IN3070
IN4070

Autumn 2021

Exercises for the Course

Logic

Solution 2.0



Solutions to the exercises not in last year's set.

Exercise 2.0 (Induction on formulas)

If you are not used to proving properties by induction on (the complexity of) formulas, do the following exercises. Such a proof proceeds by:

- *Base case.* Show property holds for atomic formulas.
- *Induction hypothesis.* Property holds for formulas A, B.
- *Induction step.* Show property holds for $\neg A$, $A \vee B$, $A \wedge B$, and $A \to B$.

1. Say that a *positive* formula is a formula in which negation ($\neg$) does not occur (anywhere). Let $\mathcal{I}$ be the interpretation which assigns all atoms the truth value T. Show that for all positive formulas A, $v_{\mathcal{I}}(A) = T$. (No case $\neg A$ here, since that would produce a non-positive formula).

   Base case A is an atomic formula p. Then $v_{\mathcal{I}}(A) = \mathcal{I}(A) = T$.

   Induction Hypothesis For formulas B, C, we have $v_{\mathcal{I}}(C) = v_{\mathcal{I}}(B) = T$.

   Induction Step We do the three connectives we have to check in turn:

   $A = B \wedge C$ By induction hypothesis, $v_{\mathcal{I}}(C) = v_{\mathcal{I}}(B) = T$. Then $v_{\mathcal{I}}(B \wedge C) = T$ by the definition of $v_{\mathcal{I}}$.

   $A = B \vee C$ By induction hypothesis, $v_{\mathcal{I}}(C) = v_{\mathcal{I}}(B) = T$. Then $v_{\mathcal{I}}(B \vee C) = T$ by the definition of $v_{\mathcal{I}}$.

   $A = B \to C$ By induction hypothesis, $v_{\mathcal{I}}(C) = v_{\mathcal{I}}(B) = T$. Then $v_{\mathcal{I}}(B \to C) = T$ by the definition of $v_{\mathcal{I}}$.

   Done.

2. Let p and q be two atoms. For a formula A, let $A[q/p]$ be the formula obtained by replacing every occurrence of p in A by q. Let $\mathcal{I}$ be an interpretation such that $\mathcal{I}(p) = \mathcal{I}(q)$. Show that for all A, $v_{\mathcal{I}}(A) = v_{\mathcal{I}}(A[q/p])$

   Note: We'll take it as obvious here that $(\neg A)[q/p] = \neg(A[q/p])$, $(A \wedge B)[q/p] = A[q/p] \wedge B[q/p]$, etc. If we wanted to be stricter we would define $A[q/p]$ by induction.

   Base case A is an atomic formula. If $A = p$ then $A[q/p] = q$. Then $v_{\mathcal{I}}(A) = \mathcal{I}(p) = \mathcal{I}(q) = v_{\mathcal{I}}(A[q/p])$. If $A = r$ with $r \neq p$, then $A[q/p] = r = A$, and the result follows trivially.

   Induction Hypothesis For formulas B, C, we have $v_{\mathcal{I}}(B) = v_{\mathcal{I}}(B[q/p])$ and $v_{\mathcal{I}}(C) = v_{\mathcal{I}}(C[q/p])$.

Induction Step We do the three connectives we have to check in turn:

$A = B \wedge C$**:** $v_{\mathcal{I}}(B \wedge C) = T$ iff $v_{\mathcal{I}}(B) = T$ and $v_{\mathcal{I}}(C) = T$ iff (by I.H) $v_{\mathcal{I}}(B[q/p]) = T$ and $v_{\mathcal{I}}(C[q/p]) = T$ iff $v_{\mathcal{I}}(B[q/p] \wedge C[q/p]) = T$ iff $v_{\mathcal{I}}((B \wedge C)[q/p]) = T$.

$A = B \vee C$**:** $v_{\mathcal{I}}(B \vee C) = T$ iff $v_{\mathcal{I}}(B) = T$ or $v_{\mathcal{I}}(C) = T$ iff (by I.H) $v_{\mathcal{I}}(B[q/p]) = T$ or $v_{\mathcal{I}}(C[q/p]) = T$ iff $v_{\mathcal{I}}(B[q/p] \vee C[q/p]) = T$ iff $v_{\mathcal{I}}((B \vee C)[q/p]) = T$.

$A = B \to C$**:** $v_{\mathcal{I}}(B \to C) = T$ iff $v_{\mathcal{I}}(B) = F$ or $v_{\mathcal{I}}(C) = T$ iff (by I.H) $v_{\mathcal{I}}(B[q/p]) = F$ or $v_{\mathcal{I}}(C[q/p]) = T$ iff $v_{\mathcal{I}}(B[q/p] \to C[q/p]) = T$ iff $v_{\mathcal{I}}((B \to C)[q/p]) = T$.

Done.

Exercise 2.1 **(Missing cases)**

The lecture left several cases in the lemma for the soundness proof, the lemma for the completeness proof, and the alternative completeness proof as exercises. Fill those out. (Do at least the ones in the soundness proof, and the rest if you still feel that you would benefit from it.)

We sketch the missing cases in the soundness lemma, doing falsifying upwards. If you want more details or are wondering about a case in the completeness proof, give us a word and it will be filled out.

$\vee$**-left:** If the interpretation makes (all formulas in Γ true and) $A \vee B$ true, it musk make either A or B true.

$\vee$**-right:** If the interpretation makes $A \vee B$ false, it must make both A and B false.

$\to$**-right** If the interpretation makes $A \to B$ false, it must make A true and B false.

$\neg$**-left** If the interpretation makes $\neg A$ true it must make A false.

$\neg$**-right** If the interpretation makes $\neg A$ false it must make A true.