Exercises for the Course

# Logic

Week 9

## Exercise 9.1 (Atomic Cut and Unit Propagation)

Using atomic cut and unit propagation, show that the following formulas (sets of clauses) are unsatisfiable, or give a satisfying substitution

1. $(\neg p \vee r \vee \neg s) \wedge (p \vee q \vee r \vee \neg s) \wedge (\neg p \vee \neg t) \wedge \neg r \wedge (p \vee s) \wedge (p \vee r \vee t) \wedge (s \vee t)$
2. $\{\{p, \neg q\}, \{\neg p, q\}, \{q, \neg r\}, \{s\}, \{\neg s, \neg q, \neg r\}, \{s, r\}\}$
3. $\{\{p, q, s, t\}, \{p, s, \neg t\}, \{q, \neg s, t\}, \{p, \neg s, \neg t\}, \{p, \neg q\}, \{\neg r, \neg p\}, \{r\}\}$

### Solution

**1.** For formatting reasons, we write the branches separately.

Root: $(\neg p \vee r \vee \neg s) \wedge (p \vee q \vee r \vee \neg s) \wedge (\neg p \vee \neg t) \wedge \neg r \wedge (p \vee s) \wedge (p \vee r \vee t) \wedge (s \vee t)$

We start with unit propagation of $\neg r$, which gives:

$(\neg p \vee \neg s) \wedge (p \vee q \vee \neg s) \wedge (\neg p \vee \neg t) \wedge \neg r \wedge (p \vee s) \wedge (p \vee t) \wedge (s \vee t)$

Atomic cut on $p$:

1. $p$. Unit propagation (simplification) gives:
   $(\neg s) \wedge (\neg t) \wedge \neg r \wedge (s \vee t)$
   Now unit propagation of $\neg s$ gives:
   $(\neg s) \wedge (\neg t) \wedge \neg r \wedge (t)$
   We can now close the branch with $t$ and $\neg t$.
2. $\neg p$. Unit propagation (simplification) gives:
   $(q \vee \neg s) \wedge \neg r \wedge (s) \wedge (t) \wedge (s \vee t)$
   Now unit propagation of $s$ gives:
   $(q) \wedge \neg r \wedge (s) \wedge (t)$
   We get a satisfying assignment $\neg p, q, \neg r, s, t$.

**2.**

Root: $\{\{p, \neg q\}, \{\neg p, q\}, \{q, \neg r\}, \{s\}, \{\neg s, \neg q, \neg r\}, \{s, r\}\}$

Unit propagation of $s$ gives:
$\{\{p, \neg q\}, \{\neg p, q\}, \{q, \neg r\}, \{s\}, \{\neg q, \neg r\}\}$

Atomic cut on $p$:

1. $p$. Unit propagation gives:
   $\{\{q\}, \{q, \neg r\}, \{s\}, \{\neg q, \neg r\}\}$
   Unit propagation of $q$:
   $\{\{q\}, \{s\}, \{\neg r\}\}$
   A satisfying assignment is $p, q, s, \neg r$.

2. $\neg p$. Unit propagation gives:
   $\{\{\neg q\}, \{q, \neg r\}, \{s\}, \{\neg q, \neg r\}\}$
   Unit propagation of $\neg q$:
   $\{\{\neg q\}, \{\neg r\}, \{s\}\}$
   So we get a second satisfying assignment $\neg p, \neg q, \neg r, s$

**3.**

Root: $\{\{p, q, s, t\}, \{p, s, \neg t\}, \{q, \neg s, t\}, \{p, \neg s, \neg t\}, \{p, \neg q\}, \{\neg r, \neg p\}, \{r\}\}$

Unit propagation of $r$:
$\{\{p, q, s, t\}, \{p, s, \neg t\}, \{q, \neg s, t\}, \{p, \neg s, \neg t\}, \{p, \neg q\}, \{\neg p\}, \{r\}\}$

Unit propagation of $\neg p$:
$\{\{q, s, t\}, \{s, \neg t\}, \{q, \neg s, t\}, \{\neg s, \neg t\}, \{\neg q\}, \{\neg p\}, \{r\}\}$

Unit propagation of $\neg q$:
$\{\{s, t\}, \{s, \neg t\}, \{\neg s, t\}, \{\neg s, \neg t\}, \{\neg q\}, \{\neg p\}, \{r\}\}$

Atomic cut on $s$:

1. $s$. Unit propagation:
   $\{\{t\}, \{\neg t\}, \{\neg q\}, \{\neg p\}, \{r\}\}$
   Branch closed with $t$ and $\neg t$.

2. $\neg s$. Unit propagation:
   $\{\{t\}, \{\neg t\}, \{\neg q\}, \{\neg p\}, \{r\}\}$
   Branch closed with $t$ and $\neg t$.

## Exercise 9.2 **(Properties of Simplification)**

Prove that the simplification rules are sound.

a) Given a propositional formula $A$ that contains a subformula $B$, and let $B \equiv B'$ (logical equivalence). Prove $A$ is logically equivalent to the result of replacing $B$ by $B'$ in $A$. *Hint: use structural induction on $A$*

b) Given propositional formulae $A$ and $B$ and an interpretation $\mathcal{I}$ with $\mathcal{I} \models B$. Prove that $\mathcal{I} \models A$ if and only if $\mathcal{I} \models A[B]$. *Hint: for the replacement of $B$ by true or false, use structural induction on $A$ similarly to part $A$. For the simplification steps, use part a).*

c) Prove that the simplification rule

   preserves falsifiability upwards.

**Solution**

**Part a)** We prove by structural induction on $A$ that the result $A'$ of replacing $B$ by $B'$ in $A$ is equivalent to $A$. We assume for this proof that only one particular occurrence of $B$ is replaced. It is also possible to show this for the result of replacing *all* occurrences of $B$ by $B'$.

Base case: $A$ is an atomic formula, containing $B$. $A$ must equal $B$ and $A'$ = $B'$. Then $v_{\mathcal{I}}(A) = v_{\mathcal{I}}(B) = v_{\mathcal{I}}(B') = v_{\mathcal{I}}(A')$.

Induction step $A = \neg A_1$, where $A$ contains $B$.

Case $\neg$-1: $A = B$. Then $A'$ = $B'$ and $v_{\mathcal{I}}(A) = v_{\mathcal{I}}(B) = v_{\mathcal{I}}(B') = v_{\mathcal{I}}(A')$, like in the base case.

Case $\neg$-2: $A_1$ contains $B$. Let $A'_1$ be the result of replacing $B$ ny $B'$ in $A_1$. Then $A' = \neg A'_1$.

The induction hypothesis tells us that $A_1$ is logically eqivalent to $A'_1$.

$v_{\mathcal{I}}(A) = T$ iff $v_{\mathcal{I}}(A_1) = F$ iff $v_{\mathcal{I}}(A'_1) = F$ (I.H.) iff $v_{\mathcal{I}}(A') = T$

Induction step $A = A_1 \wedge A_2$, where $A$ contains $B$.

Case $\wedge$-1: $A = B$. Same as before.

Case $\wedge$-2: $A_1$ contains $B$. Let $A'_1$ be the result of replacing $B$ ny $B'$ in $A_1$. Then $A' = A'_1 \wedge A_2$.

The induction hypothesis tells us that $A_1$ is logically eqivalent to $A'_1$.

$v_{\mathcal{I}}(A) = T$ iff [$v_{\mathcal{I}}(A_1) = T$ and $v_{\mathcal{I}}(A_2) = T$] iff [$v_{\mathcal{I}}(A'_1) = T$ and $v_{\mathcal{I}}(A_2) = T$, by the I.H.] iff $v_{\mathcal{I}}(A') = T$

Case $\wedge$-3: $A_2$ contains $B$: symmetric to case $\wedge$-2.

The inductions steps for $\vee$ and $\rightarrow$ are essentially the same as for $\wedge$.

**Part b)** Let $\mathcal{I} \models B$. Let $A^*$ be the formula that results from replacing all occurrences of $B$ by $true$ in $A$. We first show by structural induction on $A$ that $v_{\mathcal{I}}(A) = v_{\mathcal{I}}(A^*)$.

Base case: $A$ is an atomic formula.

If $A = B$, then $A^* = true$ and $v_{\mathcal{I}}(A) = v_{\mathcal{I}}(B) = T = v_{\mathcal{I}}(true) = v_{\mathcal{I}}(A^*)$.

Otherwise, $A^* = A$ and $v_{\mathcal{I}}(A) = v_{\mathcal{I}}(A^*)$.

Induction step $A = \neg A_1$.

If $A = B$, the proof is as before.

Otherwise $A^* = \neg A^*_1$, and

$v_{\mathcal{I}}(A) = T$ iff $v_{\mathcal{I}}(A_1) = F$ iff (by ind. hyp.) $v_{\mathcal{I}}(A^*_1) = F$ iff $v_{\mathcal{I}}(A^*) = T$.

Induction step $A = A_1 \wedge A_2$, where $A$ contains $B$..

If $A = B$, the proof is as before.

Otherwise $A^* = A^*_1 \wedge A^*_2$

$v_{\mathcal{I}}(A) = T$ iff [$v_{\mathcal{I}}(A_1) = T$ and $v_{\mathcal{I}}(A_2) = T$] iff [$v_{\mathcal{I}}(A^*_1) = T$ and $v_{\mathcal{I}}(A^*_2) = T$, by the I.H.] iff $v_{\mathcal{I}}(A^*) = T$

The inductions steps for $\vee$ and $\rightarrow$ are essentially the same as for $\wedge$.

The proof for $\mathcal{I} \models \neg B$ where $B$ is replaced by $false$ is similar.

To get from $A^*$ to $A[B]$, we have to apply a sequence of replacements, each of which replaces a subformula by a logically equivalent one. Due to part a) the resulting formula $A[B]$ will be equivalent to the starting point $A^*$.

**Part c)**

If an interpretation $\mathcal{I}$ falsifies the conclusion of the rule, then $\mathcal{I} \models B$, $\mathcal{I} \models A$, $\mathcal{I} \models \Gamma$, and $\mathcal{I} \not\models D$ for any $D \in \Delta$.

Due to part c), we then now that $\mathcal{I} \models A[B]$, and therefore $\mathcal{I}$ falsifies the premisse of the rule.

## Exercise 9.3 **(Pure Literal Elimination)**

Given a set of clauses $S$, a literal $L$ is *pure* for $S$ if it occurs only positively or only negatively. I.e. if there are no clauses $C, D \in S$ with $L \in C$ and $\overline{L} \in D$.

In a refutation procedure like DPLL or resolution, if there is a pure literal $L$, it is OK to remove all clauses that contain $L$, and this is why:

Let $S' := \{C \in S \mid L \notin C\}$ be the set of clauses in $S$ that do not contain $L$. Show that if $L$ is pure, then $S$ is satisfiable if and only if $S'$ is satisfiable.

*Hint: it is easiest to consider two cases, depending on whether $L$ is an atom like $p$ or a negated atom like $\neg p$. In both cases, if $L$ is pure, it is easy to find an interpretation that satisfies all clauses containing $L$.*

**Solution**

We show that there is an interpretation $\mathcal{I}$ with $\mathcal{I} \models S$ iff there is an interpretation $\mathcal{I}'$ with $\mathcal{I}' \models S'$.

The direction from left to right is easy: Since $S' \subseteq S$, any interpretation that satisfies all clauses in $S$ satisfies all clauses in $S'$.

For the direction from right to left, let $\mathcal{I}' \models S'$ and note that $S'$ does not contain the pure literal $L$.

Assume for now that the pure literal $L$ is an atom that occurs only positively in $S$. Let $\mathcal{I}$ by the interpretation that interprets $L$ as true, and all other propositional symbols like $\mathcal{I}'$.

Then $\mathcal{I} \models S'$ since $L$ does not occur in $S'$ and $\mathcal{I}$ is identical to $\mathcal{I}'$ on all other propositional symbols.

On the other hand, $\mathcal{I} \models S \setminus S'$ since $\mathcal{I}$ satisfies any clause $L \vee C$ that contains $L$ positively.

Together, $\mathcal{I} \models S$.

If the pure literal is a negated atom $\neg p$, we let $\mathcal{I}$ interpret $p$ as false, so $\neg p$ as true, and use the same argument.