IN3070
IN4070

Autumn 2020

Some solutions to exercises

# Logic

Week 11



## Exercise 12.1 (Intuitionistic Validity)

Prove, using the Kripke semantics and forcing relation, that the following formulae are valid in intuitionistic logic:

1. $p \to \neg\neg p$
2. $(p \wedge \neg q) \to \neg(p \to q)$
3. $(\exists x\, \neg p(x)) \to (\neg \forall x\, p(x))$

### Answer

First, a useful fact. To prove that $w_0 \Vdash A \to B$ for *all* worlds $w_0$, we have to show for all $w_0$ that $w \Vdash B$ for all worlds $w$ with $(w_0, w) \in R$ and $w \Vdash A$. That's the Kripke semantics for implication. Since all worlds $w$ are reachable from some other world (e.g. from $w$ itself), this is the same as showing for all $w \in W$ with $w \Vdash A$ that $w \Vdash B$. No need to worry about $w_0$ as long as we are interested in all $w_0$.

**1.** Let $\mathcal{I}_J = ((W, R), \{(D^w, \iota^w)\}_{w \in W})$ be a J-structure. As just discussed, we have to show that $w \Vdash \neg\neg p$ for all $w$ with $w \Vdash p$. Let $w$ be an arbitrary but fixed world with $w \Vdash p$.

To show $w \Vdash \neg\neg p$, we show that $v \nVdash \neg p$ for all $v$ with $(w, v) \in R$ (semantics of negation). So let $v$ be an arbitrary but fixed such world. Interpretations increase, and $w \Vdash p$, so also $v \Vdash p$. To show that $v \nVdash \neg p$, it suffices to show that there is a world reachable from $v$ which forces $p$. Remember that $R$ is reflexive, so $(v, v) \in R$, so $v$ is such a world that is reachable from $v$ that forces $p$.

**2.** Let $\mathcal{I}_J = ((W, R), \{(D^w, \iota^w)\}_{w \in W})$ be a J-structure. As discussed, we have to show that $w \Vdash \neg(p \to q)$ for all $w$ with $w \Vdash p \wedge \neg q$. Let $w$ be an arbitrary but fixed world with $w \Vdash p \wedge \neg q$.

The semantics tells us that $w \Vdash p$ (1) and $v \nVdash q$ for all $v$ reachable from $w$. (2)

To show $w \Vdash \neg(p \to q)$, we have to show that $v \nVdash p \to q$ for all $v$ reachable from $w$. Let $v$ be an arbitray but fixed such world. Interpretations increase, so from (1) it follows that $v \Vdash p$ (3). Also, from (2) we know that that $v \nVdash q$ (4).

To show $v \nVdash p \to q$, we have to find a world reachable from $v$ that forces $p$ but not $q$. $v$ itself is such a world, as (3) and (4) show.

**3.** Let $\mathcal{I}_J = ((W, R), \{(D^w, \iota^w)\}_{w \in W})$ be a J-structure. As discussed, we have to show that $w \Vdash \neg\forall x\, p(x)$ for all $w$ with $w \Vdash \exists x\, \neg p(x)$. Let $w$ be an arbitrary but fixed world with $w \Vdash \exists x\, \neg p(x)$.

The semantics tells us that there is a domain element $c \in D^w$ with $w \Vdash \neg p(c)$, and therefore $v \nVdash p(c)$ for every world $v$ reachable from $w$. (1)

To show $w \Vdash \neg\forall x\, p(x)$, we have to show that $u \nVdash \forall x\, p(x)$ for all worlds $u$ reachable from $w$. Let $u$ be an arbitrary but fixed such world.

From (1) we already know that $u \nVdash p(c)$. Since $(u, u) \in R$, we therefore have a combination of successor world and domain element such that the semantics of $\forall$ tells us that $u \nVdash \forall x\, p(x)$.

## Exercise 12.2 **(Intuitionistic Sequent Calculus)**

Prove the validity of the three formulae in 12.1 using the intuitionistic sequent calculus LJ.

**Solution**

**1**

$$\dfrac{\dfrac{\dfrac{\dfrac{}{p, \neg p \Longrightarrow p}\,\text{ax}}{p, \neg p \Longrightarrow}\,\neg\text{-l}}{p \Longrightarrow \neg\neg p}\,\neg\text{-r}}{\Longrightarrow p \to \neg\neg p}\,\to\text{-r}$$

**2**

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p, p \to q \Longrightarrow p}\,\text{ax} \quad \dfrac{}{p, q \Longrightarrow q}\,\text{ax}}{p, p \to q \Longrightarrow q}}{p, \neg q, p \to q \Longrightarrow}\,\neg\text{-l}}{p, \neg q \Longrightarrow \neg(p \to q)}\,\neg\text{-r}}{p \wedge \neg q \Longrightarrow \neg(p \to q)}\,\wedge\text{-l}}{\Longrightarrow (p \wedge \neg q) \to \neg(p \to q)}\,\to\text{-r}$$

**3**

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p(c), \forall x\, p(x) \Longrightarrow p(c)}\,\text{ax}}{\forall x\, p(x) \Longrightarrow p(c)}\,\forall\text{-l}}{\neg p(c), \forall x\, p(x) \Longrightarrow}\,\neg\text{-l}}{\neg p(c) \Longrightarrow \neg\forall x\, p(x)}\,\neg\text{-r}}{\exists x\, \neg p(x) \Longrightarrow \neg\forall x\, p(x)}\,\exists\text{-l}}{\Longrightarrow (\exists x\, \neg p(x)) \to (\neg\forall x\, p(x))}\,\to\text{-r}$$

## Exercise 12.3 **(Intuitionistic Countermodel)**

Show, using a falsifying J-structure, that the following formula is *not* valid in intutionistic logic:

$$\neg(p \wedge q) \to \neg p \vee \neg q$$

What happens if we try to prove the formula using LJ?

**Solution**

We can approach this systematically. If there is a J-structure that falsifies the formula, then it has a world $w_0$ with

$$w_0 \nVdash \neg(p \wedge q) \to \neg p \vee \neg q$$

It follows that there is a world $w$ with $(w_0, w) \in R$ with

$$w \Vdash \neg(p \wedge q)\ (\dagger) \quad \text{and} \quad w \nVdash \neg p \vee \neg q\ (\ddagger)$$

Now from $(\ddagger)$, we get that

$$w \nVdash \neg p \quad \text{and} \quad w \nVdash \neg q$$

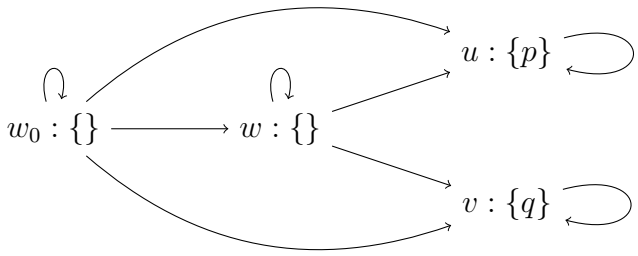
This means that there are worlds $u$ and $v$ with $(w, u) \in R$ and $(w, v) \in R$ and

$$u \Vdash p \quad \text{and} \quad v \Vdash q$$

$(\dagger)$ tells us that *none* of the successor worlds of $w$ forces $p \wedge q$. From this we know that $u \neq v$ and

$$u \nVdash q \quad \text{and} \quad v \nVdash p$$

The resulting sructure looks like this:




In all successors of $w$ (i.e. $u$, $v$, $w$), either $p$ or $q$ is *not* forced. I.e. none of the worlds forces $p \wedge q$. And therefore $w \Vdash \neg(p \wedge q)$. If the implication were valid, then we would also have $w \Vdash \neg p \vee \neg q$. But $w \nVdash \neg p$ because $u \Vdash p$. And $w \nVdash \neg q$ because $v \Vdash q$.

A proof attempt in LJ would start as follows

$$\frac{\neg(p \wedge q) \Longrightarrow \neg p \vee \neg q}{\Longrightarrow \neg(p \wedge q) \to \neg p \vee \neg q} \to\text{-r}$$

At this point the $\neg$-l rule would overwrite the succedent:

$$\dfrac{\dfrac{\dfrac{\neg(p \wedge q) \Longrightarrow p \qquad \neg(p \wedge q) \Longrightarrow q}{\neg(p \wedge q) \Longrightarrow p \wedge q}\ \wedge\text{-r}}{\neg(p \wedge q) \Longrightarrow \neg p \vee \neg q}\ \neg\text{-l}}{\Longrightarrow \neg(p \wedge q) \to \neg p \vee \neg q}\ \to\text{-r}$$

These leaf sequents are not valid classically, so they are certainly not valid intiuitionistically either. This is a dead end.

If we instead go for $\vee$-r, we have to decide for one of the disjunctions.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p, \neg(p \wedge q) \Longrightarrow p}\,\text{ax} \qquad p, \neg(p \wedge q) \Longrightarrow q}{p, \neg(p \wedge q) \Longrightarrow p \wedge q}\,\wedge\text{-r}}{p, \neg(p \wedge q) \Longrightarrow}\,\neg\text{-l}}{\neg(p \wedge q) \Longrightarrow \neg p}\,\neg\text{-r}}{\neg(p \wedge q) \Longrightarrow \neg p \vee \neg q}\,\vee\text{-r}_1}{\Longrightarrow \neg(p \wedge q) \rightarrow \neg p \vee \neg q}\,\rightarrow\text{-r}$$

There is no way to close the right branch. If we had stared with $\vee$-r$_2$, the left branch would have remained open.