Exercises for the Course

Logic

Week 9

Exercise 9.1 (Atomic Cut and Unit Propagation)

Using atomic cut and unit propagation, show that the following formulas (sets of clauses) are unsatisfiable, or give a satisfying substitution

1. $(\neg p \vee r \vee \neg s) \wedge (p \vee q \vee r \vee \neg s) \wedge (\neg p \vee \neg t) \wedge \neg r \wedge (p \vee s) \wedge (p \vee r \vee t) \wedge (s \vee t)$
2. $\{\{p, \neg q\}, \{\neg p, q\}, \{q, \neg r\}, \{s\}, \{\neg s, \neg q, \neg r\}, \{s, r\}\}$
3. $\{\{p, q, s, t\}, \{p, s, \neg t\}, \{q, \neg s, t\}, \{p, \neg s, \neg t\}, \{p, \neg q\}, \{\neg r, \neg p\}, \{r\}\}$

Solution

1. For formatting reasons, we write the branches separately.

Root: $(\neg p \vee r \vee \neg s) \wedge (p \vee q \vee r \vee \neg s) \wedge (\neg p \vee \neg t) \wedge \neg r \wedge (p \vee s) \wedge (p \vee r \vee t) \wedge (s \vee t)$

We start with unit propagation of $\neg r$, which gives:

$(\neg p \vee \neg s) \wedge (p \vee q \vee \neg s) \wedge (\neg p \vee \neg t) \wedge \neg r \wedge (p \vee s) \wedge (p \vee t) \wedge (s \vee t)$

Atomic cut on p:

1. p. Unit propagation (simplification) gives:
   $(\neg s) \wedge (\neg t) \wedge \neg r \wedge (s \vee t)$
   Now unit propagation of $\neg s$ gives:
   $(\neg s) \wedge (\neg t) \wedge \neg r \wedge (t)$
   We can now close the branch with t and $\neg t$.
2. $\neg p$. Unit propagation (simplification) gives:
   $(q \vee \neg s) \wedge \neg r \wedge (s) \wedge (t) \wedge (s \vee t)$
   Now unit propagation of s gives:
   $(q) \wedge \neg r \wedge (s) \wedge (t)$
   We get a satisfying assignment $\neg p, q, \neg r, s, t$.

2.

Root: $\{\{p, \neg q\}, \{\neg p, q\}, \{q, \neg r\}, \{s\}, \{\neg s, \neg q, \neg r\}, \{s, r\}\}$

Unit propagation of s gives:
$\{\{p, \neg q\}, \{\neg p, q\}, \{q, \neg r\}, \{s\}, \{\neg q, \neg r\}\}$

Atomic cut on p:

1. p. Unit propagation gives:
   $\{\{q\}, \{q, \neg r\}, \{s\}, \{\neg q, \neg r\}\}$
   Unit propagation of q:
   $\{\{q\}, \{s\}, \{\neg r\}\}$
   A satisfying assignment is $p, q, s, \neg r$.

2. $\neg p$. Unit propagation gives:
   $\{\{\neg q\}, \{q, \neg r\}, \{s\}, \{\neg q, \neg r\}\}$
   Unit propagation of $\neg q$:
   $\{\{\neg q\}, \{\neg r\}, \{s\}\}$
   So we get a second satisfying assignment $\neg p, \neg q, \neg r, s$

3.

Root: $\{\{p, q, s, t\}, \{p, s, \neg t\}, \{q, \neg s, t\}, \{p, \neg s, \neg t\}, \{p, \neg q\}, \{\neg r, \neg p\}, \{r\}\}$

Unit propagation of r:
$\{\{p, q, s, t\}, \{p, s, \neg t\}, \{q, \neg s, t\}, \{p, \neg s, \neg t\}, \{p, \neg q\}, \{\neg p\}, \{r\}\}$

Unit propagation of $\neg p$:
$\{\{q, s, t\}, \{s, \neg t\}, \{q, \neg s, t\}, \{\neg s, \neg t\}, \{\neg q\}, \{\neg p\}, \{r\}\}$

Unit propagation of $\neg q$:
$\{\{s, t\}, \{s, \neg t\}, \{\neg s, t\}, \{\neg s, \neg t\}, \{\neg q\}, \{\neg p\}, \{r\}\}$

Atomic cut on s:

1. s. Unit propagation:
   $\{\{t\}, \{\neg t\}, \{\neg q\}, \{\neg p\}, \{r\}\}$
   Branch closed with t and $\neg t$.

2. $\neg s$. Unit propagation:
   $\{\{t\}, \{\neg t\}, \{\neg q\}, \{\neg p\}, \{r\}\}$
   Branch closed with t and $\neg t$.

Exercise 9.2 **(Properties of Simplification)**

Prove that the simplification rules are sound.

a) Given a propositional formula A that contains a subformula B, and let $B \equiv B'$ (logical equivalence). Prove A is logically equivalent to the result of replacing B by B' in A. *Hint: use structural induction on A*

b) Given propositional formulae A and B and an interpretation $\mathcal{I}$ with $\mathcal{I} \models B$. Prove that $\mathcal{I} \models A$ if and only if $\mathcal{I} \models A[B]$. *Hint: for the replacement of B by true or false, use structural induction on A similarly to part A. For the simplification steps, use part a).*

c) Prove that the simplification rule

$$\frac{B,\ A[B],\ \Gamma \Longrightarrow \Delta}{B,\ A,\ \Gamma \Longrightarrow \Delta}$$

preserves falsifiability upwards.

Solution

Part a) We prove by structural induction on A that the result A' of replacing B by B' in A is equivalent to A. We assume for this proof that only one particular occurrence of B is replaced. It is also possible to show this for the result of replacing *all* occurrences of B by B'.

Base case: A is an atomic formula, containing B. A must equal B and A' = B'. Then $v_{\mathcal{I}}(A) = v_{\mathcal{I}}(B) = v_{\mathcal{I}}(B') = v_{\mathcal{I}}(A')$.

Induction step $A = \neg A_1$, where A contains B.

Case $\neg$-1: $A = B$. Then A' = B' and $v_{\mathcal{I}}(A) = v_{\mathcal{I}}(B) = v_{\mathcal{I}}(B') = v_{\mathcal{I}}(A')$, like in the base case.

Case $\neg$-2: A_1 contains B. Let A'_1 be the result of replacing B ny B' in A_1. Then $A' = \neg A'_1$.

The induction hypothesis tells us that A_1 is logically eqivalent to A'_1.

$v_{\mathcal{I}}(A) = T$ iff $v_{\mathcal{I}}(A_1) = F$ iff $v_{\mathcal{I}}(A'_1) = F$ (I.H.) iff $v_{\mathcal{I}}(A') = T$

Induction step $A = A_1 \wedge A_2$, where A contains B.

Case $\wedge$-1: $A = B$. Same as before.

Case $\wedge$-2: A_1 contains B. Let A'_1 be the result of replacing B ny B' in A_1. Then $A' = A'_1 \wedge A_2$.

The induction hypothesis tells us that A_1 is logically eqivalent to A'_1.

$v_{\mathcal{I}}(A) = T$ iff [$v_{\mathcal{I}}(A_1) = T$ and $v_{\mathcal{I}}(A_2) = T$] iff [$v_{\mathcal{I}}(A'_1) = T$ and $v_{\mathcal{I}}(A_2) = T$, by the I.H.] iff $v_{\mathcal{I}}(A') = T$

Case $\wedge$-3: A_2 contains B: symmetric to case $\wedge$-2.

The inductions steps for $\vee$ and $\rightarrow$ are essentially the same as for $\wedge$.

Part b) Let $\mathcal{I} \models B$. Let A^* be the formula that results from replacing all occurrences of B by $true$ in A. We first show by structural induction on A that $v_{\mathcal{I}}(A) = v_{\mathcal{I}}(A^*)$.

Base case: A is an atomic formula.

If $A = B$, then $A^* = true$ and $v_{\mathcal{I}}(A) = v_{\mathcal{I}}(B) = T = v_{\mathcal{I}}(true) = v_{\mathcal{I}}(A^*)$.

Otherwise, $A^* = A$ and $v_{\mathcal{I}}(A) = v_{\mathcal{I}}(A^*)$.

Induction step $A = \neg A_1$.

If $A = B$, the proof is as before.

Otherwise $A^* = \neg A^*_1$, and

$v_{\mathcal{I}}(A) = T$ iff $v_{\mathcal{I}}(A_1) = F$ iff (by ind. hyp.) $v_{\mathcal{I}}(A^*_1) = F$ iff $v_{\mathcal{I}}(A^*) = T$.

Induction step $A = A_1 \wedge A_2$, where A contains B..

If $A = B$, the proof is as before.

Otherwise $A^* = A^*_1 \wedge A^*_2$

$v_{\mathcal{I}}(A) = T$ iff [$v_{\mathcal{I}}(A_1) = T$ and $v_{\mathcal{I}}(A_2) = T$] iff [$v_{\mathcal{I}}(A_1^*) = T$ and $v_{\mathcal{I}}(A_2^*) = T$, by the I.H.] iff $v_{\mathcal{I}}(A^*) = T$

The inductions steps for $\vee$ and $\rightarrow$ are essentially the same as for $\wedge$.

The proof for $\mathcal{I} \models \neg B$ where B is replaced by $false$ is similar.

To get from A^* to $A[B]$, we have to apply a sequence of replacements, each of which replaces a subformula by a logically equivalent one. Due to part a) the resulting formula $A[B]$ will be equivalent to the starting point A^*.

Part c)

If an interpretation $\mathcal{I}$ falsifies the conclusion of the rule, then $\mathcal{I} \models B$, $\mathcal{I} \models A$, $\mathcal{I} \models \Gamma$, and $\mathcal{I} \not\models D$ for any $D \in \Delta$.

Due to part c), we then now that $\mathcal{I} \models A[B]$, and therefore $\mathcal{I}$ falsifies the premisse of the rule.

Exercise 9.3 **(Pure Literal Elimination)**

Given a set of clauses S, a literal L is *pure* for S if it occurs only positively or only negatively. I.e. if there are no clauses $C, D \in S$ with $L \in C$ and $\overline{L} \in D$.

In a refutation procedure like DPLL or resolution, if there is a pure literal L, it is OK to remove all clauses that contain L, and this is why:

Let $S' := \{C \in S \mid L \notin C\}$ be the set of clauses in S that do not contain L. Show that if L is pure, then S is satisfiable if and only if S' is satisfiable.

Hint: it is easiest to consider two cases, depending on whether L is an atom like p or a negated atom like $\neg p$. In both cases, if L is pure, it is easy to find an interpretation that satisfies all clauses containing L.

Solution

We show that there is an interpretation $\mathcal{I}$ with $\mathcal{I} \models S$ iff there is an interpretation $\mathcal{I}'$ with $\mathcal{I}' \models S'$.

The direction from left to right is easy: Since $S' \subseteq S$, any interpretation that satisfies all clauses in S satisfies all clauses in S'.

For the direction from right to left, let $\mathcal{I}' \models S'$ and note that S' does not contain the pure literal L.

Assume for now that the pure literal L is an atom that occurs only positively in S. Let $\mathcal{I}$ by the interpretation that interprets L as true, and all other propositional symbols like $\mathcal{I}'$.

Then $\mathcal{I} \models S'$ since L does not occur in S' and $\mathcal{I}$ is identical to $\mathcal{I}'$ on all other propositional symbols.

On the other hand, $\mathcal{I} \models S \setminus S'$ since $\mathcal{I}$ satisfies any clause $L \vee C$ that contains L positively.

Together, $\mathcal{I} \models S$.

If the pure literal is a negated atom $\neg p$, we let $\mathcal{I}$ interpret p as false, so $\neg p$ as true, and use the same argument.