2), p(U_3) \implies q(fa)}{\forall x p(x), p(U_1), p(U_2) \implies q(fa)} \quad \vdots$$
  - ▶  $\tau(U_1) = a$ , and
  - ▶  $\tau(U_{i+1}) = f\tau(U_i)$ .
- ▶ We apply  $\tau$  to  $\mathcal{B}$ .
- ▶ We now have  $p(t) \in \tau(\mathcal{B})$  for all terms  $t$  in the Herbrand universe of  $\tau(\mathcal{B})$ .
- ▶ Therefore, the Herbrand model generated from  $\tau(\mathcal{B})$  will satisfy  $\forall x p(x)$ .
- ▶ We call  $\tau$  a **fair substitution**.

## Fair Substitutions (cont.)

**Definition 4.2 (Fair substitution).**

Let  $\pi$  be a fair limit derivation, and let  $\mathcal{B}$  be a branch of  $\pi$ . Let  $\sigma$  be a substitution.

- ▶  $\sigma$  is **fair w.r.t. a  $\gamma$ -formulae**  $\forall/\exists x A$  on  $\mathcal{B}$  if for all terms  $t$  in the Herbrand-universe of  $\mathcal{B}$ , there is a  $\gamma$  application on  $\mathcal{B}$  that introduces  $A[x \setminus U]$  such that  $\sigma(U) = t$ .
- ▶  $\sigma$  is **fair w.r.t. the branch**  $\mathcal{B}$  if  $\sigma$  is fair w.r.t. all  $\gamma$ -formulae in  $\mathcal{B}$ .
- ▶  $\sigma$  is **fair w.r.t. the derivation**  $\pi$  if  $\sigma$  is fair w.r.t. every branch of  $\pi$ .

## Model existence for free variable LK – Proof

- ▶ Assume that  $\Gamma \implies \Delta$  is **not** provable in free-variable LK.
  - ▶ For all derivations from the root sequent  $\Gamma \implies \Delta$ , there is **no** substitution that closes all branches.
- ▶ We construct a **fair limit derivation**  $\pi$  with root sequent  $\Gamma \implies \Delta$ .
- ▶ We construct a substitution  $\sigma$  that is **fair** w.r.t.  $\pi$  and that replaces all free variables in  $\pi$  by ground terms.
- ▶ Since  $\sigma$  does not close  $\pi$ , there must be a branch  $\mathcal{B}$  in  $\pi$  that is not closed by  $\sigma$  (by König's Lemma).
- ▶ We apply  $\sigma$  to all formulae on  $\mathcal{B}$ . Note that  $\sigma(\mathcal{B})$  only contains **closed** formulae. We can now apply the model existence theorem for ground LK.
- ▶ Note that  $\Gamma \implies \Delta$  is **closed**. The Herbrand model generated from  $\sigma(\mathcal{B})$  will therefore falsify  $\Gamma \implies \Delta$  independently of  $\sigma$ .