IN3070/4070 – Logic – Autumn 2019

Lecture 7: Free-variable calculi

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Today's Plan

- Introduction
- The Free-variable Sequent Calculus
- Soundness
- Completeness

Outline

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- Soundness
- Completeness

Smullyan's categories: $\alpha/\beta/\gamma/\delta$

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Apply for all terms t, e.g.

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Introduce new constant c, e.g.

$$\frac{\Gamma, A[x\backslash c], \implies \Delta}{\Gamma, \exists x\, A \implies \Delta} \exists\text{-left}$$

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$$\underset{1}{a}, \underset{2}{fa}, \underset{3}{ga}, \underset{4}{ffa}, \underset{5}{fga}, \ldots, \underset{i}{fffa}, \ldots$$

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$$\frac{\dfrac{\dfrac{\forall x p(x), p(a), \ldots, p(fffa) \implies p(fffa), q(ga)}{\vdots}}{\forall x p(x), p(a) \implies p(fffa), q(ga)}}{\forall x p(x) \implies p(fffa), q(ga)}$$

$a,$	$fa,$	$ga,$	$ffa,$	$fga,$	$\ldots,$	$fffa,$	$\ldots$
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- Substitute terms for free variables that make leaves into axioms
- Which substitutions can we use to make leaves into axioms?
- That's a unification problem!

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- This will ensure soundness of the δ-rule no matter how the free variables are instantiated.

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- By *new*, we mean that it does not previously occur in the derivation.

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Definition 2.2 (δ-rules in free-variable LK).

The δ-rules in free-variable LK *are:*

$$\frac{\Gamma, A[x \backslash f(\vec{U})] \implies \Delta}{\Gamma, \exists x\, A \implies \Delta}\ \exists\text{-}left \qquad \frac{\Gamma \implies \Delta, A[x \backslash f(\vec{U})]}{\Gamma \implies \Delta, \forall x\, A}\ \forall\text{-}right$$

f is a new Skolem function
$\vec{U} = U_1, \ldots, U_n$ are the free variables occurring in $\exists x\, A$.

- By *new*, we mean that f does not previously occur in the derivation.

Rules and Derivations

Definition 2.3 (Rules of free-variable LK).

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- Free-variable derivations are defined inductively:
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- So we still allow *only closed formulae* in the root sequent.
- Formulae later on are not necessarily closed.

Examples of free-variable derivations

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$$\forall x\, p(x) \implies \exists x\, p(x)$$

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Examples of free-variable derivations

$$\frac{\forall x\, p(x), p(U) \implies \exists x\, p(x)}{\forall x\, p(x) \implies \exists x\, p(x)}$$

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$\exists$-right can not introduce U, since it already occurs in the derivation.

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$\forall$-left on the two branches can not introduce the same free variable, for the same reason as above.

Examples of free-variable derivations

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$$\forall x\,(p(x) \lor q(x)) \implies \forall x\, p(x), \forall x\, q(x)$$

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$$\dfrac{\dfrac{\forall x \ldots, p(U) \implies \forall x\, p(x), \forall x\, q(x) \qquad \forall x \ldots, q(U) \implies \forall x\, p(x), \forall x\, q(x)}{\forall x\, (p(x) \lor q(x)), p(U) \lor q(U) \implies \forall x\, p(x), \forall x\, q(x)}}{\forall x\, (p(x) \lor q(x)) \implies \forall x\, p(x), \forall x\, q(x)}$$

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\dfrac{
\dfrac{\forall x \ldots, p(U) \implies p(a), \forall x\, q(x)}{\forall x \ldots, p(U) \implies \forall x\, p(x), \forall x\, q(x)}
\qquad
\forall x \ldots, q(U) \implies \forall x\, p(x), \forall x\, q(x)
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- Every δ-rule application has to introduce a new Skolem symbol, i.e. one that does not yet occur in the derivation.

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- In this derivation, we don't know which symbols occur in the conclusion until we have instantiated U!

Examples of Objects that are *not* Derivations

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$$\dfrac{\dfrac{p(x) \Longrightarrow p(a)}{p(x) \Longrightarrow \forall y\, p(y)}\ \forall\text{-right}}{\Longrightarrow p(x) \to \forall y\, p(y)}\ \to\text{-right}$$

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The two δ-applications introduce the same Skolem symbol.

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- *σ closes π if σ closes all leaf sequents of π.*

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- A ground substitution is a substitution σ such that $\sigma(x)$ is closed (contains no variables) for all $x \in \mathcal{V}$.
- We restrict closing substitutions to be ground, since this makes the soundness proof *slightly* simpler.
- We will later see that this requirement can be dropped to allow arbitrary closing substitutions.

Examples

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Let π *be the derivation*

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and let $\sigma = \{U \backslash a, V \backslash a\}$.

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- σ *closes the leaf sequent:* $\sigma(p(U)) = p(a) = \sigma(p(V))$.

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- *σ closes π, since it closes the only leaf sequent.*

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- σ *closes the leaf sequent:* $\sigma(p(U)) = p(a) = \sigma(p(V))$.
- σ *closes* π, *since it closes the only leaf sequent.*
- *Thus,* $\langle \pi, \sigma \rangle$ *is a proof for the sequent* $\forall x\, p(x) \implies \exists x\, p(x)$.

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and let $\sigma = \{U \backslash a, V \backslash a\}$.

- *σ closes the leaf sequent: $\sigma(p(U)) = p(a) = \sigma(p(V))$.*
- *σ closes π, since it closes the only leaf sequent.*
- *Thus, $\langle \pi, \sigma \rangle$ is a proof for the sequent $\forall x\, p(x) \implies \exists x\, p(x)$.*

Note:

With our current definition of free-variable LK, e.g. $\langle \pi, \sigma' \rangle$ where $\sigma' = \{U \backslash V\}$ is not a proof, since σ' is not ground.

Example (2).

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Let π *be the derivation*

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$$
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- *Is the root sequent valid. . . ?*

Outline

- Introduction
- The Free-variable Sequent Calculus
- **Soundness**
- Completeness

Soundess Proof: Basic Idea & Difficulty

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- Example:

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- Remember: new constant was important for soundness of δ rule.
- Defined a new interpretation $\mathcal{I}'$ that is the same as $\mathcal{I}$ for old formulae.
- Now we have $f(\vec{U})$... but how will that help?

Extending the Interpretation

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Extending the Interpretation (cont.)

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- Let $U_1, \dots U_m$ be the free variables in $\exists/\forall x_{i+1}\, A_{i+1}$.
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  - If $v_{\mathcal{I}_i}(\alpha, \exists x_{i+1}\, A_{i+1}) = T$

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- Case 1: $\exists x_{i+1}\, A_{i+1}$ occurs in an antecedent
  - If $v_{\mathcal{I}_i}(\alpha, \exists x_{i+1}\, A_{i+1}) = T$, pick one $d \in D$ with $v_{\mathcal{I}_i}(\alpha\{x{\leftarrow}d\}, A_{i+1}) = T$

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  - If $v_{\mathcal{I}_i}(\alpha, \exists x_{i+1}\, A_{i+1}) = T$, pick one $d \in D$ with $v_{\mathcal{I}_i}(\alpha\{x \leftarrow d\}, A_{i+1}) = T$, and define $f_{i+1}^{\iota_{i+1}}(d_1, \ldots, d_m) := d$

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Extending the Interpretation (cont.)

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  - If $v_{\mathcal{I}_i}(\alpha, \exists x_{i+1}\, A_{i+1}) = T$, pick one $d \in D$ with $v_{\mathcal{I}_i}(\alpha\{x {\leftarrow} d\}, A_{i+1}) = T$, and define $f_{i+1}^{\iota_{i+1}}(d_1, \ldots, d_m) := d$
  - If $v_{\mathcal{I}_i}(\alpha, \exists x_{i+1}\, A_{i+1}) = F$, define $f_{i+1}^{\iota_{i+1}}(d_1, \ldots, d_m)$ to an arbitrary domain element.

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  - If $v_{\mathcal{I}_i}(\alpha, \exists x_{i+1}\, A_{i+1}) = F$, define $f_{i+1}^{\iota_{i+1}}(d_1, \ldots, d_m)$ to an arbitrary domain element.
- Case 2: $\forall x_{i+1}\, A_{i+1}$ occurs in a succedent
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Extending the Interpretation (cont.)

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$\Gamma \implies \Delta$ *is called σ-falsifiable if there is an interpretation that σ-falsifies it.*

Preservation of σ-Falsifiability

Definition 3.2.

An LK*-rule* θ *preserves σ-falsifiability (upwards) by an inprerretation* $\mathcal{I}$ *if whenever the conclusion* w *of an instance* $\frac{w_1 \cdots w_n}{w}$ *of* θ *is σ-falsified by* $\mathcal{I}$*, then also at least one of the premises* w_i *is σ-falsified by* $\mathcal{I}$*.*

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- The proofs for the propositional connectives are almost identical. We show $\rightarrow$-left.
- It remains to show that also the $\forall$ and $\exists$ rules preserve σ-falsifiability.

Proof for →-left

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$$\frac{\Gamma \implies A, \Delta \qquad \Gamma, B \implies \Delta}{\Gamma, A \to B \implies \Delta} \to\text{-left}$$

□

Proof for $\rightarrow$-left

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- Assume that $\mathcal{I}_n$ σ-falsifies the conclusion.

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- Assume that $\mathcal{I}_n$ σ-falsifies the conclusion.
- Then $\mathcal{I}_n$ satisfies $\sigma(\Gamma) \cup \{\sigma(A \rightarrow B)\}$

□

Proof for →-left

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$$\dfrac{\Gamma \implies A, \Delta \qquad \Gamma, B \implies \Delta}{\Gamma, A \to B \implies \Delta} \to\text{-left}$$

- Assume that $\mathcal{I}_n$ σ-falsifies the conclusion.
- Then $\mathcal{I}_n$ satisfies $\sigma(\Gamma) \cup \{\sigma(A \to B)\}$ and falsifies all formlae in $\sigma(\Delta)$.

□

Proof for $\rightarrow$-left

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$$\frac{\Gamma \implies A, \Delta \qquad \Gamma, B \implies \Delta}{\Gamma, A \rightarrow B \implies \Delta} \rightarrow\text{-left}$$

- Assume that $\mathcal{I}_n$ σ-falsifies the conclusion.
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  (1) $\mathcal{I}_n \not\models \sigma(A)$, or

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□

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  (1) $\mathcal{I}_n \not\models \sigma(A)$, or
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- Assume that $\mathcal{I}_n$ σ-falsifies the conclusion.
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  (2) $\mathcal{I}_n \models \sigma(B)$.
- In case (1), $\mathcal{I}_n$ σ-falsifies the left premisse.
- In case (2), $\mathcal{I}_n$ σ-falsifies the right premisse.

□

Proof: $\forall$-left preserves σ-falsifiability

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$$\frac{\Gamma, \forall x\, A, A[x \backslash U] \implies \Delta}{\Gamma, \forall x\, A \implies \Delta}\ \forall\text{-left}$$

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Proof: $\forall$-left preserves σ-falsifiability

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- Assume that $\mathcal{I}_n = (D, \iota)$ σ-falsifies the conclusion $\Gamma, \forall x\, A \implies \Delta$.

Proof: $\forall$-left preserves σ-falsifiability

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- $\mathcal{I}_n$ makes all formulae in $\sigma(\Gamma) \cup \{\sigma(\forall x A)\}$ true and all formulae in $\sigma(\Delta)$ false.

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- Since $\mathcal{I}_n \models \sigma(\forall x\, A)$, and $\sigma(\forall x\, A) = \forall x\, \sigma_x(A)$,

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- It suffices to show that $\mathcal{I}_n \models \sigma(A[x \backslash U])$. Then, the premiss is falsified by $\mathcal{I}_n$.
- Since $\mathcal{I}_n \models \sigma(\forall x\, A)$, and $\sigma(\forall x\, A) = \forall x\, \sigma_x(A)$, we know that $v_{\mathcal{I}_n}(\alpha\{x \leftarrow d\}, \sigma_x(A)) = T$ for all $d \in D$ and any α.

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- In particular, $v_{\mathcal{I}_n}(\alpha\{x \leftarrow v_{\mathcal{I}_n}(\alpha, \sigma(U))\}, \sigma_x(A)) = T$

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- In particular, $v_{\mathcal{I}_n}(\alpha\{x \leftarrow v_{\mathcal{I}_n}(\alpha, \sigma(U))\}, \sigma_x(A)) = T$
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- Since $\mathcal{I}_n \models \sigma(\forall x\, A)$, and $\sigma(\forall x\, A) = \forall x\, \sigma_x(A)$, we know that $v_{\mathcal{I}_n}(\alpha\{x \leftarrow d\}, \sigma_x(A)) = T$ for all $d \in D$ and any α.
- In particular, $v_{\mathcal{I}_n}(\alpha\{x \leftarrow v_{\mathcal{I}_n}(\alpha, \sigma(U))\}, \sigma_x(A)) = T$
- By the substitution lemma: $v_{\mathcal{I}_n}(\alpha, \sigma_x(A)[x\backslash\sigma(U)]) = T$
- Which is the same as $v_{\mathcal{I}_n}(\alpha, \sigma(A[x\backslash U])) = T$
- And therefore: $\mathcal{I}_n \models \sigma(A[x\backslash U])$.

Proof: $\exists$-left preserves σ-falsifiability

Proof: ∃-left preserves σ-falsifiability

$$\frac{\Gamma, A[x\backslash f(\vec{U}] \implies \Delta}{\Gamma, \exists x\, A \implies \Delta}\ \exists\text{-left}$$

f is a function that does not occur in the conculsion and $\vec{U}$ are the free variables in A.

Proof: ∃-left preserves σ-falsifiability

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- Define a variable assignment α with $\alpha(U) = v_{\mathcal{I}_n}(\sigma(U))$ for all U in $\vec{U}$.

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- And therefore $\mathcal{I}_n \models \sigma(A[x\backslash f(\vec{U})])$

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- By construction of the Skolem extension, $v_{\mathcal{I}_n}(\alpha, A[x\backslash f(\vec{U})]) = T$
- And therefore $\mathcal{I}_n \models \sigma(A[x\backslash f(\vec{U})])$
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Substitution Lemma for Multiple Variables

A variant of the Substitution Lemma, for multiple variables, but only ground substitutions

Lemma 3.2.

Let $\mathcal{I}$ be an interpretation, and let σ be a ground substitution.

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Proof.

Exercise. □

Soundness for free-variable LK

Similarly to closed LK, we show the following lemmas:

1. All LK-rules preserve σ-falsifiability upwards for all Skolem extensions
2. An LK-derivation with a σ-falsifiable root sequent has at least one σ-falsifiable leaf sequent
3. If σ closes a leaf $\Gamma \implies \Delta$, then $\sigma(\Gamma \implies \Delta)$ is valid

Where the notion of σ-falsifiability will be defined.

Finally, we use these lemmas to show the soundness theorem.

Existence of a σ-falsifiable leaf sequent

Lemma 3.3.

If the root sequent $\mathcal{I}$ of an an LK*-derivation is σ-falsifiable, then at least one of the leaf sequents is σ-falsifiable.*

- If $\sigma(\Gamma \implies \Delta)$ is falsified by $\mathcal{I}$, it is also falsified by the Skolem extension $\mathcal{I}_n$.
- We can inductively use that σ-falsifiability by Skolem extensions is preserved.

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Validity of closed leaves

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- Therefore, the same formula $\sigma(B)$ is satisfied in the succedent.

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- Any interpretation that satisfies the antecedent satisfies $\sigma(A)$.
- Therefore, the same formula $\sigma(B)$ is satisfied in the succedent.
- Thus $\sigma(\Gamma \implies \Delta)$ is valid.

Theorem 3.1 (Soundness).

The free-variable sequent calculus LK *is sound.*

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Proof.

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- Let $\langle \pi, \sigma \rangle$ be a proof for $\Gamma \implies \Delta$.
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- $\Gamma \implies \Delta$ is closed so $\mathcal{I}$ falsifies $\sigma(\Gamma \implies \Delta)$.

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- $\Gamma \implies \Delta$ is σ-falsifiable.

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- One of the leaf sequents $\Gamma' \implies \Delta'$ is σ-falsifiable (Lemma)

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- $\Gamma \implies \Delta$ is σ-falsifiable.
- One of the leaf sequents $\Gamma' \implies \Delta'$ is σ-falsifiable (Lemma)
- But if σ closes π then $\sigma(\Gamma' \implies \Delta')$ must be valid.

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Proof.

- Let $\langle \pi, \sigma \rangle$ be a proof for $\Gamma \implies \Delta$.
- Assume for the sake of contradiction that $\Gamma \implies \Delta$ is not valid, but falsifiable by some $\mathcal{I}$.
- $\Gamma \implies \Delta$ is closed so $\mathcal{I}$ falsifies $\sigma(\Gamma \implies \Delta)$.
- $\Gamma \implies \Delta$ is σ-falsifiable.
- One of the leaf sequents $\Gamma' \implies \Delta'$ is σ-falsifiable (Lemma)
- But if σ closes π then $\sigma(\Gamma' \implies \Delta')$ must be valid.
- Contradiction!

□

Outline

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Remember:

- All formulae in $\Gamma \implies \Delta$ are closed
- No need to worry about a closing substitution at this point.

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  - *"All possible rule applications have been tried on all open branches."*
- Since $\Gamma \implies \Delta$ is not provable, the limit derivation must contain an open branch $\mathcal{B}$ (by König's lemma).

Reminder: model existence for ground LK, cont.

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  - For n-ary relation symbols p on $\mathcal{B}$: $\langle t_1, \ldots, t_n \rangle \in p^{\iota} \;\Leftrightarrow\; p(t_1, \ldots, t_n) \in \mathcal{B}^{\top}$

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- *Remember: the Herbrand universe of a branch is the set of all closed terms that can be constructed from the constant and function symbols on the branch.*
- We then show by structural induction on formulae in $\mathcal{B}$ that $\mathcal{I}$ satisfies all formulae in $\mathcal{B}^{\top}$ and falsifies all formulae in $\mathcal{B}^{\perp}$.
- It follows that $\mathcal{I}$ falsifies $\Gamma \implies \Delta$, since $\Gamma \subseteq \mathcal{B}^{\top}$ og $\Delta \subseteq \mathcal{B}^{\perp}$.

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- The γ-rules of free-variable LK introduce free variables instead of terms, so we need a new definition of fair limit derivations.
- We have to choose the substitution in such a way that a grounded open branch will have the same properteis w.r.t. γ-formlae as in ground LK.

Fairness for free-variable LK

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A limit derivation is fair if each branch has the following properties:

1. *If an α, β, or δ formula occurs, then the corresponding* LK *rule is applied to the formula on that branch.*
2. *If a γ formula occurs, then the $\forall$-left, resp. $\exists$-right rules are applied to the formula on that branch infinitely often on that branch, introducing infinitely many free variables.*

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$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\forall x\, p(x), p(U_1), p(U_2), p(U_3) \implies q(fa)}}{\forall x\, p(x), p(U_1), p(U_2) \implies q(fa)}}{\forall x\, p(x), p(U_1) \implies q(fa)}}{\forall x\, p(x) \implies q(fa)}$$

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- The Herbrand universe of $\sigma(\mathcal{B})$ is $a, fa, ffa, fffa, \ldots$ There are terms t in the Herbrand universe such that $p()t)$ is not on $\sigma(\mathcal{B})$, e.g. $t = fa$.
- The Herbrand model generated from $\sigma(\mathcal{B})$ will therefore not make $\forall x\, p(x)$ true.

Fair Substitutions (cont.)

- Recursively define τ such that

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  - $\tau(U_1) = a$, and
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- Therefore, the Herbrand model generated from $\tau(\mathcal{B})$ will satisfy $\forall x\, p(x)$.
- We call τ a fair substitution.

Fair Substitutions (cont.)

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Let π be a fair limit derivation, and let $\mathcal{B}$ be a branch of π. let σ be a substitution.

- σ *is fair w.r.t. a γ-formulae* $\forall/\exists x\, A$ *on* $\mathcal{B}$ *if*

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- *σ is fair w.r.t. the derivation π if*

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- *σ is fair w.r.t. the branch $\mathcal{B}$ if σ is fair w.r.t. all γ-formlae in $\mathcal{B}$.*
- *σ is fair w.r.t. the derivation π if σ is fair w.r.t. every branch of π.*

Model existence for free variable LK – Proof

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- Assume that $\Gamma \implies \Delta$ is not provable in free-variable LK.

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- Note that $\Gamma \implies \Delta$ is closed. The Herbrand model generated from $\sigma(\mathcal{B})$ will therefore falsify $\Gamma \implies \Delta$ independently of σ.