IN3070
IN4070

Autumn 2019

Solutions to Exercises for the Course

Logic

Week 3



Exercise 3.1 (Variables & Assignments)

a) Which variables occur free in the following formulae? Which variables occur bound?

b) Apply the following substitution to both formulae: $\{x\backslash a, y\backslash f(a,b), z\backslash g(x,c)\}$.

c) Is the substitution capture-free for these formulae?

1. $\forall x\,(p(y) \land \forall y\,(p(x,z) \to q(y)))$
2. $\forall x\,\exists x\,\forall y\,(p(x) \lor \exists y\,q(y)) \to r(x)$

Solution

a) The free variables are colored red, the bound are colored green.

1. $\forall x(p(y) \land \forall y(p(x,z) \to q(y)))$
   y occurs free in $p(y)$, z in $p(x,z)$. x occurs bound in $p(x,z)$, y in $q(y)$.
2. $\forall x\,\exists x\,\forall y\,(p(x) \lor \exists y\,q(y)) \to r(x)$
   x occurs free in $r(x)$. x occurs bound in $p(x)$, and y in $q(y)$.

b) Let $\sigma = \{x\backslash a, y\backslash f(a,b), z\backslash g(x,c)\}$.

1.
$$\begin{aligned}&\sigma(\forall x(p(y) \land \forall y(p(x,z) \to q(y))))\\ &= \forall x(p(f(a,b)) \land \forall y(p(x,g(x,c)) \to q(y)))\end{aligned}$$

2.
$$\begin{aligned}&\sigma(\forall x\,\exists x\,\forall y\,(p(x) \lor \exists y\,q(y)) \to r(x))\\ &= \forall x\,\exists x\,\forall y\,(p(x) \lor \exists y\,q(y)) \to r(a)\end{aligned}$$

c) It is not capture-free for formula 1, since x is bound in $g(x,c)$. It is capture-free for formula 2.

Exercise 3.2 (Formalization & Interpretations)

a) "If there is a man in town who shaves all the men in town who do not shave themselves, then some man in town shaves himself." Formalize these statements by a single first-order formula F_S.

Solution

We use a binary predicate s, where $s(x, y)$ denotes that x shaves y.

"y is a man who does not shave himself" — $\neg s(y, y)$

"If y does not shave himself, then x shaves y" — $\neg s(y, y) \to s(x, y)$

"x shaves all the men in town who do not shave themselves" — $\forall y\,(\neg s(y, y) \to s(x, y))$

"there is a man in town who shaves all the men in town who do not shave themselves" — $\exists x \forall y\,(\neg s(y, y) \to s(x, y))$

So all together:

$$\exists x \forall y\,(\neg s(y, y) \to s(x, y)) \;\to\; \exists z\, s(z, z)$$

b) What is the value of the term $(5 + 3) * (8 - 5)$ under the interpretation $\mathcal{I} = (\mathbb{N}, \iota)$ with $+^\iota = *$, $*^\iota = -$, $-^\iota = \div$ (division), $3^\iota = 8$, $5^\iota = 6$, $8^\iota = 36$.

Solution To reduce the confusion between the symbols in our alphabet and the usual mathematical ones, we colour the terms red.

$$\begin{aligned}((5 + 3) * (8 - 5))^\iota &= (5 + 3)^\iota - (8 - 5)^\iota \\ &= (5^\iota * 3^\iota) - (8^\iota \div 5^\iota) \\ &= (6 * 8) - (36 \div 6) \\ &= 48 - 6 \\ &= 42\end{aligned}$$

c) Show that the following formulae are satisfiable or invalid (or both) by providing a model and/or a counter-model (i.e. an interpretation that falsifies the formula).

1. $\forall x\, p(f(x), a) \to \exists x\, p(g(x), x)$
2. $\forall y\, q(y, b) \to \exists x\, q(a, x)$

Solution

1. $\forall x\, p(f(x), a) \to \exists x\, p(g(x), x)$

   The formula is satisfiable. The following is a model:

$$\begin{aligned}\mathcal{I} &= (D, \iota) & D &= \{1, 2\} \\ f^\iota = g^\iota &= \{(1, 1), (2, 2)\} & p^\iota &= \emptyset \\ a^\iota &= 1\end{aligned}$$

This is a model because p is not true for any pair of domain elements, so $v_{\mathcal{I}}(\forall x\, p(f(x), a)) = F$, and therefore the implication is true.

The formula is also invalid. The following is a countermodel:

$$\begin{aligned} \mathcal{I} &= (D, \iota) & D &= \{1, 2\} \\ f^\iota(1) &= f^\iota(2) = 1 & p^\iota &= \{(1, 1)\} \\ g^\iota(1) &= g^\iota(2) = 2 & a^\iota &= 1 \end{aligned}$$

2. $\forall y\, q(y, b) \to \exists x\, q(a, x)$

   The formula is satisfiable. The following is a model:

$$\begin{aligned} \mathcal{I} &= (D, \iota) & D &= \{1\} \\ a^\iota &= 1 & b^\iota &= 1 \\ q^\iota &= \{(1, 1)\} \end{aligned}$$

   The formula is valid. Suppose that $\mathcal{I}$ falsifies the formula. Then $\mathcal{I}$ satisfies $\forall y\, q(y, b)$ and falsifies $\exists x\, q(a, x)$. For all domain elements d, $(d, b^\iota) \in q^\iota$. Then it must also be true that $(a^\iota, b^\iota) \in q^\iota$, so $\exists x\, q(a, x)$ is satisfied.

Exercise 3.3 (Sequent Calculus LK & Eigenvariables)

Prove the validity of the following formulae in the sequent calculus LK. If you cannot find a proof in LK then povide a counter-model.

1. The first-order formula F_S from Exercise 3.2 a).
2. $\forall x\, \exists y\, p(x, y) \to \forall u\, \exists v\, p(u, v)$
3. $\forall x\, \exists y\, p(x, y) \to \exists v\, \forall u\, p(u, v)$

Solution for 3.3.1

Intuition: some man shaves all men who do not shave them selves. Assume that this man does not shave himself. Then he would be among the men who do not shave themseles, and therefore he would have to shave himself anyway. Contradiction. So he does shave himself.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\forall y \ldots,\ s(a,a) \implies s(a,a),\ \exists z \ldots}\text{ axiom}}{\forall y \ldots \implies \neg s(a,a),\ s(a,a),\ \exists z \ldots}\neg\text{-right} \qquad \dfrac{}{\forall y \ldots,\ s(a,a) \implies s(a,a),\ \exists z \ldots}\text{ axiom}}{\forall y \ldots,\ \neg s(a,a) \to s(a,a) \implies s(a,a),\ \exists z \ldots}\to\text{-left}}{\forall y \ldots,\ \neg s(a,a) \to s(a,a) \implies \exists z\, s(z,z)}\exists\text{-right}}{\forall y\, (\neg s(y,y) \to s(a,y)) \implies \exists z\, s(z,z)}\forall\text{-left}}{\exists x \forall y\, (\neg s(y,y) \to s(x,y)) \implies \exists z\, s(z,z)}\exists\text{-left}}{\implies \exists x \forall y\, (\neg s(y,y) \to s(x,y)) \to \exists z\, s(z,z)}\to\text{-right}$$

Solution for 3.3.2 The two halves of this formula are equal up to renaming of the quantified variables. But LK has no rule that allows us to rename variables, so we have to 'unravel' both formulae down to the atoms. An $\forall$-right application will provide a constant that we can use to instantiate an $\forall$ formula on the left, and $\exists$-left applications give the constants for the $\exists$ formulae on the right.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\forall x \ldots, p(a,b) \implies \exists v \ldots, p(a,b)}{\forall x \ldots, p(a,b) \implies \exists v\, p(a,v)}\,\exists\text{-right}}{\forall x \ldots, \exists y\, p(a,y) \implies \exists v\, p(a,v)}\,\exists\text{-left}}{\forall x\, \exists y\, p(x,y) \implies \exists v\, p(a,v)}\,\forall\text{-left}}{\forall x\, \exists y\, p(x,y) \implies \forall u\, \exists v\, p(u,v)}\,\forall\text{-right}}{\implies \forall x\, \exists y\, p(x,y) \to \forall u\, \exists v\, p(u,v)}\,\to\text{-right}$$

Solution for 3.3.3

If we try to construct a proof, we get the following:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\forall x \ldots, p(b,c) \implies p(b,a), \exists v \ldots}{\forall x \ldots, \exists y\, p(b,y) \implies p(b,a), \exists v \ldots}\,\exists\text{-left}}{\forall x\, \exists y\, p(x,y) \implies p(b,a), \exists v \ldots}\,\forall\text{-left}}{\forall x\, \exists y\, p(x,y) \implies \forall u\, p(u,a), \exists v \ldots}\,\forall\text{-right}}{\forall x\, \exists y\, p(x,y) \implies \exists v\, \forall u\, p(u,v)}\,\exists\text{-right}}{\implies \forall x\, \exists y\, p(x,y) \to \exists v\, \forall u\, p(u,v)}\,\to\text{-right}$$

We could continue, using $\exists$-right with the new constant c. But we will never be able to close the branch, since repeated applications of $\exists$-left application in the antecedent and $\forall$-right in the succedent keep producing new constants that prevent the atoms on the left and the right from becoming equal.

The formula $\forall x\, \exists y\, p(x,y) \to \exists v\, \forall u\, p(u,v)$ is invalid. Countermodel:

$$\mathcal{I} = (D, \iota) \qquad\qquad D = \{1,2\}$$
$$p^\iota = \{(1,2),(2,1)\}$$

$\mathcal{I}$ satisfies $\forall x\, \exists y\, p(x,y)$ because $v_\mathcal{I}(\alpha\{x \leftarrow 1\}\{y \leftarrow 2\}, p(x,y)) = T$ and $v_\mathcal{I}(\alpha\{x \leftarrow 2\}\{y \leftarrow 1\}, p(x,y)) = T$. $\mathcal{I}$ falsifies $\exists v\, \forall u\, p(u,v)$ because there exists no domain element d such that $(1,d) \in p^\iota$ and $(2,d) \in p^\iota$.

Exercise 3.4 **(Symmetry of LK)**

Let A be a formula containing only $\forall$, $\exists$ and the connectives $\neg$, $\vee$, and $\wedge$. The dual formula A' of A is obtained by exchanging $\forall$ and $\exists$, and exchanging $\vee$ and $\wedge$. Prove that $\vdash A$ iff $\vdash \neg A'$.

Hint: try to explain how to transform an LK proof for A into a proof for $\neg A'$, and vice versa

Solution

After applying $\neg$-right to $\implies \neg A'$, we have the sequent $A' \implies$.

Each LK rule has a dual rule which does the same, except that the antecedent and succedent are swapped. For instance, $\wedge$-left has the dual rule $\vee$-right, etc. To transform the proof for A into a proof for $\neg A'$, apply the dual rule to $A' \implies$ for each rule application to A.

Exercise 3.5 (Variable Assignments and Closed Formulas)

The term value, resp. truth value of a *closed* term t, resp. formula A in an interpretation $\mathcal{I} = (D, \iota)$ is independent of the variable assignment. I.e. if α and β are two variable assignments for $\mathcal{I}$, and t is closed, then

$$v_{\mathcal{I}}(\alpha, t) = v_{\mathcal{I}}(\beta, t)$$

and if A is closed then

$$v_{\mathcal{I}}(\alpha, A) = v_{\mathcal{I}}(\beta, A)$$

Prove these facts by structural induction on t and A.

Solution

Most of this proof looks straightforward, using structural induction. However, the case for $v_{\mathcal{I}}(\alpha, \exists x\, A) = v_{\mathcal{I}}(\beta, \exists x\, A)$ breaks down: $v_{\mathcal{I}}(\alpha\{x \leftarrow d\}, A) = v_{\mathcal{I}}(\beta\{x \leftarrow d\}, A)$ does not follow from the induction hypothesis, since A is not necessarily closed, and the IH (like our theorem) talks only about closed formulae.

What we have to do in such cases is to *strengthen* or *generalise* the inductive statement, and show something more. We will show that the variable assignment is only important for the free variables of t resp. A.

> If t is a term (resp. A a formula) with free variables V and α and β are two variable assignments for $\mathcal{I}$ with
>
> $$\alpha(x) = \beta(x) \quad \text{for all } x \in V$$
>
> then and t is closed, then
>
> $$v_{\mathcal{I}}(\alpha, t) = v_{\mathcal{I}}(\beta, t)$$
>
> $$v_{\mathcal{I}}(\alpha, A) = v_{\mathcal{I}}(\beta, A)$$

For closed terms and formulae, $V = \emptyset$, and the equalities hold for arbitrary α and β so the statement in the exercise is a consequence of this stronger statement.

Proof for $v_{\mathcal{I}}(\alpha, t) = v_{\mathcal{I}}(\beta, t)$:

Induction base: if t is a constant a:

$$v_{\mathcal{I}}(\alpha, a) = a^{\iota} = v_{\mathcal{I}}(\beta, a)$$

If t is a variable x, then x is free in t so $\alpha(x) = \beta(x)$.

$$v_{\mathcal{I}}(\alpha, x) = \alpha(x) = \beta(x) = v_{\mathcal{I}}(\beta, x)$$

Induction hypothesis:

If α and β agree on all variables in t_i

$$v_{\mathcal{I}}(\alpha, t_i) = v_{\mathcal{I}}(\beta, t_i)$$

Let f be an n-ary function symbol and $t = f(t_0, \ldots, t_n)$. Then the free variables of t are all the free variables in all subterms t_i. Thus, if α and β are equal on all free variables of t, then they are also equal on all free variables of the subterms t_i. Then

$$\begin{aligned}
& v_{\mathcal{I}}(\alpha, f(t_0, \ldots, t_n)) \\
&= f^{\iota}(v_{\mathcal{I}}(\alpha, t_0), \ldots, v_{\mathcal{I}}(\alpha, t_n)) && \textit{by definition of term values} \\
&= f^{\iota}(v_{\mathcal{I}}(\beta, t_0), \ldots, v_{\mathcal{I}}(\beta, t_n)) && \textit{by induction hypothesis} \\
&= v_{\mathcal{I}}(\beta, f(t_0, \ldots, t_n)) && \textit{by definition of term values}
\end{aligned}$$

Proof for $v_{\mathcal{I}}(\alpha, A) = v_{\mathcal{I}}(\beta, A)$:

Induction base: $A = p$ or $A = p(t_0, \ldots, t_n)$ f. As for compound terms, if α and β are equal on all free variables of A, then they are also equal on all free variables of the subterms t_i. If $p^{\iota} = T$ then $v_{\mathcal{I}}(\alpha, p) = T = v_{\mathcal{I}}(\beta, p)$, otherwise $v_{\mathcal{I}}(\alpha, p) = F = v_{\mathcal{I}}(\beta, p)$.

By model semantics, $v_{\mathcal{I}}(\alpha, p(t_0, \ldots, t_n)) = T$ iff $(v_{\mathcal{I}}(\alpha, t_0), \ldots, v_{\mathcal{I}}(\alpha, t_n)) \in p^{\iota}$. We already proved that $(v_{\mathcal{I}}(\alpha, t_0), \ldots, v_{\mathcal{I}}(\alpha, t_n)) = (v_{\mathcal{I}}(\beta, t_0), \ldots, v_{\mathcal{I}}(\beta, t_n)$, so $v_{\mathcal{I}}(\alpha, p(t_0, \ldots, t_n)) = v_{\mathcal{I}}(\beta, p(t_0, \ldots, t_n))$.

Induction hypothesis: If A and B are formulae, and α and β agree on their free variables, then $v_{\mathcal{I}}(\alpha, A) = v_{\mathcal{I}}(\beta, A)$ and $v_{\mathcal{I}}(\alpha, B) = v_{\mathcal{I}}(\beta, B)$.

Now, we need to show this for $\neg A$, $A \wedge B$, $A \vee B$, $A \to B$, $\exists x\, A$, and $\forall x\, A$.

Suppose $v_{\mathcal{I}}(\alpha, A) = T$. By the induction hypothesis, $v_{\mathcal{I}}(\beta, A) = T$ also. By model semantics, $v_{\mathcal{I}}(\alpha, \neg A) = F = v_{\mathcal{I}}(\beta, \neg A)$. Now suppose $v_{\mathcal{I}}(\alpha, A) = F$. By the induction hypothesis, $v_{\mathcal{I}}(\beta, A) = F$. By model semantics, $v_{\mathcal{I}}(\alpha, \neg A) = T = v_{\mathcal{I}}(\beta, \neg A)$. The proof is similar for $A \wedge B$, $A \vee B$, and $A \to B$, if we remember that the free variables of A and B are also free in these formulae.

Proof for $\exists x\, A$:

$v_{\mathcal{I}}(\alpha, \exists x\, A) = T$ iff $v_{\mathcal{I}}(\alpha\{x \leftarrow d\}, A) = T$ for some $d \in D$. The formula A might have the free variable x in addition to the free variables of $\exists x\, A$, so if α and β agree on all free variables V of $\exists x\, A$

$$\alpha(x) = \beta(x) \quad \text{for all } x \in V,$$

this does not necessarily hold for all variables $V \cup \{x\}$ of A. We do however have

$$\alpha\{x \leftarrow d\}(x) = \beta\{x \leftarrow d\}(x) \quad \text{for all } x \in V \cup \{x\},$$

where the modification takes care of the additional variable x.

We may thus apply the induction hypothesis to conclude that $v_{\mathcal{I}}(\alpha\{x \leftarrow d\}, A) = v_{\mathcal{I}}(\beta\{x \leftarrow d\}, A)$. Similarly for $\forall x\, A$.