Exercises for the Course

Logic

Week 6



Exercise 6.1 (Resolution Calculus)

Show that the following set of clauses is unsatisfiable, using resolution.

$$\{\{p, q\}, \{\neg p, r\}, \{\neg p, \neg r\}, \{p, \neg q\}\}$$

Solution

1. $\{p, q\}$
2. $\{\neg p, r\}$
3. $\{\neg p, \neg r\}$
4. $\{p, \neg q\}$
5. $\{p\}$ – Resolvent of 1. and 4.
6. $\{\neg p\}$ – Resolvent of 2. and 3.
7. $\square$ – Resolvent of 5. and 6.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\{\{p, q\}, \{\neg p, r\}, \{\neg p, \neg r\}, \{p, \neg q\}, \{p\}, \{\neg p\}, \square\}}\text{ axiom}}{\{\{p, q\}, \{\neg p, r\}, \{\neg p, \neg r\}, \{p, \neg q\}, \{p\}, \{\neg p\}\}}\text{ resolution}}{\{\{p, q\}, \{\neg p, r\}, \{\neg p, \neg r\}, \{p, \neg q\}, \{p\}\}}\text{ resolution}}{\{\{p, q\}, \{\neg p, r\}, \{\neg p, \neg r\}, \{p, \neg q\}\}}\text{ resolution}$$

Exercise 6.2 (CNF and Resolution Calculus)

Show that the first two formulae are valid and the third formula is unsatisfiable by using the resolution calculus. Remember that you may have to negate some of the formulae before transforming them to clause form.

a) $(p \to (q \to r)) \to (p \wedge q \to r)$

b) $(((\neg p \vee q) \to q) \wedge (q \to r) \wedge \neg r) \to p$

c) $\neg p \wedge (\neg q \vee r) \wedge (\neg p \to (q \wedge \neg r))$

Solutions

a) Since we want to prove that the formula is valid, we negate it and prove that the resulting set of clauses is unsatisfiable.

$$
\begin{aligned}
& \neg((p \to (q \to r)) \to (p \wedge q \to r)) \\
\equiv\ & \neg(\neg(\neg p \vee (\neg q \vee r)) \vee (\neg(p \wedge q) \vee r)) && \textit{Using } A \to B \equiv \neg A \vee B \\
\equiv\ & \neg\neg(\neg p \vee (\neg q \vee r)) \wedge \neg(\neg(p \wedge q) \vee r) && \textit{Using } \neg(A \vee B) \equiv \neg A \wedge \neg B \\
\equiv\ & \neg\neg(\neg p \vee (\neg q \vee r)) \wedge \neg\neg(p \wedge q) \wedge \neg r && \textit{Using } \neg(A \vee B) \equiv \neg A \wedge \neg B \\
\equiv\ & (\neg p \vee (\neg q \vee r)) \wedge (p \wedge q) \wedge \neg r && \textit{Using } \neg\neg A \equiv A
\end{aligned}
$$

The resulting set of clauses is

$$\{\{\neg p,\ \neg q,\ r\},\ \{p\},\ \{q\},\ \{\neg r\}\}$$

1. $\{\neg p,\ \neg q,\ r\}$
2. $\{p\}$
3. $\{q\}$
4. $\{\neg r\}$
5. $\{\neg p, \neg q\}$ – Resolvent of 1. and 4.
6. $\{\neg p\}$ – Resolvent of 3. and 5.
7. $\square$ – Resolvent of 2. and 6.

b) Again, we need to prove that the negation is unsatisfiable.

$$
\begin{aligned}
& \neg((((\neg p \vee q) \to q) \wedge (q \to r) \wedge \neg r) \to p) \\
\equiv\ & (p \vee q) \wedge (\neg q \vee q) \wedge (\neg q \vee r) \wedge \neg r \wedge \neg p
\end{aligned}
$$

he resulting set of clauses is

$$\{\{p,\ q\},\ \{\neg q,\ q\},\ \{\neg q,\ r\},\ \{\neg r\},\ \{\neg p\}\}$$

1. $\{p,\ q\}$
2. $\{\neg q,\ q\}$ – Since this clause is valid, we could remove it if we wanted to
3. $\{\neg q,\ r\}$
4. $\{\neg r\}$
5. $\{\neg p\}$
6. $\{q\}$ – Resolvent of 1. and 5.
7. $\{\neg q\}$ – Resolvent of 3. and 4.

8. $\square$ – Resolvent of 6. and 7.

c) Since we want to prove that this is *unsatisfiable*, we don't need to negate this formula.

$$\begin{aligned}
&\neg p \wedge (\neg q \vee r) \wedge (\neg p \rightarrow (q \wedge \neg r)) \\
\equiv\ &\neg p \wedge (\neg q \vee r) \wedge (p \vee (q \wedge \neg r)) && \textit{Converted to NNF} \\
\equiv\ &\neg p \wedge (\neg q \vee r) \wedge (p \vee q) \wedge (p \vee \neg r) && \textit{Using } A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)
\end{aligned}$$

The resulting set of clauses is

$$\{\{\neg p\},\ \{\neg q,\ r\},\ \{p,\ q\},\ \{p,\ \neg r\}\}$$

1. $\{\neg p\}$
2. $\{\neg q,\ r\}$
3. $\{p,\ q\}$
4. $\{p,\ \neg r\}$
5. $\{p,\ r\}$ – Resolvent of 2. and 3
6. $\{p\}$ – Resolvent of 4. and 5
7. $\square$ – Resolvent of 1. and 6.

Exercise 6.2 **(Positive and Negative clauses)**

A *positive clause* is one where none of the atoms is negated, e.g. $\{p, q\}$. A *negative clause* is one where all of the atoms are negated, e.g. $\{\neg p, \neg q, \neg r\}$.

(The empty clause counts as both positive and negative.)

Show that every unsatisfiable clause set S contains at least one negative clause and at least one positive clause.

Hint: there is a very simple interpretation that satisfies all clauses that are not positive. And another, also very simple interpretation that satisfies all clauses that are not negative.

Solution

Proof idea: Prove the contrapositive of the statement.

If S contains no negative clauses, or S contains no positive clauses, then S is satisfiable.

We can split this statement into two cases:

(i) If S contains no negative clauses, then S is satisfiable.

(ii) If S contains no positive clauses, then S is satisfiable.

Proof for (i):

Suppose S contains no negative clauses. Then every clause in S contains at least one non-negated atom p. We construct an interpretation $\mathcal{I} \models S$ that makes all atoms true.

Proof for (ii):

Very similar. This time, each clause in S contains at least one negated atom $\neg p$. We construct an interpretation $\mathcal{I} \models S$ that makes all atoms p false, thus satisfying $\neg p$.