IN3070
IN4070

Autumn 2020

Solutions to Exercises for the Course

Logic

Week 10



Exercise 10.1

(Modal Sequent Calculus)

Using the sequent calculus from the lecture, prove

1. $\Box(p \to q) \to (\Box p \to \Box q)$
2. $\Diamond(p \vee q) \to (\Diamond p \vee \Diamond q)$
3. $(\Diamond p \vee \Diamond q) \to \Diamond(p \vee q)$
4. $((p \to \Box p) \wedge (\Diamond \neg p)) \to \neg p$

Solution

1.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{2:p, 1:\Box(p\to q), 1:\Box p, 1R2 \Longrightarrow 2:p, 2:q}\,\text{ax} \quad \dfrac{}{2:q, 2:p, 1:\Box(p\to q), 1:\Box p, 1R2 \Longrightarrow 2:q}\,\text{ax}}{2:p\to q, 2:p, 1:\Box(p\to q), 1:\Box p, 1R2 \Longrightarrow 2:q}\,{\to}\text{-l}}{2:p\to q, 1:\Box(p\to q), 1:\Box p, 1R2 \Longrightarrow 2:q}\,\Box\text{-l}}{1:\Box(p\to q), 1:\Box p, 1R2 \Longrightarrow 2:q}\,\Box\text{-l}}{1:\Box(p\to q), 1:\Box p \Longrightarrow 1:\Box q}\,\Box\text{-r}}{1:\Box(p\to q) \Longrightarrow 1:\Box p \to \Box q}\,{\to}\text{-r}}{\Longrightarrow 1:\Box(p\to q) \to (\Box p \to \Box q)}\,{\to}\text{-r}$$

2.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{2:p, 1R2 \Longrightarrow 2:p, 1:\Diamond p, 1:\Diamond q}\,\text{ax}}{2:p, 1R2 \Longrightarrow 1:\Diamond p, 1:\Diamond q}\,\Diamond\text{-r} \quad \dfrac{\dfrac{}{2:q, 1R2 \Longrightarrow 2:q, 1:\Diamond p, 1:\Diamond q}\,\text{ax}}{2:q, 1R2 \Longrightarrow 1:\Diamond p, 1:\Diamond q}\,\Diamond\text{-r}}{2:p\vee q, 1R2 \Longrightarrow 1:\Diamond p, 1:\Diamond q}\,\vee\text{-l}}{1:\Diamond(p\vee q) \Longrightarrow 1:\Diamond p, 1:\Diamond q}\,\Diamond\text{-l}}{1:\Diamond(p\vee q) \Longrightarrow 1:(\Diamond p \vee \Diamond q)}\,\vee\text{-r}}{\Longrightarrow 1:\Diamond(p\vee q) \to (\Diamond p \vee \Diamond q)}\,{\to}\text{-r}$$

3.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{2:p, 1R2 \Longrightarrow 2:p, 2:q, 1:\Diamond(p\vee q)}\,\text{ax}}{2:p, 1R2 \Longrightarrow 2:p\vee q, 1:\Diamond(p\vee q)}\,\vee\text{-r}}{2:p, 1R2 \Longrightarrow 1:\Diamond(p\vee q)}\,\Diamond\text{-r}}{1:\Diamond p \Longrightarrow 1:\Diamond(p\vee q)}\,\Diamond\text{-l} \quad \dfrac{\dfrac{\dfrac{}{2:q, 1R2 \Longrightarrow 2:p, 2:q, 1:\Diamond(p\vee q)}\,\text{ax}}{2:q, 1R2 \Longrightarrow 2:p\vee q, 1:\Diamond(p\vee q)}\,\vee\text{-r}}{2:q, 1R2 \Longrightarrow 1:\Diamond(p\vee q)}\,\Diamond\text{-r}}{1:\Diamond q \Longrightarrow 1:\Diamond(p\vee q)}\,\Diamond\text{-l}}{1:\Diamond p \vee \Diamond q \Longrightarrow 1:\Diamond(p\vee q)}\,\vee\text{-l}}{\Longrightarrow 1:(\Diamond p \vee \Diamond q) \to \Diamond(p\vee q)}\,{\to}\text{-r}$$

4.

$$\dfrac{\dfrac{}{1:p,1:\Diamond\neg p \Longrightarrow 1:p}\,\text{ax} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{1:p,2:p,1:\Box p,1R2 \Longrightarrow 2:p}\,\text{ax}}{1:p,2:p,1:\Box p,2:\neg p,1R2 \Longrightarrow}\,\neg\text{-l}}{1:p,1:\Box p,2:\neg p,1R2 \Longrightarrow}\,\Box\text{-l}}{1:p,1:\Box p,1:\Diamond\neg p \Longrightarrow}\,\Diamond\text{-l}}{\dfrac{1:p,1:(p\to\Box p),1:\Diamond\neg p \Longrightarrow}{\dfrac{1:p\to\Box p,1:\Diamond\neg p \Longrightarrow 1:\neg p}{\dfrac{1:p\to\Box p\wedge\Diamond\neg p \Longrightarrow 1:\neg p}{\Longrightarrow 1:((p\to\Box p)\wedge(\Diamond\neg p))\to\neg p}\,\to\text{-r}}\,\wedge\text{-l}}\,\neg\text{-r}}\,\to\text{-l}$$

Exercise 10.2 (Model Semantics)

a) Let A be a *valid* formula of propositional logic, i.e. it is true in all propositional interpretations.

Show that $\Box A$ is valid in modal logic **K**.

Show that $\Diamond A$ is *not* valid in modal logic **K**.

b) Prove (using the model semantics) that the formula $\Box p \to \Box\Box p$ is satisfied at all worlds of any Kripke structure if the accessibility relation R is *transitive*.

c) Fix a frame $F = (W, R)$. Show that if all Kripke models that have F as their underlying frame satisfy $\Box p \to \Box\Box p$ at all worlds, then the accessibility relation R is transitive.

d) Construct a Kripke model $((W, R)\{\mathcal{I}_w\}_{w\in W})$ which satisfies $\Box p \to \Box\Box p$ at all worlds where R is not transitive.

(e) Let A be a modal propositional formula such that A is not satisfied at any world in any Kripke model.

Show that $\Diamond A$ is not satisfied at any world in any Kripke model.

Show that there is a Kripke model with a world such that $w \Vdash \Box A$.

(f) Show the last claim on slide 23 (note that there are two things to prove):

- if $U \models^L A$ then $U \models^G A$; the opposite direction does not hold

Solution

a)

Let A be a valid propositional formula. It is therefore true in any propositional interpretation, and in particular $u \Vdash A$ in *any* world u of any Kripke model. This implies in particular that $u \Vdash A$ in any world u *reachable* from any world w. Therefore $w \Vdash \Box A$ for any world w of any Kripke model, which means that $\Box A$ is valid.

$\Diamond A$ is not valid: take a Kripke model in which not every world has an accessible world. E.g. R could be the empty relation. Let w be a world that has no u that is accessible from w. Then $w \nVdash \Diamond A$.

b)

Let w be an arbitrary world of a Kripke model with a transitive accessibility relation R. To show that $w \Vdash \Box p \to \Box\Box p$, we may assume that $w \Vdash \Box p$ (†), and have to show that $w \Vdash \Box\Box p$.

The latter means that $v \Vdash \Box p$ for all v such that wRv. This in turn means that $u \Vdash p$ for all u with vRu. In other words, we have to show that $u \Vdash p$ for all worlds u such that wRv and vRu for some v. Due to the transitivity, this always means that also wRu. 2 We now use our assumption (†) for u, which tells us that $u \Vdash p$, which is what we had to prove.

c). Let three worlds u, w, v of W be given, and assume $R(u, w)$ and $R(w, v)$. We need to show that $R(u, v)$. Define an interpretation of p by: for any world $t \in W$, $t \Vdash p$ iff $r(u, t)$. Then, automatically, $u \Vdash \Box p$. By assumption, $u \Vdash \Box\Box p$. Now, $R(u, w)$, so that means that $w \Vdash \Box p$. And, $R(w, v)$, so that means that $v \Vdash p$. But by the definition of the interpretation of p, that means that $R(u, v)$.

d). Let W consist of three worlds u, w, v such that $R(u, w)$, $R(w, v)$, and $R(v, v)$ (but not $R(u, v)$, so it is not transitive). Define an interpretation of p by setting p to be false at all three worlds. Since all worlds relate to some world, $\Box p$ is false at all worlds. And then $\Box p \to \Box\Box p$ is forced at all worlds.

e). For $\Diamond A$ to be forced at a world w, there must exist a world v such that $R(w, v)$ and $v \Vdash A$. But no such world exists.

However, for $\Box A$ to be forced at a world w, A must be forced at all worlds that w relates to. If w does not relate to any worlds, this condition is satisfied. So a frame with a single world and empty accessibility relation will do the trick.

f). Recall that U locally implies A if for all worlds in all modal interpretations, if the world forces all formulas in U then it forces A. And that U globally implies A if for all modal interpretations, if all formulas in U are forced at all worlds in the interpretation, then A is forced at all worlds of the interpretation.

Suppose U locally implies A, and let a modal interpretation $((W, R)\{\mathcal{I}_w\}_{w \in W})$ be given. Assume $w \Vdash F$ for all worlds $w \in W$ and all $F \in U$. Then, for any $w \in W$, $w \Vdash F$ for all $F \in U$, and since U locally implies A, this means that $w \Vdash A$. Hence all worlds in W force A.

To show that the opposite direction does not hold, we construct a counterexample. Let $U = \{p\}$, for p a propositional variable, and $A = \Box p$. Then U does not locally imply A, since we have Kripke models where p is true at a world, but not at a related world. But U globally implies A. For if p is forced at all worlds in a Kripke model, then all worlds in that Kripke model will also force $\Box p$.

Exercise 10.3 **(A Logic of Knowledge)**

Here is an attempt at formalising the idea of "knowledge."

We assume that our knowledge is determined by what we have *observed* about the world. E.g. the temperature, the colour of a light, etc. Let O be a set of possible observations, W a set of possible states of the world, and $o : W \to O$ an observation function that tells us for each state

of the world what we have observed about it.

E.g. in the scenario where we may or may not have looked outside, and it may or may not rain, the set W would contain four worlds, according to the four possibilities. O would contain only three possible observations: NotLooked, LookedAndRain, LookedAndNotRain.

We can now say that two worlds $u, v \in W$ appear the same to us if our observations are the same, $o(u) = o(v)$. E.g. the two worlds where we have not looked outside appear the same because the observation is NotLooked for both of them

Now we can say that we "know" A in a world $w \in W$ if A is true in all worlds that fit our observations i.e., A is true in all worlds v with $o(u) = o(v)$.

a) Define an accessibility relation $R \subseteq W \times W$ such that the modal semantics of $\Box A$ is the same is this notion of knowledge.

b) Which of the properties of frames (reflexive, serial, transitive, etc.) we saw in the lecture does R have? Which of the modal logics listed in the lecture (K, D, T,...) corresponds to this notion of knowledge?

Solution

a)

We can define $(u, v) \in R :\Leftrightarrow o(u) = o(v)$.

Then $u \Vdash \Box A$ if $v \Vdash A$ holds in all worlds with $o(u) = o(v)$ – i.e. all worlds that have the same observations as u.

b)

The relation R defined by $(u, v) \in R :\Leftrightarrow o(u) = o(v)$ is easily seen to be reflexive, symmetric, and transitive: it is an equivalence relation on the set of worlds. The frame is thus an **S5** frame.

Converesely, any **S5** frame, i.e. one where R is an equivalence relation can be defined through a suitably chosen o: Let

$$o(u) := \{v \in W \mid uRv\} \quad ,$$

i.e. $o(u)$ is the equivalence class of u. With this definition, we get that

$$(u, v) \in R \quad \text{if and only if} o(u) = o(v) \quad .$$

So the modal logic **S5** expresses exatly the notion of knowledge that is based on the equality of observations.