Solutions to Exercises for the Course

Logic

Week 5



Exercise 5.1 (Substitution and Unification)

a) Let $\sigma = \{u\backslash y, y\backslash f(a), x\backslash g(u)\}$, $\theta = \{x\backslash f(g(y)), y\backslash u, z\backslash f(y)\}$ and $E = p(x, f(y), g(u), z)$. Compute $\sigma\theta$ using the Proposition on slide 11 of lecture 6. Show that $(\sigma\theta)E = \sigma(\theta(E))$.

b) Unify the following pairs of atomic formulae, i.e. try to find substitutions such that

$$\sigma_1(\, p(a, x, f(g(y)))\,) = \sigma_1(\, p(z, g(f(a)), f(z))\,),$$
$$\sigma_2(\, p(f(y), z, y)\,) = \sigma_2(\, p(z, h(z, u), f(u))\,),$$
$$\sigma_3(\, p(f(x, y), y, g(x))\,) = \sigma_3(\, p(f(v, v), z, z)\,).$$

Solution

a) This is the proposition:

Let $\sigma = \{x_1\backslash s_1, \ldots, x_n\backslash s_n\}$ and $\tau = \{y_1\backslash t_1, \ldots, y_k\backslash t_k\}$. Then

$$\tau\sigma = \{x_1\backslash\tau(s_1), \ldots, x_n\backslash\tau(s_n), z_1\backslash\tau(z_1), \ldots, z_m\backslash\tau(z_m)\}$$

where $z_1, \ldots, z_m$ are the variables amongst $y_1, \ldots, y_k$ that are *not* amongst $x_1, \ldots, x_n$.

$\sigma = \{u\backslash y, y\backslash f(a), x\backslash g(u)\}$, $\theta = \{x\backslash f(g(y)), y\backslash u, z\backslash f(y)\}$ and $E = p(x, f(y), g(u), z)$.

For the given σ and θ, to compute $\sigma\theta$, we need to find the variables touched by σ but not by θ. That is the case only for u.

$$\begin{aligned}\sigma\theta &= \{x\backslash\sigma(f(g(y))), y\backslash\sigma(u), z\backslash\sigma(f(y)), u\backslash\sigma(u)\}\\ &= \{x\backslash f(g(f(a))), y\backslash y, z\backslash f(f(a)), u\backslash y\}\\ &= \{x\backslash f(g(f(a))), z\backslash f(f(a)), u\backslash y\}\end{aligned}$$

We can now apply first θ and then σ to E, as follows:

$$\begin{aligned}\sigma(\theta(E)) &= \sigma(\theta(p(x, f(y), g(u), z)))\\ &= \sigma(p(f(g(y)), f(u), g(u), f(y)))\\ &= p(f(g(f(a))), f(y), g(y), f(f(a)))\end{aligned}$$

On the other hand, applying the composed substitution, we get

$$\begin{aligned}(\sigma\theta)(E) &= (\sigma\theta)(p(x, f(y), g(u), z))\\ &= p(f(g(f(a))), f(y), g(y), f(f(a)))\end{aligned}$$

We see that we get the same result in both cases.

b)

1:

$$\sigma_1(\, p(a, x, f(g(y)))\,) = \sigma_1(\, p(z, g(f(a)), f(z))\,)$$

Looking at the first subterm, we see that we have to assign $z \backslash a$. It remains to unify

$$p(a, x, f(g(y))) \quad \text{and} \quad p(a, g(f(a)), f(a))$$

To unify the second subterms, we apply $[x \backslash g(f(a)]$. It remains to unify

$$p(a, g(f(a)), f(g(y))) \quad \text{and} \quad p(a, g(f(a)), f(a))$$

The third subterms cannot be unified, since a and $g(y)$ cannot be unified.

2:

$$\sigma_2(\, p(f(y), z, y)\,) = \sigma_2(\, p(z, h(z, u), f(u))\,)$$

To unify the first subterms, we have to assign $[z \backslash f(y)]$. It remains to unify

$$p(f(y), f(y), y) \quad \text{and} \quad p(f(y), h(f(y), u), f(u))$$

The second subterms have f and h as top function symbols, so they cannot be unified.

3:

$$\sigma_3(\, p(f(x, y), y, g(x))\,) = \sigma_3(\, p(f(v, v), z, z)\,)$$

To unify the leftmost subterms (first argument of first argument), we have to assign $[x \backslash v]$ (or vice versa.) It remains to unify

$$p(f(v, y), y, g(v)) \quad \text{and} \quad p(f(v, v), z, z)$$

To unify the next subterms (second argument of first argument), we have to assign $[y \backslash v]$ (or vice versa). It remains to unify

$$p(f(v, v), v, g(v)) \quad \text{and} \quad p(f(v, v), z, z)$$

To unify the second subterms, we have to assign $[v \backslash z]$ (or vice versa). It remains to unify

$$p(f(z, z), z, g(z)) \quad \text{and} \quad p(f(z, z), z, z)$$

To unify the third subterms, we have to unify $g(z)$ and z, which is not possible ('occurs check'). So the terms are not unifiable.

Exercise 5.2 (DNF)

A formula is in disjunctive normal form (DNF) iff it is a disjunction of conjunctions of literals. Show that every formula can be transformed to an equivalent one in DNF.

Solution

The exercise should have said "every propositional formula" — since we would otherwise have to get rid of quantifiers.

For a propositional formula, first compute the Negation Normal Form. Then apply the distributivity equivalence

$$A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$$

until there are no more disjunctions below conjunctions. The formula is then in disjunctive normal form.

Exercise 5.3 (Propositional Clause Form)

Transform the following set of formulae to clause form:

$$\{p, p \to ((q \vee r) \wedge \neg(q \wedge r)), p \to ((s \vee t) \wedge \neg(s \wedge t)), s \to q, \neg r \to t, t \to s\}$$

Solution

One by one formula can be transformed into one or more clauses.

- $\{p\}$ is a clause
- $p \to ((q \vee r) \wedge \neg(q \wedge r)) \equiv \neg p \vee ((q \vee r) \wedge \neg(q \wedge r)) \equiv \neg p \vee ((q \vee r) \wedge (\neg q \vee \neg r))$ $\equiv (\neg p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r))$ Corresponding to two clauses $\{\neg p, q, r\}$ and $\{\neg p, \neg q, \neg r\}$
- Similarly, $p \to ((s \vee t) \wedge \neg(s \wedge t))$ corresponds to two clauses $\{\neg p, s, t\}$ and $\{\neg p, \neg s, \neg t\}$
- The last three forumlae correspond to $\{\neg s, q\}$, $\{r, t\}$, $\{\neg t, s\}$, respectively.

Exercise 5.4 **(First-order Clause Form)**

Transform each of the following formulas to clausal form:

$$\forall x(p(x) \to \exists y\, q(y))$$
$$\forall x \forall y(\exists z\, p(z) \land \exists u(q(x,u) \to \exists v\, q(y,v)))$$
$$\exists x(\neg \exists y\, p(y) \to \exists z(q(z) \to r(x)))$$

Solutions

1.

- Formula: $\forall x(p(x) \to \exists y\, q(y))$
- NNF: $\forall x(\neg p(x) \lor \exists y\, q(y))$
- Prenex CNF: $\forall x \exists y(\neg p(x) \lor q(y))$
- Skolemisation: $\forall x(\neg p(x) \lor q(f(x)))$
- Clause form: $\{\{\neg p(x), q(f(x))\}\}$

2.

- Formula: $\forall x \forall y(\exists z\, p(z) \land \exists u(q(x,u) \to \exists v\, q(y,v)))$
- NNF: $\forall x \forall y(\exists z\, p(z) \land \exists u(\neg q(x,u) \lor \exists v\, q(y,v)))$
- Prenex CNF: $\forall x \forall y \exists z \exists u \exists v(p(z) \land (\neg q(x,u) \lor q(y,v)))$
- Skolemisation: $\forall x \forall y(p(f(x,y)) \land (\neg q(x,g(x,y)) \lor q(y,h(x,y))))$
- Clause Form: $\{\{p(f(x,y))\},\ \{\neg q(x,g(x,y)), q(y,h(x,y))\}\}$

3.

- Formula: $\exists x(\neg \exists y\, p(y) \to \exists z(q(z) \to r(x)))$
- NNF: $\exists x(\exists y\, p(y) \lor \exists z(\neg q(z) \lor r(x)))$
- Prenex CNF: $\exists x \exists y \exists z(p(y) \lor \neg q(z) \lor r(x))$
- Skolemisation: $p(b) \lor \neg p(c) \lor r(a)$
- Clause Form: $\{\{p(b), \neg p(c), r(a)\}\}$

Exercise 5.5 **(Equivalence vs. Equisatisfiability)**

Skolemisation does not necessarily produce a logically equivalent formula. In this exercise, you prove that the logical consequence between the formula $\forall x \exists y\, A(x, y)$ and its Skolemisation holds in one direction, but not the other.

a) $\forall x\, A(x, f(x)) \models \forall x \exists y\, A(x, y)$

b) $\forall x \exists y\, A(x, y) \not\models \forall x\, A(x, f(x))$.

Solution

a) Let $\mathcal{I} = (D, \iota)$ be an interpretation with $\mathcal{I} \models \forall x\, A(x, f(x))$. Then, for any $d \in D$, we know that $v_{\mathcal{I}}(\{x \leftarrow d\}, A(x, f(x))) = T$, so by the Substitution Lemma, $v_{\mathcal{I}}(\{x \leftarrow d, y \leftarrow f^{\iota}(d)\}, A(x, y)) = T$. It follows in particular that for every $d \in D$, there is a $c \in D$ (namely $c = f^{\iota}(d)$) with $v_{\mathcal{I}}(\{x \leftarrow d, y \leftarrow c\}, A(x, y)) = T$, and so by model semantics $v_{\mathcal{I}}(\{x \leftarrow d\}, \exists y\, A(x, y)) = T$. Since this holds for arbitray $d \in D$, we get $\mathcal{I} \models \forall x \exists y\, A(x, y)$.

b) We give an interpretation $\mathcal{I} = (D, \iota)$ that satisfies $\forall x \exists y\, A(x, y)$, but not $\forall x\, A(x, f(x))$. That shows that the entailment does not hold.

Let $D = \{a, b\}$, $f^{\iota}(a) = a$, $f^{\iota}(b)$, and $A^{\iota} = \{\langle a, b\rangle, \langle b, a\rangle\}$

Then A relates both elements of D respectively to the other element. However since f is interpreted as the identity function, A does not relate any element $d \in D$ to $f^{\iota}(d)$.