IN3070
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Autumn 2020

Some solutions to exercises

Logic

Week 11



Exercise 12.1 (Intuitionistic Validity)

Prove, using the Kripke semantics and forcing relation, that the following formulae are valid in intuitionistic logic:

1. $p \to \neg\neg p$
2. $(p \wedge \neg q) \to \neg(p \to q)$
3. $(\exists x\, \neg p(x)) \to (\neg \forall x\, p(x))$

Answer

First, a useful fact. To prove that $w_0 \Vdash A \to B$ for *all* worlds w_0, we have to show for all w_0 that $w \Vdash B$ for all worlds w with $(w_0, w) \in R$ and $w \Vdash A$. That's the Kripke semantics for implication. Since all worlds w are reachable from some other world (e.g. from w itself), this is the same as showing for all $w \in W$ with $w \Vdash A$ that $w \Vdash B$. No need to worry about w_0 as long as we are interested in all w_0.

1. Let $\mathcal{I}_J = ((W, R), \{(D^w, \iota^w)\}_{w \in W})$ be a J-structure. As just discussed, we have to show that $w \Vdash \neg\neg p$ for all w with $w \Vdash p$. Let w be an arbitrary but fixed world with $w \Vdash p$.

To show $w \Vdash \neg\neg p$, we show that $v \nVdash \neg p$ for all v with $(w, v) \in R$ (semantics of negation). So let v be an arbitrary but fixed such world. Interpretations increase, and $w \Vdash p$, so also $v \Vdash p$. To show that $v \nVdash \neg p$, it suffices to show that there is a world reachable from v which forces p. Remember that R is reflexive, so $(v, v) \in R$, so v is such a world that is reachable from v that forces p.

2. Let $\mathcal{I}_J = ((W, R), \{(D^w, \iota^w)\}_{w \in W})$ be a J-structure. As discussed, we have to show that $w \Vdash \neg(p \to q)$ for all w with $w \Vdash p \wedge \neg q$. Let w be an arbitrary but fixed world with $w \Vdash p \wedge \neg q$.

The semantics tells us that $w \Vdash p$ (1) and $v \nVdash q$ for all v reachable from w. (2)

To show $w \Vdash \neg(p \to q)$, we have to show that $v \nVdash p \to q$ for all v reachable from w. Let v be an arbitray but fixed such world. Interpretations increase, so from (1) it follows that $v \Vdash p$ (3). Also, from (2) we know that that $v \nVdash q$ (4).

To show $v \nVdash p \to q$, we have to find a world reachable from v that forces p but not q. v itself is such a world, as (3) and (4) show.

3. Let $\mathcal{I}_J = ((W, R), \{(D^w, \iota^w)\}_{w \in W})$ be a J-structure. As discussed, we have to show that $w \Vdash \neg\forall x\, p(x)$ for all w with $w \Vdash \exists x\, \neg p(x)$. Let w be an arbitrary but fixed world with $w \Vdash \exists x\, \neg p(x)$.

The semantics tells us that there is a domain element $c \in D^w$ with $w \Vdash \neg p(c)$, and therefore $v \nVdash p(c)$ for every world v reachable from w. (1)

To show $w \Vdash \neg\forall x\, p(x)$, we have to show that $u \nVdash \forall x\, p(x)$ for all worlds u reachable from w. Let u be an arbitrary but fixed such world.

From (1) we already know that $u \nVdash p(c)$. Since $(u, u) \in R$, we therefore have a combination of successor world and domain element such that the semantics of $\forall$ tells us that $u \nVdash \forall x\, p(x)$.

Exercise 12.2 **(Intuitionistic Sequent Calculus)**

Prove the validity of the three formulae in 12.1 using the intuitionistic sequent calculus LJ.

Solution

1

$$\dfrac{\dfrac{\dfrac{\dfrac{}{p, \neg p \Longrightarrow p}\,\text{ax}}{p, \neg p \Longrightarrow}\,\neg\text{-l}}{p \Longrightarrow \neg\neg p}\,\neg\text{-r}}{\Longrightarrow p \to \neg\neg p}\,\to\text{-r}$$

2

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p, p \to q \Longrightarrow p}\,\text{ax} \quad \dfrac{}{p, q \Longrightarrow q}\,\text{ax}}{p, p \to q \Longrightarrow q}}{p, \neg q, p \to q \Longrightarrow}\,\neg\text{-l}}{p, \neg q \Longrightarrow \neg(p \to q)}\,\neg\text{-r}}{p \wedge \neg q \Longrightarrow \neg(p \to q)}\,\wedge\text{-l}}{\Longrightarrow (p \wedge \neg q) \to \neg(p \to q)}\,\to\text{-r}$$

3

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p(c), \forall x\, p(x) \Longrightarrow p(c)}\,\text{ax}}{\forall x\, p(x) \Longrightarrow p(c)}\,\forall\text{-l}}{\neg p(c), \forall x\, p(x) \Longrightarrow}\,\neg\text{-l}}{\neg p(c) \Longrightarrow \neg\forall x\, p(x)}\,\neg\text{-r}}{\exists x\, \neg p(x) \Longrightarrow \neg\forall x\, p(x)}\,\exists\text{-l}}{\Longrightarrow (\exists x\, \neg p(x)) \to (\neg\forall x\, p(x))}\,\to\text{-r}$$

Exercise 12.3 **(Intuitionistic Countermodel)**

Show, using a falsifying Kripke-structure, that the following formulas are *not* valid in intutionistic logic:

1. $\neg\neg p \to p$
2. $\neg(p \wedge q) \to \neg p \vee \neg q$
3. What happens if we try to prove the formulas using LJ?

Solution

Countermodel for formula 1:

The model of slide 17 will work for this one. $w_0 \Vdash \neg\neg p$ iff neither $w_0 \Vdash \neg p$ nor $w_1 \Vdash \neg p$. And that is the case, since $w_1 \Vdash p$, hence both w_0 and w_1 relate to a world (w_1) in which p is forced.

But $w_0 \nVdash p$.

Countermodel and LJ-discussion of formula 2:

We can approach this systematically. If there is a J-structure that falsifies the formula, then it has a world w_0 with

$$w_0 \nVdash \neg(p \wedge q) \rightarrow \neg p \vee \neg q$$

It follows that there is a world w with $(w_0, w) \in R$ with

$$w \Vdash \neg(p \wedge q)\ (\dagger) \quad \text{and} \quad w \nVdash \neg p \vee \neg q\ (\ddagger)$$

Now from $(\ddagger)$, we get that

$$w \nVdash \neg p \quad \text{and} \quad w \nVdash \neg q$$

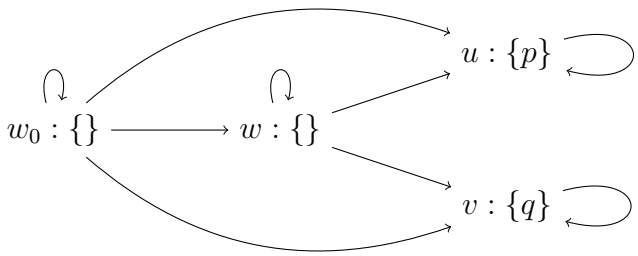
This means that there are worlds u and v with $(w, u) \in R$ and $(w, v) \in R$ and

$$u \Vdash p \quad \text{and} \quad v \Vdash q$$

$(\dagger)$ tells us that *none* of the successor worlds of w forces $p \wedge q$. From this we know that $u \neq v$ and

$$u \nVdash q \quad \text{and} \quad v \nVdash p$$

The resulting sructure looks like this:



In all successors of w (i.e. u, v, w), either p or q is *not* forced. I.e. none of the worlds forces $p \wedge q$. And therefore $w \Vdash \neg(p \wedge q)$. If the implication were valid, then we would also have $w \Vdash \neg p \vee \neg q$. But $w \nVdash \neg p$ because $u \Vdash p$. And $w \nVdash \neg q$ because $v \Vdash q$.

A proof attempt in LJ would start as follows

$$\frac{\neg(p \wedge q) \Longrightarrow \neg p \vee \neg q}{\Longrightarrow \neg(p \wedge q) \rightarrow \neg p \vee \neg q} \rightarrow\text{-r}$$

At this point the $\neg$-l rule would overwrite the succedent:

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg(p \wedge q) \Longrightarrow p \qquad \neg(p \wedge q) \Longrightarrow q}{\neg(p \wedge q) \Longrightarrow p \wedge q}\,\wedge\text{-r}}{\neg(p \wedge q) \Longrightarrow \neg p \vee \neg q}\,\neg\text{-l}}{\Longrightarrow \neg(p \wedge q) \rightarrow \neg p \vee \neg q}\,\rightarrow\text{-r}$$

These leaf sequents are not valid classically, so they are certainly not valid intiuitionistically either. This is a dead end.

If we instead go for $\vee$-r, we have to decide for one of the disjunctions.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p, \neg(p \wedge q) \Longrightarrow p}\,\text{ax} \qquad p, \neg(p \wedge q) \Longrightarrow q}{p, \neg(p \wedge q) \Longrightarrow p \wedge q}\,\wedge\text{-r}}{p, \neg(p \wedge q) \Longrightarrow}\,\neg\text{-l}}{\neg(p \wedge q) \Longrightarrow \neg p}\,\neg\text{-r}}{\neg(p \wedge q) \Longrightarrow \neg p \vee \neg q}\,\vee\text{-r}_1}{\Longrightarrow \neg(p \wedge q) \rightarrow \neg p \vee \neg q}\,\rightarrow\text{-r}$$

There is no way to close the right branch. If we had stared with $\vee$-r$_2$, the left branch would have remained open.

Exercise 12.4 **(Properties of Kripke models)**

Show the monotonicity property for Kripke models using induction on formulas. That is, show that for all Kripke models, for every formula A and for all worlds w, v, if $w \Vdash A$ and $R(w, v)$, then $v \Vdash A$.

First, recall that *interpretations only "increase"*, i.e. for all $w, v \in W$ with $(w, v) \in R$:

a. $a^{\iota^w} = a^{\iota^v}$ for every constant a

b. $f^{\iota^w} \subseteq f^{\iota^v}$ for every function f

c. $p^{\iota^w} = T$ implies $p^{\iota^v} = T$ for every $p \in \mathcal{P}^0$

d. $p^{\iota^w} \subseteq p^{\iota^v}$ for every predicate $p \in \mathcal{P}^n$ with $n > 0$

Solution

Let a Kripke model I with worlds w and v such that $R(w, v)$ be given. By induction on A:

Base case , $A = p$, for p a propositional variable: $w \Vdash p$ for $p \in \mathcal{P}^0$ iff $p^{\iota^w} = T$ implies $p^{\iota^v} = T$ iff $w \Vdash p$.

Base case , $A = p(t_1, ..., t_n)$ for $p \in \mathcal{P}^n$, $n > 0$: $w \Vdash p(t_1, ..., t_n)$ iff $(v_I(w, t_1), ..., v_I(w, t_n)) \in p^{\iota^w}$ implies $(v_I(v, t_1), ..., v_I(v, t_n)) \in p^{\iota^v}$ iff $v \Vdash p(t_1, ..., t_n)$.

Conjunction , $A = B \wedge C$. $w \Vdash B \wedge C$ iff $w \Vdash B$ and $w \Vdash C$ implies (by induction hypothesis) that $v \Vdash B$ and $v \Vdash C$.

Disjunction Similar to conjunction.

Existential , $A = \exists x B$. $w \Vdash \exists x B$ iff $w \Vdash B[x \backslash d]$, for some $d \in D^w$, implies $v \Vdash B[x \backslash d]$ (by I.H. and since $D^w \subseteq D^v$) implies $w \Vdash \exists x B$.

Implication , $A = B \to C$. Let $u \geq v$ and assume $u \Vdash B$. By transitivity of R, $u \geq w$ Since $w \Vdash B \to C$, we have $u \Vdash C$. So $v \Vdash B \to C$. (We did not need the I.H. for this one.)

Negation , $A = \neg B$. Similar to implication.

Forall , $A = \forall x B$. Similar to implication.