IN3070
IN4070

Autumn 2020

Solutions to Exercises for the Course

Logic

Week 10



Exercise 10.1

(Modal Sequent Calculus)

Using the sequent calculus from the lecture, prove

1. $\Box(p \to q) \to (\Box p \to \Box q)$
2. $\Diamond(p \vee q) \to (\Diamond p \vee \Diamond q)$
3. $(\Diamond p \vee \Diamond q) \to \Diamond(p \vee q)$
4. $((p \to \Box p) \wedge (\Diamond \neg p)) \to \neg p$

Solution

1.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{2:p, 1:\Box(p\to q), 1:\Box p, 1R2 \Longrightarrow 2:p, 2:q}\,\text{ax} \quad \dfrac{}{2:q, 2:p, 1:\Box(p\to q), 1:\Box p, 1R2 \Longrightarrow 2:q}\,\text{ax}}{2:p\to q, 2:p, 1:\Box(p\to q), 1:\Box p, 1R2 \Longrightarrow 2:q}\,{\to}\text{-l}}{2:p\to q, 1:\Box(p\to q), 1:\Box p, 1R2 \Longrightarrow 2:q}\,\Box\text{-l}}{1:\Box(p\to q), 1:\Box p, 1R2 \Longrightarrow 2:q}\,\Box\text{-l}}{1:\Box(p\to q), 1:\Box p \Longrightarrow 1:\Box q}\,\Box\text{-r}}{1:\Box(p\to q) \Longrightarrow 1:\Box p \to \Box q}\,{\to}\text{-r}}{\Longrightarrow 1:\Box(p\to q) \to (\Box p \to \Box q)}\,{\to}\text{-r}$$

2.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{2:p, 1R2 \Longrightarrow 2:p, 1:\Diamond p, 1:\Diamond q}\,\text{ax}}{2:p, 1R2 \Longrightarrow 1:\Diamond p, 1:\Diamond q}\,\Diamond\text{-r} \quad \dfrac{\dfrac{}{2:q, 1R2 \Longrightarrow 2:q, 1:\Diamond p, 1:\Diamond q}\,\text{ax}}{2:q, 1R2 \Longrightarrow 1:\Diamond p, 1:\Diamond q}\,\Diamond\text{-r}}{2:p\vee q, 1R2 \Longrightarrow 1:\Diamond p, 1:\Diamond q}\,\vee\text{-l}}{1:\Diamond(p\vee q) \Longrightarrow 1:\Diamond p, 1:\Diamond q}\,\Diamond\text{-l}}{1:\Diamond(p\vee q) \Longrightarrow 1:(\Diamond p \vee \Diamond q)}\,\vee\text{-r}}{\Longrightarrow 1:\Diamond(p\vee q) \to (\Diamond p \vee \Diamond q)}\,{\to}\text{-r}$$

3.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{2:p, 1R2 \Longrightarrow 2:p, 2:q, 1:\Diamond(p\vee q)}\,\text{ax}}{2:p, 1R2 \Longrightarrow 2:p\vee q, 1:\Diamond(p\vee q)}\,\vee\text{-r}}{2:p, 1R2 \Longrightarrow 1:\Diamond(p\vee q)}\,\Diamond\text{-r}}{1:\Diamond p \Longrightarrow 1:\Diamond(p\vee q)}\,\Diamond\text{-l} \quad \dfrac{\dfrac{\dfrac{}{2:q, 1R2 \Longrightarrow 2:p, 2:q, 1:\Diamond(p\vee q)}\,\text{ax}}{2:q, 1R2 \Longrightarrow 2:p\vee q, 1:\Diamond(p\vee q)}\,\vee\text{-r}}{2:q, 1R2 \Longrightarrow 1:\Diamond(p\vee q)}\,\Diamond\text{-r}}{1:\Diamond q \Longrightarrow 1:\Diamond(p\vee q)}\,\Diamond\text{-l}}{1:\Diamond p \vee \Diamond q \Longrightarrow 1:\Diamond(p\vee q)}\,\vee\text{-l}}{\Longrightarrow 1:(\Diamond p \vee \Diamond q) \to \Diamond(p\vee q)}\,{\to}\text{-r}$$

4.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{1:p,2:p,1:\Box p,1R2 \Longrightarrow 2:p}\text{ ax}}{1:p,2:p,1:\Box p,2:\neg p,1R2 \Longrightarrow}\neg\text{-l}}{1:p,1:\Box p,2:\neg p,1R2 \Longrightarrow}\Box\text{-l}}{1:p,1:\Box p,1:\Diamond\neg p \Longrightarrow}\Diamond\text{-l} \qquad \dfrac{}{1:p,1:\Diamond\neg p \Longrightarrow 1:p}\text{ ax}}{1:p,1:(p\to\Box p),1:\Diamond\neg p \Longrightarrow}\to\text{-l}}{1:p\to\Box p,1:\Diamond\neg p \Longrightarrow 1:\neg p}\neg\text{-r}}{1:p\to\Box p\wedge\Diamond\neg p \Longrightarrow 1:\neg p}\wedge\text{-l}}{\Longrightarrow 1:((p\to\Box p)\wedge(\Diamond\neg p))\to\neg p}\to\text{-r}$$

Exercise 10.2 (Model Semantics)

a) Let A be a *valid* formula of propositional logic, i.e. it is true in all propositional interpretations.

Show that $\Box A$ is valid in modal logic $\mathbf{K}$.

Show that $\Diamond A$ is *not* valid in modal logic $\mathbf{K}$.

b) Prove (using the model semantics) that the formula $\Box p \to \Box\Box p$ is satisfied at all worlds of any Kripke structure if the accessibility relation R is *transitive*.

Solution

a)

Let A be a valid propositional formula. It is therefore true in any propositional interpretation, and in particular $u \Vdash A$ in *any* world u of any Kripke model. This implies in particular that $u \Vdash A$ in any world u *reachable* from any world w. Therefore $w \Vdash \Box A$ for any world w of any Kripke model, which means that $\Box A$ is valid.

$\Diamond A$ is not valid: take a Kripke model in which not every world has an accessible world. E.g. R could be the empty relation. Let w be a world that has no u that is accessible from w. Then $w \nVdash \Diamond A$.

b)

Let w be an arbitrary world of a Kripke model with a transitive accessibility relation R. To show that $w \Vdash \Box p \to \Box\Box p$, we may assume that $w \Vdash \Box p$ (†), and have to show that $w \Vdash \Box\Box p$.

The latter means that $v \Vdash \Box p$ for all v such that wRv. This in turn means that $u \Vdash p$ for all u with vRu. In other words, we have to show that $u \Vdash p$ for all worlds u such that wRv and vRu for some v. Due to the transitivity, this always means that also wRu. 2 We now use our assumption (†) for u, which tells us that $u \Vdash p$, which is what we had to prove.

Exercise 10.3 (A Logic of Knowledge)

Here is an attempt at formalising the idea of "knowledge."

We assume that our knowledge is determined by what we have *observed* about the world. E.g.

the temperature, the colour of a light, etc. Let O be a set of possible observations, W a set of possible states of the world, and $o : W \to O$ an observation function that tells us for each state of the world what we have observed about it.

E.g. in the scenario where we may or may not have looked outside, and it may or may not rain, the set W would contain four worlds, according to the four possibilities. O would contain only three possible observations: NotLooked, LookedAndRain, LookedAndNotRain.

We can now say that two worlds $u, v \in W$ appear the same to us if our observations are the same, $o(u) = o(v)$. E.g. the two worlds where we have not looked outside appear the same because the observation is NotLooked for both of them

Now we can say that we "know" A in a world $w \in W$ if A is true in all worlds that fit our observations i.e., A is true in all worlds v with $o(u) = o(v)$.

a) Define an accessibility relation $R \subseteq W \times W$ such that the modal semantics of $\Box A$ is the same is this notion of knowledge.

b) Which of the properties of frames (reflexive, serial, transitive, etc.) we saw in the lecture does R have? Which of the modal logics listed in the lecture (K, D, T,...) corresponds to this notion of knowledge?

Solution

a)

We can define $(u, v) \in R :\Leftrightarrow o(u) = o(v)$.

Then $u \Vdash \Box A$ if $v \Vdash A$ holds in all worlds with $o(u) = o(v)$ – i.e. all worlds that have the same observations as u.

b)

The relation R defined by $(u, v) \in R :\Leftrightarrow o(u) = o(v)$ is easily seen to be reflexive, symmetric, and transitive: it is an equivalence relation on the set of worlds. The frame is thus an **S5** frame.

Converesely, any **S5** frame, i.e. one where R is an equivalence relation can be defined through a suitably chosen o: Let

$$o(u) := \{v \in W \mid uRv\} \quad ,$$

i.e. $o(u)$ is the equivalence class of u. With this definition, we get that

$$(u, v) \in R \quad \text{if and only if} o(u) = o(v) \quad .$$

So the modal logic **S5** expresses exatly the notion of knowledge that is based on the equality of observations.