Grading Guidelines
IN3070/IN4070
Autumn 2022

The maximum number of marks for the whole exam was 100.

The minimum number of marks required for each grade was as follows: For E, min. 26. For D, min. 42. For C, min. 50. For B, min. 62. For A, min. 80.

These boundaries are based on an alignment of the delivered work with the grade definitions given here:

`https://www.uio.no/studier/eksamen/karakterer/`

The final grade for each candidate is based on an evaluation of the delivered work as a whole, and may therefore in some cases deviate from the boundaries given above.

Question 1 – Sequent Calculi LK and LJ

Prove the validity of the following formulae using the given calculus. Note that the first two formulas are to be proven in the classical logic LK, while the last two are to be proven in the intuitionistic logic LJ.

In this task you can submit a hand-written answer. Use the sketching paper handed to you in the exam room for this. See instructions in the link below the task bar.

A) $(p \wedge (q \vee r)) \to ((\neg p \vee \neg q) \to (p \wedge r))$, using propositional LK [5 marks]

Answer:

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$p, q \vee r \;\Rightarrow\; p, p \wedge r$}
\RightLabel{$\neg$-l}
\UnaryInfC{$p, q \vee r, \neg p \;\Rightarrow\; p \wedge r$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$p, q \;\Rightarrow\; q, p \wedge r$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$p, r \;\Rightarrow\; q, p$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$p, r \;\Rightarrow\; q, r$}
\RightLabel{$\wedge$-r}
\BinaryInfC{$p, r \;\Rightarrow\; q, p \wedge r$}
\RightLabel{$\vee$-l}
\BinaryInfC{$p, q \vee r \;\Rightarrow\; q, p \wedge r$}
\RightLabel{$\neg$-l}
\UnaryInfC{$p, q \vee r, \neg q \;\Rightarrow\; p \wedge r$}
\RightLabel{$\vee$-l}
\BinaryInfC{$p, q \vee r, \neg p \vee \neg q \;\Rightarrow\; p \wedge r$}
\RightLabel{$\wedge$-l}
\UnaryInfC{$p \wedge (q \vee r), \neg p \vee \neg q \;\Rightarrow\; p \wedge r$}
\RightLabel{$\to$-r}
\UnaryInfC{$p \wedge (q \vee r) \;\Rightarrow\; (\neg p \vee \neg q) \to (p \wedge r)$}
\RightLabel{$\to$-r}
\UnaryInfC{$\Rightarrow\; (p \wedge (q \vee r)) \to ((\neg p \vee \neg q) \to (p \wedge r))$}
\end{prooftree}
$$

Grading: The order of rule applications can differ. But there have to be four branches. 1 mark for each, and one for correct rule labels.

B) $\neg\forall x\, p(x) \to \exists x\, \neg p(x)$, using first-order LK [5 marks]

Answer:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p(c) \;\Rightarrow\; p(c),\, \exists x\, \neg p(x)}\;\text{ax}}{\Rightarrow\; p(c),\, \neg p(c),\, \exists x\, \neg p(x)}\;\neg\text{-r}}{\Rightarrow\; p(c),\, \exists x\, \neg p(x)}\;\exists\text{-r},\ [x \backslash c]}{\Rightarrow\; \forall x\, p(x),\, \exists x\, \neg p(x)}\;\forall\text{-r}}{\neg\forall x\, p(x) \;\Rightarrow\; \exists x\, \neg p(x)}\;\neg\text{-l}}{\Rightarrow\; \neg\forall x\, p(x) \to \exists x\, \neg p(x)}\;\to\text{-r}$$

Grading: 2 marks per correct quantifier rule, 1 for the correct propositional reasoning.

C) $\neg\neg(p \vee \neg p)$, using propositional LJ [6 marks]

Answer:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p, \neg(p \vee \neg p) \;\Rightarrow\; p}\;\text{ax}}{p, \neg(p \vee \neg p) \;\Rightarrow\; p \vee \neg p}\;\vee\text{-r}_1}{p, \neg(p \vee \neg p) \;\Rightarrow}\;\neg\text{-l}}{\neg(p \vee \neg p) \;\Rightarrow\; \neg p}\;\neg\text{-r}}{\neg(p \vee \neg p) \;\Rightarrow\; p \vee \neg p}\;\vee\text{-r}_2}{\neg(p \vee \neg p) \;\Rightarrow}\;\neg\text{-l}}{\Rightarrow\; \neg\neg(p \vee \neg p)}\;\neg\text{-r}$$

Grading:

- 6 marks for a correct proof
- minus 2 for forgetting to keep a copy in ¬-left
- minus 3 for more than one formula in a succedent

D) $\neg\exists x\, p(x) \to \forall x\, \neg p(x)$ using first-order LJ [6 marks]

Answer:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p(c),\, \neg\exists x\, p(x) \;\Rightarrow\; p(c)}\;\text{ax}}{p(c),\, \neg\exists x\, p(x) \;\Rightarrow\; \exists x\, p(x)}\;\exists\text{-r}}{p(c),\, \neg\exists x\, p(x) \;\Rightarrow}\;\neg\text{-l}}{\neg\exists x\, p(x) \;\Rightarrow\; \neg p(c)}\;\neg\text{-r}}{\neg\exists x\, p(x) \;\Rightarrow\; \forall x\, \neg p(x)}\;\forall\text{-r}}{\Rightarrow\; \neg\exists x\, p(x) \to \forall x\, \neg p(x)}\;\to\text{-r}$$

Grading:

- 6 marks for a correct proof
- minus 1 for forgetting to keep a copy in $\neg$-left
- minus 3 for more than one formula in a succedent, e.g. by keeping a copy in $\exists$-r.

Question 2 – Intuitionistic Logic

Consider the formula: $A = \neg\neg p \vee \neg p$

A is valid in classical logic but not in intuitionistic logic.

A) Explain, by pointing to the differences between the calculi, why A can be proven using LK, but not using LJ. [5 marks]

Answer: The LJ calculus only permits one formula in the succedent of a sequent. In particular, the $\vee$-right rule requires choosing one of the disjuncts to keep in the succedent, while LK keeps boths disjuncts.

Starting from the sequent

$$\Rightarrow \neg\neg p \vee \neg p$$

LK and LJ allow no other rule than $\vee$-right to be applied. In the case of LJ, this gives us either the sequent

$$\Rightarrow \neg\neg p$$

or

$$\Rightarrow \neg p$$

and none of these lead to a closed proof.

On the other hand, in LK, we obtain

$$\Rightarrow \neg\neg p, \neg p$$

which can be closed by several applications of the $\neg$ rules.

Grading:

- Max. 1 mark for an answer that doesn't refer to the calculus rules (e.g. only saying that LK is sound and LJ is complete)
- 2 marks for pointing to the difference in $\vee$-r rules.
- 3 more marks for a complete explanation.

B) Show that A is not valid in intuitionistic logic using the model semantics. I.e. construct an intuitionistic structure with a world w such that $w \nVdash \neg\neg p \vee \neg p$ [7 marks]

In this task you can submit a hand-written answer. Use the sketching paper handed to you in the exam room for this. See instructions in the link below the task bar.

Answer: We can systematically derive conditions on the intutionistic structure.

If

$$w \nVdash \neg\neg p \vee \neg p \tag{1}$$

then by the model semantics for $\vee$,

$$w \nVdash \neg\neg p \tag{2}$$

and

$$w \nVdash \neg p \tag{3}$$

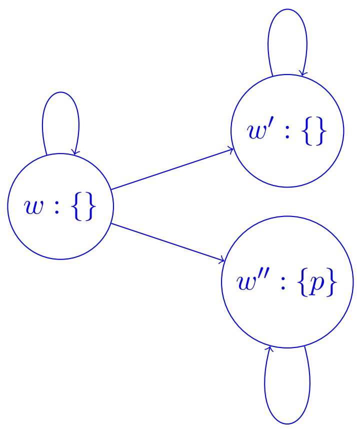
From (2) and the semantics of negation it follows that there is a world w' reachable from w with

$$w' \Vdash \neg p \tag{4}$$

and from (3) and the semantics of negation it follows that there is a world w'' reachable from w with

$$w'' \Vdash p \tag{5}$$

(4) tells us that neither w' nor any world reachable from w' forces p. This means that w' cannot be the same world as w'' (which forces p as we saw in (5)) or w (from which w'' is reachable). This gives us the following structure:



We can now double-check the semantics and verify that that (4) and (5) hold. Also due to w', (2) holds, and due to w'' (3) holds, which together gives (1).

Grading:

- 2 marks for arguing about the forcing relation
- 1 mark for treating the model semantics of $\vee$ correctly
- 2 marks for treating the model semantics of $\neg$ correctly
- 2 marks for correctly introducing the successor worlds
- -2 marks for a wrong model, like e.g. one that is not monotonic.
- max. 3 marks for giving *only* a correct model, but with no explanation

Question 3 – The Eigenvariable Condition

The so called «eigenvariable condition» for some of the rules of the first-order sequent calculus requires to substitute a «new» constant for a quantified variable, i.e. one that does not occur in the conclusion.

You can submit a hand-written answer. Use the sketching paper handed to you in the exam room for this. See instructions in the link below the task bar.

A) What would happen without this condition, i.e. if you could substitute any constant in the rules $\exists$-left and $\forall$-right? Would the calculus become unsound? Incomplete? Neither? Both?

Answer: The calculus would become unsound, as we show in B) It would remain complete, since any proof of LK (where the eigenvariable condition is satisified for all rule applications where it is applicable) is also a proof in the calculus where this condition is not required.

B) Give an example derivation that demonstrates what you wrote in A), i.e. a valid formula that cannot be proven without the eigenvariable condition and/or an invalid formula that can be proven, depending on your answer to A).

Answer: Without the eigenvariable condition, we could construct a proof for $\forall x\, p(x)$, by showing that p holds not for an arbitrary value, but only for one specific value. E.g., we can prove $p(c) \to \forall x\, p(x)$:

$$\dfrac{\dfrac{\dfrac{}{p(c) \;\Rightarrow\; p(c)}\text{ax}}{p(c) \;\Rightarrow\; \forall x\, p(x)}\forall\text{-r without ev condition!}}{\Rightarrow\; p(c) \to \forall x\, p(x)}\to\text{-r}$$

Similarly, we could construct a proof for $\exists x\, p(x) \to \forall x\, p(x)$ or $\exists x\, p(x) \to p(c)$ (using a wrong $\exists$-l rule)

[14 marks total]

Grading:

- 3 marks for the right answer about soundness

- 3 marks for the right answer about completeness
- 8 marks for a correct example

Question 4 – An inductive proof

Consider modal logic formulas that consist only of an atomic formula ($p, q, r, \ldots$) and an arbitrary nesting of $\Box$ and $\Diamond$ operators. E.g. p, $\Box\Diamond q$, $\Diamond\Box\Box p$ are examples of such formulas. However, logical connectives are not allowed. Formally, let S be the set of modal logic formulas inductively defined as follows:

- $p \in S$ for every propositional variable p
- $\Box A \in S$ for every $A \in S$
- $\Diamond A \in S$ for every $A \in S$

Show by structural induction on S that all formulas in S are satisfiable. [12 marks]

Hint: there is a single Kripke structure that satisfies all of these formulas. In fact, there is such a Kripke structure with only a single world.

Answer: Consider, for instance, any reflexive Kripke frame where the interpretation at any world makes all propositional variables true. We show by induction on formulas in S that all formulas are true at all worlds in such a Kripke structure. The induction hypothesis (IH) is that a formula $A \in S$ is true in all worlds of the structure; in the induction steps, we need to show that the IH implies that also $\Box A$ and $\Diamond A$ are true in all worlds.

- Base case: All propositional variables are true at all worlds, by assumption.
- Induction step for $\Box A$: Let w be a world. Let v be a world such that wRv. By Induction Hypothesis, $v \vDash A$. Thus $w \vDash \Box A$.
- Induction step for $\Diamond A$: Let w be a world. wRw, since R is reflexive. By Induction Hypothesis, $w \vDash A$. Thus $w \vDash \Diamond A$.

Note that reflexivity is not essential here, the thing we need for the $\Diamond$-step is that all worlds relate to some world.

Grading:

- 2 marks for a correct Kripke structure
- 2 marks for a correct structural induction argument:
  - 1 mark for an explicit induction hypothesis

  - 1 mark for the general structure
- 2 mark for the base case
- 3 marks for each induction step:
  - 1 mark for referring to the IH and the semantics
  - 2 marks for the general reasoning about the semantics

Question 5 – The Substitution Lemma

Prove the Substitution Lemma for first order terms, i.e.

for all interpretations I, assignments α for I, variables x, and terms s, t, $v_I(\alpha, t[x\backslash s]) = v_I(\alpha', t)$ where $\alpha' = \alpha\{x \leftarrow v_I(\alpha, s)\}$. [12 marks]

Hints:

- *You should prove this by structural induction over the term t.*
- *Your proof will need one base case each for constants, the variable x, and variables other than x*
- *Substitution is defined inductively as follows*
  - $x[x\backslash s] = s$
  - $y[x\backslash s] = y$ *for all variables* $y \neq x$
  - $a[x\backslash s] = a$ *for all constants* a
  - $f(t_1, \ldots, t_n)[x\backslash s] = f(t_1[x\backslash s], \ldots, t_n[x\backslash s])$
- *The modified variable assignment* $\alpha\{x \leftarrow d\}$ *is defined by*
  - $\alpha\{x \leftarrow d\}(y) = \alpha(y)$ *for all variables* $y \neq x$
  - $\alpha\{x \leftarrow d\}(x) = d$

Answer: Write $I = (D, \iota)$. By induction on t:

- Base case:
  - $t = x$: $v_I(\alpha, x[x\backslash s]) = v_I(\alpha, s) = \alpha'(x) = v_I(\alpha', x)$ (since $\alpha' = \alpha\{x \leftarrow v_I(\alpha, s)\}$).
  - $t = y$, where $y \neq x$: $v_I(\alpha, y[x\backslash s]) = v_I(\alpha, y) = \alpha(y) = \alpha'(y) = v_I(\alpha', y)$.
  - $t = a$, where a is a constant symbol: $v_I(\alpha, a[x\backslash s]) = v_I(\alpha, a) = a^\iota = v_I(\alpha', a)$

- Induction step: $t = f(t_1, \ldots, t_n)$, where the induction hypothesis is that the statement holds for all subterms t_i.

$$
\begin{aligned}
v_I(\alpha, f(t_1, \ldots, t_n)[x\backslash s]) &= v_I(\alpha, f(t_1[x\backslash s], \ldots, t_n[x\backslash s])) && (1)\\
&= f^\iota(v_I(\alpha, t_1[x\backslash s], \ldots, v_I(\alpha, t_n[x\backslash s])) && (2)\\
&= f^\iota(v_I(\alpha', t_1), \ldots, v_I(\alpha', t_n)) && (3)\\
&= v_I(\alpha', f(t_1, \ldots, t_n)) && (4)
\end{aligned}
$$

where the Induction Hypothesis is used n times in step (3).

Grading:

- 2 marks for a correct structural induction argument:
  - 1 mark for an explicit induction hypothesis
  - 1 mark for the general structure
- 2 marks for each of the three base cases
- 4 marks for the induction step:
  - 1 mark for referring to the IH
  - 3 marks for the general reasoning about the semantics

Question 6 – Soundness and completeness for LK

In this task you can submit a hand-written answer. Use the sketching paper handed to you in the exam room for this. See instructions in the link below the task bar.

A) Is the statement "If a first-order formula has a counter model, then it is not provable in the LK calculus" equivalent to the soundness or the completeness theorem for the LK calculus? Explain. [2 marks]

Answer: Soundness. The statement is the contrapositive of the Soundness Theorem.

Grading: 1 mark for soundness, 1 mark for the explanation.

B) Consider the formula $\forall x(p(x) \vee q(x)) \rightarrow \forall x p(x) \vee \forall x q(x)$.

- Construct a fair derivation of the formula in the LK calculus.
- Construct a counter model of the formula from an open branch of the derivation, using the Herbrand universe of the branch.

[10 marks]

Answer:

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\quad
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}{\ldots, p(a), p(b) \vee q(b) \;\Rightarrow\; p(b), q(a)}\,\vee\text{-l}
\quad
\dfrac{}{\ldots, q(a), p(b) \vee q(b) \;\Rightarrow\; p(b), q(a)}\,\text{ax}
}{\forall x(p(x) \vee q(x)), p(a) \vee q(a), p(b) \vee q(b) \;\Rightarrow\; p(b), q(a)}\,\vee\text{-l}
}{\forall x(p(x) \vee q(x)), p(a) \vee q(a) \;\Rightarrow\; p(b), q(a)}\,\forall\text{-l}
}{\forall x(p(x) \vee q(x)) \;\Rightarrow\; p(b), q(a)}\,\forall\text{-l}
}{\forall x(p(x) \vee q(x)) \;\Rightarrow\; \forall x p(x), q(a)}\,\forall\text{-r}
}{\forall x(p(x) \vee q(x)) \;\Rightarrow\; \forall x p(x), \forall x q(x)}\,\forall\text{-r}
}{\forall x(p(x) \vee q(x)) \;\Rightarrow\; \forall x p(x) \vee \forall x q(x)}\,\vee\text{-r}
}{\Rightarrow\; (\forall x(p(x) \vee q(x)) \to \forall x p(x) \vee \forall x q(x))}\,\to\text{-r}
$$

There is exactly one branch which does not end in an axiom. Call it X. The Herbrand universe of the branch X is $\{a, b\}$.

The derivation is fair, see slide 25 of lecture 5 (but essentially because we have applied every rule that we can, and in particular $\forall$-l for every term in the Herbrand universe of the branch).

We define an interpretation $I = (D, \iota)$ from X as follows. $D := \{a, b\}$, the Herbrand universe of X. Since $p(a)$ and $q(b)$ occur on the left in the top sequent of X, we set $p^\iota = \{a\}$ and $q^\iota = \{b\}$.

We see that $I \vDash \forall x(p(x) \vee q(x))$, as all elements are either p or q. But neither $I \vDash \forall x p(x)$ nor $I \vDash \forall x q(x)$, as $I \nvDash p(b)$ and $I \nvDash q(a)$.

Grading:

- 4 marks for the derivation
- 2 marks for the domain, i.e. the Herbrand universe
- 4 marks for the correct interpretation

Question 7 – Classical Semantics and LJ

Note: The semantics in the following is always the usual one for classical logic, defined in terms of interpretations . Although the intuitionistic calculus LJ is mentioned, you need at no point here use or consider Kripke semantics, nor will you need to remember the LJ rules.

A) The ¬-left rule of the first-order LJ calculus is

$$
\frac{\Gamma, \neg A \Rightarrow A}{\Gamma, \neg A \Rightarrow D}
$$

Show that this rule can be added to the LK calculus (i.e. the one for classical logic) by showing either that it preserves validity downwards or falsifiability upwards. [8 marks]

Answer: We show preservation of falsifiability upwards. Let $\mathcal{I}$ be an interpretation, and assume that $\mathcal{I}$ falsifies $\Gamma, \neg A \Rightarrow D$. Then $\mathcal{I}$ satisfies all formulas in Γ as well as $\neg A$. Hence it falsifies A, so it falsifies $\Gamma, \neg A \Rightarrow A$. The downwards direction is similar: Suppose that $\mathcal{I}$ makes $\Gamma, \neg A \Rightarrow A$ true. It cannot simultaneously satisfy both A and $\neg A$, so the only possibility is that it makes a formula in the antecedent false, either one in Γ or $\neg A$. But then $\mathcal{I}$ also satisfies $\Gamma, \neg A \Rightarrow D$, irrespective of what truth value $\mathcal{I}$ assigns to D.

Grading:

- max 4 marks for confused notion of falsifiability

B) The other rules of the LJ calculus also preserve validity downwards w.r.t the classical logic semantics. You do not need to show this. Explain, based on this observation, why any formula which is provable in the LJ calculus must also be provable in the LK calculus. [4 marks]

Answer: Let σ be any sequent, and assume that σ is provable in LJ. Since the rules of LJ preserve validity downwards, and the axioms are valid, the sequent σ is valid. By completeness of LK, σ is provable in LK.

Grading:

- subtract 1 mark for forgetting to mention the completeness of LK.

C) Show that the LJ calculus is not complete w.r.t. the classical logic semantics. (You may, in your answer, refer to other questions in this exam even if you have not answered them.) [4 marks]

Answer: Question 2 above provides us with a formula which is valid in classical semantics but which can not be proven in LJ. Hence LJ is not complete with respect to classical semantics.

Grading: