IN3070
IN4070

Autumn 2022

Exercises for the Course

Logic

Week 13



Exercise 11.1 (DL Formalisation)

Given

- concepts Cat, Dog, $Human$, $Crazy$
- a role $hasOwner$

Write $\mathcal{ALC}$ description logic axioms that express the following:

1. Nothing is both a Cat and a Dog
2. Every Dog has at least one owner who is a $Human$
3. Anybody who owns a Cat is $Crazy$

Hint: don't invent syntax. DL has concept expressions and $\sqsubseteq$ axioms, and nothing else. In particular, no variables. But it can help to formalise in first-order logic and try to look at the correspondence between first-order and DL shown in the lecture.

Solution

1. $\mathsf{Cat} \sqcap \mathsf{Dog} \sqsubseteq \bot$ or $\mathsf{Cat} \sqsubseteq \neg\mathsf{Dog}$ or $\mathsf{Dog} \sqsubseteq \neg\mathsf{Cat}$

2. $\mathsf{Dog} \sqsubseteq \exists\mathsf{hasOwner}.\mathsf{Human}$

3. This one is a bit tricky. Owners of cats might be expressed by $\exists\mathsf{owns}.\mathsf{Cat}$ – but we have no role owns, only hasOwner which goes in the wrong direction. But it is possible to express this as follows:

$$\mathsf{Cat} \sqsubseteq \forall\mathsf{hasOwner}.\mathsf{Crazy}$$

Exercise 11.2 (DL Syntax and Calculus)

Consider the following set $\mathcal{K}$ of First Order logic formulas:

$$\forall x.(\mathsf{Man}(x) \leftrightarrow \mathsf{Human}(x) \wedge \mathsf{Male}(x)) \tag{1}$$
$$\forall x.(\mathsf{Parent}(x) \leftrightarrow \mathsf{Human}(x) \wedge \exists y.(\mathsf{hasChild}(x,y) \wedge \mathsf{Human}(y))) \tag{2}$$
$$\forall x.(\mathsf{Father}(x) \leftrightarrow \mathsf{Man}(x) \wedge \exists y.(\mathsf{hasChild}(x,y) \wedge \mathsf{Human}(y))) \tag{3}$$
$$\forall x.(\mathsf{GrandFather(x)} \leftrightarrow \mathsf{Man}(x) \wedge \exists y.(\mathsf{hasChild}(x,y) \wedge \mathsf{Parent}(y))) \tag{4}$$

Do the following:

- Write an $\mathcal{ALC}$ TBox $\mathcal{T}$ that is logically equivalent to $\mathcal{K}$.
- Determine whether the axiom

  $$\mathsf{GrandFather} \sqsubseteq \mathsf{Parent}$$

  is a logical consequence of $\mathcal{T}$ by applying the $\mathcal{ALC}$ calculus w.r.t. a TBox.

Solution: The following TBox $\mathcal{T}$ is equivalent to $\mathcal{K}$:

$$\begin{array}{rl}
\mathsf{Man} \sqsubseteq \mathsf{Human} \sqcap \mathsf{Male} & \mathsf{Human} \sqcap \mathsf{Male} \sqsubseteq \mathsf{Man} \\
\mathsf{Parent} \sqsubseteq \mathsf{Human} \sqcap \exists \mathsf{hasChild}.\mathsf{Human} & \mathsf{Human} \sqcap \exists \mathsf{hasChild}.\mathsf{Human} \sqsubseteq \mathsf{Parent} \\
\mathsf{Father} \sqsubseteq \mathsf{Man} \sqcap \exists \mathsf{hasChild}.\mathsf{Human} & \mathsf{Man} \sqcap \exists \mathsf{hasChild}.\mathsf{Human} \sqsubseteq \mathsf{Father} \\
\mathsf{GrandFather} \sqsubseteq \mathsf{Man} \sqcap \exists \mathsf{hasChild}.\mathsf{Parent} & \mathsf{Man} \sqcap \exists \mathsf{hasChild}.\mathsf{Parent} \sqsubseteq \mathsf{GrandFather}
\end{array}$$

The calculus inference is given below (only important steps are shown, the concept and role names are abbreviated):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dots, 2 : P \Rightarrow 2 : P, \dots \qquad \dots, 1 : H \Rightarrow 1 : H, \dots}{\dots, 2 : P, 2 : \neg P \sqcup (H \sqcap \exists hC.H) \Rightarrow 2 : H, \dots}}{\dots, 1 : \exists hC.P, 1hC2, 2 : P, 2 : C_{\mathcal{T}} \Rightarrow 1 : \exists hC : H, 2 : H, \dots}}{\dots, 1 : \exists hC.P, 1hC2, 2 : P, 2 : C_{\mathcal{T}} \Rightarrow 1 : \exists hC : H, \dots}}{\dots, 1 : H \Rightarrow 1 : H, \dots \quad \dots, 1 : P \Rightarrow 1 : P, \dots \quad \dots, 1 : \exists hC.P \Rightarrow 1 : \exists hC : H, \dots}}{\dots, 1 : \neg H \sqcup \neg hC.H \sqcup P, 1 : H, 1 : \exists hC.P \Rightarrow 1 : P, \dots}}{\dots, 1 : M \Rightarrow 1 : M, \dots \qquad \dots, 1 : H, 1 : \exists hC.P \Rightarrow 1 : P, \dots}}{\dots, 1 : \neg M \sqcup (H \sqcap Ma), 1 : M, 1 : \exists hC.P \Rightarrow 1 : P, \dots}}{\dots, 1 : GF \Rightarrow 1 : GF, \dots \qquad \dots, 1 : M, 1 : \exists hC.P \Rightarrow 1 : P, \dots}}{\dots, 1 : \neg GF \sqcup (M \sqcap \exists hC.P) \Rightarrow 1 : \neg GF, 1 : P}}{1 : C_{\mathcal{T}} \Rightarrow 1 : \neg GF \sqcup P}$$

Exercise 11.3 **(Role Satisfiability)**

We say that an atomic role R is satisfiable w.r.t. a TBox $\mathcal{T}$ if there exists a model $\mathcal{I}$ of $\mathcal{T}$ such that $R^{\mathcal{I}} \neq \emptyset$.

- Write a satisfiable $\mathcal{ALC}$-TBox such that role R is unsatisfiable w.r.t. $\mathcal{T}$.
- Reduce the problem of checking satisfiability of an atomic role w.r.t. an $\mathcal{ALC}$-TBox to the problem of concept satisfiability w.r.t. an $\mathcal{ALC}$-TBox.

Solution: The following is a satisfiable TBox with an unsatisfiable role:

$$\mathcal{T} = \{\exists R.\top \sqsubseteq \bot\}$$

To reduce role satisfiability to concept satisfiability, we only need to notice that a role R is satisfiable w.r.t. a TBox $\mathcal{T}$ if and only of the concept $\exists R.\top$ is satisfiable w.r.t. $\mathcal{T}$. Hence, given as input R and $\mathcal{T}$ we can transform R into the concept $C = \exists R.\top$ and check whether C is satisfiable w.r.t. $\mathcal{T}$. If it is, then return $true$ and otherwise return $false$.