Grading Guidelines
IN3070/IN4070/INF4171
Autumn 2019

January 13, 2020

The maximum number of marks for the whole exam was 100. The minimum number of marks required for each grade was as follows: For E, min. 40. For D, min. 50. For C, min. 57. For B, min. 70. For C, min. 85.

These boundaries are based on an alignment of the delivered work with the grade definitions given here:

https://www.uio.no/studier/eksamen/karakterer/

The final grade for each candidate is based on an evaluation of the delivered work as a whole, and may therefore in some cases deviate from the bounraries given above.

Question 1 – Notions: Soundness and Completeness

A) Briefly explain what it means that a calculus is **sound** for a given logic. [2 marks]

Answer: A calculus is sound for a logic if any formula or sequent that is provable (i.e. there is a closed derivation) is also valid.

In a refutation calculus, if there is a closed derivation, the original set of formulae, clauses, etc., is unsatisfiable.

Grading: One of these variants (proving validity or proving unsatisfiability) is enough for 2 marks.

B) Briefly explain what it means that a calculus is complete for a given logic [2 marks]

Answer: A calculus is complete for a logic if there is a proof for every valid formula. Or, in the case of a refutation calculus, for every unsatisfiable formula, set of clauses, etc.

Grading: One of these variants (proving validity or proving unsatisfiability) is enough for 2 marks.

C) It is very easy to give a calculus (e.g. for first order logic) that is sound, but not complete. It is also very easy to give a calculus that is complete but not sound. Explain! [4 marks]

Answer: A calculus without any rules, and without any axiom, does not allow any derivations. Therefore, it cannot possibly give a proof for an invalid formula (or any other formula). Therefore, it is sound.

A calculus with an axiom $\dfrac{}{\Gamma \vdash \Delta}$ hat allows proving *any* sequent, is complete, but not sound.

Grading: 2 marks for each direction.

The question doesn't specify what is meant by 'very easy,' so it is OK to give answers that add or remove rules from LK for instance.

D) When we add a rule to sound and complete calculus (e.g. for efficiency), do we have to reconsider soundness, or completeness, or both? [2 marks]

Answer: If we only add a rule, without restricting the application of the 'old' rules, completeness is preserved since a proof in the old calculus is still a proof. We do need to check soundness of the added rule though.

Grading: 1 mark for 'need to check soundness', 1 credit for 'don't need to check completeness'

E) When we remove a rule from a sound and complete calculus, or restrict its application in some way (e.g. regularity), do we have to reconsider soundness, or completeness, or both?

Answer: If we remove a rule, or restrict the application of some rules, soundness is preserved, because any proof in the restricted calculus was a proof in the original calculus. We have to check completeness.

Grading: 1 mark for 'need to check completeness', 1 credit for 'don't need to check soundness'

Question 2 – Sequent Calculus

Prove the validity of the following formulae using the given calculus. Use the hand drawing sheets

A) $(p \to (q \to r)) \to (q \to (p \to r))$ using **propositional LK** [6 marks]

Answer:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{q,\, p \Rightarrow p,\, r}\,\text{Ax} \qquad \dfrac{\dfrac{}{q,\, p \Rightarrow q,\, r}\,\text{Ax} \qquad \dfrac{}{r,\, q,\, p \Rightarrow r}\,\text{Ax}}{(q \to r),\, q,\, p \Rightarrow r}\,\to\text{-left}}{p \to (q \to r),\, q,\, p \Rightarrow r}\,\to\text{-left}}{p \to (q \to r),\, q \Rightarrow p \to r}\,\to\text{-right}}{p \to (q \to r) \Rightarrow q \to (p \to r)}\,\to\text{-right}}{\Rightarrow (p \to (q \to r)) \to (q \to (p \to r))}\,\to\text{-right}
$$

Grading: The order of rule applications can differ, but there have to be three branches. 2 marks for each. Total 1 credit subtracted for missing labels.

B) $(p(a) \land \forall x\,(p(x) \to p(f(x)))) \to p(f(f(a)))$ using **ground** first-order LK [7 marks]

Answer:

$$\dfrac{\dfrac{}{p(a),\, \forall \ldots \Rightarrow p(a),\, p(f(f(a)))}\,\text{Ax} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\cdot}{p(a),\, p(f(a)), \ldots \Rightarrow p(f(a)),\, p(f(f(a)))}\,\text{Ax} \qquad \dfrac{}{p(a),\, p(f(a)),\, p(f(f(a))) \ldots \Rightarrow p(f(f(a)))}\,\text{Ax}}{p(a),\, p(f(a)),\, p(f(a)) \to p(f(f(a))), \ldots \Rightarrow p(f(f(a)))}\,\to\text{-left}}{\vdots}\,\forall\text{-left with } f(a)}{p(a),\, p(f(a)), \forall \ldots \Rightarrow p(f(f(a)))}}{\dfrac{p(a),\, p(a) \to p(f(a)), \forall \ldots \Rightarrow p(f(f(a)))}{\dfrac{p(a),\, \forall x\,(p(x) \to p(f(x))) \Rightarrow p(f(f(a)))}{\dfrac{p(a) \land \forall x\,(p(x) \to p(f(x))) \Rightarrow p(f(f(a)))}{\Rightarrow\ (p(a) \land \forall x\,(p(x) \to p(f(x)))) \to p(f(f(a)))}\,\to\text{-right}}\,\land\text{-left}}\,\forall\text{-left with } a}\,\to\text{-left}$$

Grading:

- Propositonal structure (3 branches): 3 marks
- Instantiating with a: 2 marks
- Instantiating with $f(a)$: 2 marks
- Abbreviations (name for $\forall$ formula, ellipsis…) is OK.

C) $(\forall x\, p(x)) \to \forall y\,(p(y) \lor q(y))$ using **free variable** first-order LK [6 marks]

Answer:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{-\text{ closed with } [U\backslash a]\ -}{p(U),\, \forall x\, p(x) \Rightarrow p(a),\, q(a)}}{\forall x\, p(x) \Rightarrow p(a),\, q(a)}\,\forall\text{-left}}{\forall x\, p(x) \Rightarrow p(a) \lor q(a)}\,\lor\text{-right}}{\forall x\, p(x) \Rightarrow \forall y\,(p(y) \lor q(y))}\,\forall\text{-right}}{\Rightarrow (\forall x\, p(x)) \to \forall y\,(p(y) \lor q(y))}\,\to\text{-right}$$

Grading:

- 2 marks for a correct $\forall$-left
- 2 marks for a correct $\forall$-right
- 2 marks for giving a closing substitutionb

D) $\exists x\,(p(x) \to \forall y\, p(y))$ using ground first-order LK [7 marks]

Answer:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p(c),\, p(a) \Rightarrow p(a),\, \forall y\, p(y),\, \ldots}\ \text{Ax}}{p(c) \Rightarrow p(a),\, p(a) \to \forall y\, p(y),\, \ldots}\ \to\text{-right}}{p(c) \Rightarrow p(a),\, \exists x\,(p(x) \to \forall y\, p(y))}\ \exists\text{-right with } a}{p(c) \Rightarrow \forall y\, p(y),\, \exists x\,(p(x) \to \forall y\, p(y))}\ \forall\text{-right}}{\Rightarrow p(c) \to \forall y\, p(y),\, \exists x\,(p(x) \to \forall y\, p(y))}\ \to\text{-right}}{\Rightarrow \exists x\,(p(x) \to \forall y\, p(y))}\ \exists\text{-right with dummy const. } c$$

Grading:

- 3 marks for ∃-right with a dummy constant
- 2 marks for ∀-right introducing a new constant
- 2 marks for re-applying $\exists - right$

E) $\exists x\,(p(x) \to \forall y\, p(y))$ using **free variable** first-order LK [6 marks]

Answer:

$$\dfrac{\dfrac{\dfrac{\dfrac{-\ \text{closed with } [U \backslash a]\ -}{p(U) \Rightarrow p(a),\, \exists x \ldots}}{p(U) \Rightarrow \forall y\, p(y),\, \exists x \ldots}\ \forall\text{-right}}{\Rightarrow p(U) \to \forall y\, p(y),\, \exists x \ldots}\ \to\text{-right}}{\Rightarrow \exists x\,(p(x) \to \forall y\, p(y))}\ \exists\text{-right}$$

Grading:

- 2 marks for ∃-right with a free variable
- 2 marks for ∀-right introducing a new constant
- 2 marks for giving the closing substitution
- −2 marks for using 2 ∃-right rules, or similar.

Question 3 – Resolution

We want to formalise arithmetic of natural numbers in first order logic.

We use a constant symbol a to denote zero, and a unary function symbol f to denote the "successor" of a number, i.e.the number that is larger by one. For instance $f(a)$ denotes 1, $f(f(a))$ denotes 2, etc.

We also use a predicate sum so that $sum(x, y, z)$ denotes that $x + y = z$. For instance $sum(f(f(a), f(a), f(f(f(a))))$ means that $2 + 1 = 3$.

We now formalise the properties of addition using the following two formulae:

$$A = \forall x\, sum(a, x, x)$$
$$B = \forall x \forall y \forall z\, (sum(x, y, z) \rightarrow sum(f(x), y, f(z))$$

A) Write these two formulae as Prolog clauses. [4 marks]
Answer:

```
sum(a,X,X).
sum(f(X),Y,f(Z)) :- sum(X,Y,Z).
```

Grading:

- 1 mark per formula/clause
- 1 mark for capital variable names
- 1 mark for reversed implication `:-`
- -0.5 credit for using 0 instead of a as constant

B) Write the formulae A and B in clause form, i.e. as a set of sets of literals. You can use the minus sign for negation. [4 marks]
Answer: m

```
{{sum(a,x,x)},{-sum(x,y,z),sum(f(x),y,f(z))}}
```

Grading:

- 2 marks per clause
- lower or upper case for variables doesn't matter.

C) Use the resolution calculus to prove that

$$C = \ sum(f(f(a), f(a), f(f(f(a))))$$

is a logical consequence of the formulae A and B. In other words, show that $(A \wedge B) \rightarrow C$ is valid. Please number the clauses you derive and make it clear in each step which clauses you resolve and with which substitutions. Remember that you might have to rename variables in clauses before resolving. [8 marks]
Answer:

```
1: sum(a,x1,x1)                          formula A
2: -sum(x2,y2,z2),sum(f(x2),y2,f(z2))   formula B
3: -sum(f(f(a),f(a),f(f(f(a)))          negated formula C
4: sum(f(a),y4,f(y4))                    res of 1&2, x2\a,x1\y2,z2\y2
5: sum(f(f(a)),y5,f(f(y5)))              res of 4&2, x2\f(a),y2\y4,z2\f(y4)
6: []                                    res of 3&5, y5=f(a)
```

Or alternatively (Prolog-like, with no need for var renaming)

```
1: sum(a,x1,x1)                          formula A
2: -sum(x2,y2,z2),sum(f(x2),y2,f(z2))   formula B
3: -sum(f(f(a),f(a),f(f(f(a)))          negated formula C
4: -sum(f(a),f(a),f(f(a)))               res of 2&3, x2\fa,y2\fa,z2\ffa
5: -sum(a,f(a),f(a))                     res of 2&4, x2\a,y2\fa,z2\fa
6: []                                    res of 1&5, x1=f(a)
```

Grading:

- 2 marks for initial clauses, including negated C
- 2 marks for each of the three resolution steps
- Only 1 mark per resolution if a substitution is not specified.
- Other ways of coping with the renaming of variables are OK.

Question 4 – Classical vs. Intuitionistic Logic

In the following, there is a number of statements, that may be true in some logics, but not in others.

Reminder: validity in intuitionistic logic means that a formula is true in *every* world of *every* intuitinistic Kripke structure.

A) If A is a valid formula and B is a valid formula, then $A \wedge B$ is a valid formula
Answer: This is true for both classical and intuitionistic logic

B) If $A \wedge B$ is a valid formula, then A is a valid formula and B is a valid formula.
Answer: This is true for both classical and intuitionistic logic

C) If A is a valid formula then $A \vee B$ is a valid formula
Answer: This is true for both classical and intuitionistic logic

D) If $A \lor B$ is a valid formula then either A or B or both are valid formulas.
Answer: This is true for intuitionistic but not for classical logic. (e.g. $p \lor \neg p$ is valid but neither p nor $\neg p$ are.

E) Which of the following is true:

- Every formula valid in intuitionistic logic is also valid in classical logic.
- Every formula valid in classical logic is also valid in intuitionistic logic.

Answer: Every formula valid in intuitionistic logic is also valid in classical logic.
Grading: Automatically graded. 2 marks per correct answer, -2 marks per wrong answer, not less than 0 points for the whole question.

Question 5 – Modal Logic

Consider the modal formula: $A = \Box p \to \Diamond p$.

A) Show that it is not valid in modal logic **K** by giving a Kripke structure where A does not hold. [5 marks]
Use the sheets for hand drawings.
Answer: For a counterexample, we need to construct a Kripke structure with a word w such that p holds in every reachable successor world ($w \Vdash p$), but still there is no reachable successor world, where p holds ($w \nVdash p$). This is only possible, if w has no reachable successors. A Kripke struture with frame $(\{w\}, \emptyset)$ with one world and an empty accessibility relation does the trick. The interpretation at w doesn't matter.
Grading:

- Full score (5 marks) for a correct proof
- 2 marks for a correct description of a Kripke structure that is not a counterexample
- Between 2 or 5 for incorrect answers with some amount of correct argumentation.
- Reading $\Box p \to \Diamond p$ as $\Box(p \to \Diamond p)$ and working correctly from there: -1 mark. (Because it is much easier.)

B) Using the model semantics, show that it is valid in modal logic **T**, i.e. A holds in every world of any Kripke structure with a reflexive frame, i.e. $(x,x) \in R$ for all $x \in W$. [5 marks]

Answer: Let $w \in W$ be an arbitrary but fixed world of an arbitrary but fixed reflexive Kripke structure. If $w \Vdash \Box p$, then, since $(w,w) \in R$, in particular $w \Vdash p$, according to the model semantics. Again, due to reflexivity, w is a successor of itself, so according to the model semantics of $\Diamond$, we have $w \Vdash \Diamond p$. Together: $w \Vdash \Box p \to \Diamond p$.

Grading:

- Full score for a correct proof
- 2 marks for showing sth for an arbtirary world of an arbitrary reflexive structure.
- 2 more marks for using the model semantics to do so.
- Only 2 marks for noting that the formula holds in D according to the slides and that all T theorems are D theorems. Smart, but the question asks for an argument using the model semantics.

Question 6 – An Inductive Proof

Let A and B be two logically equivalent propositional formulae.

Let C be a propositional formula, and C' the result of replacing all occurrences of A in C by B.

E.g. $A = p \to q$ and $B = \neg q \to \neg p$ are logically equivalent.
If $C = (p \to q) \vee (q \to r)$ then $C' = (\neg q \to \neg p) \vee (q \to r)$.

Prove by structural induction on C that C and C' are logically equivalent.

You may use a hand drawing sheet, or the text field below. You can use ASCII renditions like `/\`, `/\`, `->`, `-`. Most of the cases are very similar, you do not need to spell them all out.

Answer: Let I be an arbitrary propositional interpretation. We prove by structural induction on C that $v_I(C) = v_I(C')$, where v_I is the truth value in I.

- If C is an atomic formula
  - If $C = A$, then $v_I(C) = v_I(A) = v_I(B) = v_I(C')$.
  - Otherwise $C = C'$ and $v_I(C) = v_I(C')$
- If $C = C_1 \wedge C_2$
  - If $C = A$, then $v_I(C) = v_I(A) = v_I(B) = v_I(C')$.

- Otherwise $C' = C'_1 \wedge C'_2$ where C'_1 and C'_2 are the results of replacing A by B in C_1 and C_2. By the induction hypothesis, $v_I(C'_1) = v_I(C_1)$ and $v_I(C'_2) = v_I(C_2)$. Therefore $v_I(C) = T$ iff ($v_I(C_1) = T$ and $v_I(C_2) = T$) iff ($v_I(C'_1) = T$ and $v_I(C'_2) = T$) iff $v_I(C') = T$.

- The cases for $C_1 \vee C_2$, $C = C_1 \to C_2$, $C = \neg C_1$ are all very similar.

Grading:

- 4 marks for correct structure of proof, structural induction and proving for all interpretations.
- 2 marks for correct base case
- 2 marks for a correct induction step
- 2 marks for a correct $C = A$ case (or two)
- It's OK to cover the $C = A$ case jointly for the base case and the step.
- Proofs that somehow assume that A occurs directly underneath the top symbol (i.e. no actual induction), otherwise OK: 3 marks.

Question 7 – Multiple Choice Mix

A) Which of these is a correct formulation of the substitution lemma for for terms

[2 marks]

Answer: $v_I(\alpha, t[x\backslash s]) = v_I(\alpha\{x \leftarrow v_I(\alpha, s)\}, t)$

B) Are these terms unifiable? $f(g(x), x)$ and $f(y, h(y))$

Answer: No. After unifying the first subterms with $y\backslash g(x)$, we would have to unify x and $h(g(x))$ for the second subterms, which fails (occur check).

C) The first order formula $\forall x\,(p(x) \wedge \neg p(x))$ is satisfiable, because it is true in an interpretation $\mathcal{I} = (\emptyset, \iota)$ with an empty domain.

Answer: False. The domain of an interpretation is by definition non-empty. If we allowed empty domains, we would get a different logic, with a different semantics, would need different calculi, etc.

D) From a finite number of atomic formulae (also known as propositional variables), we can construct only finitely many formulae

Answer: False. We can construct p, $p \wedge p$, $p \wedge (p \wedge p)$, etc.

It is true that only find finitely many (max. 2^n for n variables) formulae are pariwise not logically equivalent. But that was not the question.

E) If the clauses in a set S of propositional clauses contains one atomic formula only positively (e.g., only but not), then… [2 marks]

Answer: …S could be both satisfiable or unsatisfiable. E.g. $\{\{p\}\}$ contains p only positively and is satisfiable. $\{\{p\}, \{q\}, \{\neg q\}\}$ contains p only positively and is unsatisfiable.

If this scenario reminds you of something: See Exercise 9.3 on Pure Literal Elimination.

Grading: Automatically graded. 2 marks per correct answer, −2 marks per wrong answer, not less than 0 points for the whole question.