



Exercise 3.05

(Recursively defining a function)

The definition of *free variable* on slide 11 is somewhat informal. Try giving a proper, recursive definition of the function

$$\text{Free} : \mathcal{F} \rightarrow \mathcal{P}(\mathcal{V})$$

which sends a formula to the set of variables occurring free in it. ($\mathcal{P}(\mathcal{V})$ is the powerset of $\mathcal{V}$ —the set of sets of variables.)

You might want to start with defining the function that sends a term to the variables that occur in it, or you can take that as given if you couldn't be bothered to.

Exercise 3.1

(Variables & Assignments)

- a) Which variables occur free in the following formulae? Which variables occur bound?
- b) Apply the following substitution to both formulae: $\{x \backslash a, y \backslash f(a, b), z \backslash g(x, c)\}$.
- c) Is the substitution capture-free for these formulae?

- 1. $\forall x (p(y) \wedge \forall y (p(x, z) \rightarrow q(y)))$
- 2. $\forall x \exists x \forall y (p(x) \vee \exists y q(y)) \rightarrow r(x)$

Exercise 3.2

(Formalization & Interpretations)

- a) “If there is a man in town who shaves all the men in town who do not shave themselves, then some man in town shaves himself.” Formalize these statements by a single first-order formula F_S .
- b) What is the value of the term $(5 + 3) * (8 - 5)$ under the interpretation $\mathcal{I} = (\mathbb{N}, \iota)$ with $+^\iota = *$, $*^\iota = -$, $-^\iota = \div$ (division), $3^\iota = 8$, $5^\iota = 6$, $8^\iota = 36$.
- c) Show that the following formulae are satisfiable or invalid (or both) by providing a model and/or a counter-model (i.e. an interpretation that falsifies the formula).

- 1. $\forall x p(f(x), a) \rightarrow \exists x p(g(x), x)$
- 2. $\forall y q(y, b) \rightarrow \exists x q(a, x)$

Exercise 3.25

(Interpretations)

The language of arithmetic has constants $\{\bar{0}, \bar{1}, \bar{2}, \dots\}$, two function symbols of arity 2, $\{\bar{+}, \bar{\times}\}$, and a relation symbol of arity 2, $\{\bar{\leq}\}$, (and equality). We use infix notation for convenience, that is we write e.g. $x \bar{\leq} y$ instead of $\bar{\leq}(x, y)$.

Give interpretations to *falsify* the following two statements:

1. $\bar{1} + \bar{2} = \bar{3}$
2. $\forall x \exists y (x \leq y)$

Exercise 3.3 (Postponed to next week)

(Sequent Calculus LK & Eigenvariables)

Try to prove the validity of the following formulae in the sequent calculus LK. If you cannot find a proof in LK then provide a counter-model.

1. The first-order formula F_S from Exercise 3.2 a).
2. $\forall x \exists y p(x, y) \rightarrow \forall u \exists v p(u, v)$
3. $\forall x \exists y p(x, y) \rightarrow \exists v \forall u p(u, v)$

Exercise 3.4 (Postponed to next week)

(Symmetry of LK)

Let A be a formula containing only $\forall, \exists$ and the connectives $\neg, \vee$, and $\wedge$. The dual formula A' of A is obtained by exchanging $\forall$ and $\exists$, and exchanging $\vee$ and $\wedge$. Prove that $\vdash A$ iff $\vdash \neg A'$.

Hint: try to explain how to transform an LK proof for A into a proof for $\neg A'$, and vice versa

Exercise 3.5

(Variable Assignments and Closed Formulas)

The term value, resp. truth value of a *closed* term t , resp. formula A in an interpretation $\mathcal{I} = (D, \iota)$ is independent of the variable assignment. I.e. if α and β are two variable assignments for $\mathcal{I}$, and t is closed, then

$$v_{\mathcal{I}}(\alpha, t) = v_{\mathcal{I}}(\beta, t)$$

and if A is closed then

$$v_{\mathcal{I}}(\alpha, A) = v_{\mathcal{I}}(\beta, A)$$

Prove these facts by structural induction on t and A .

Hint: when you try to prove this for $\forall x A$ and $\exists x A$, you will run into the obstacle that the subformula A is not necessarily closed, so it's not possible to apply the induction hypothesis to it. You have to find and prove a more general statement about variable assignments and the free variables occurring in formulae and terms.