IN3070/4070 – Logic – Autumn 2022

Lecture 6: Unification, Normal Forms

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27th September 2022

Today's Plan

- Unification
- Normal Forms
- Negation Normal Form
- Conjunctive Normal Form
- Clausal Form
- Prenex Normal Forms
- Skolemization

Outline

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- Conjunctive Normal Form
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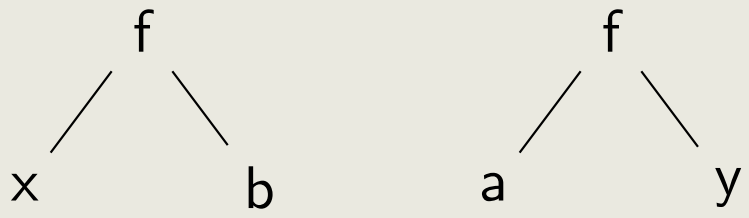

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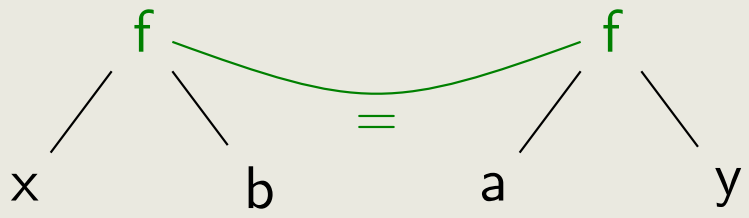

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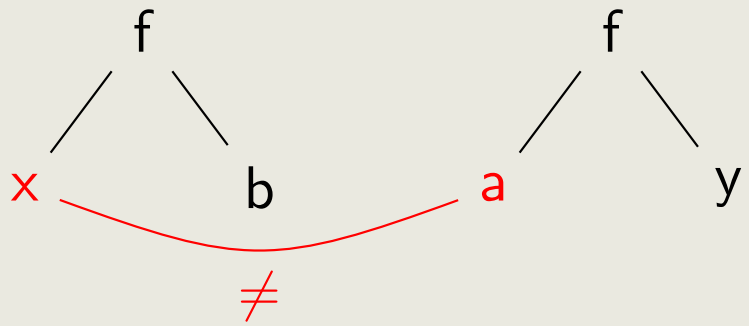

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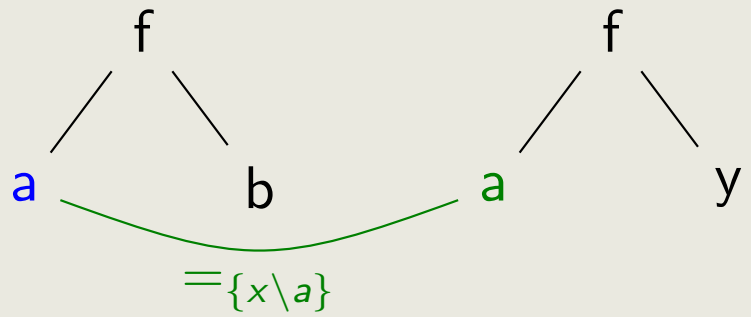

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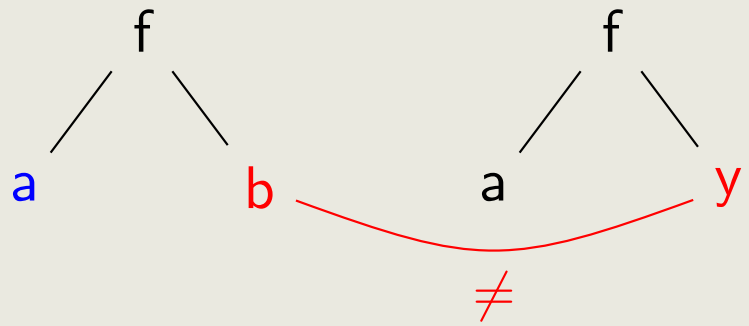

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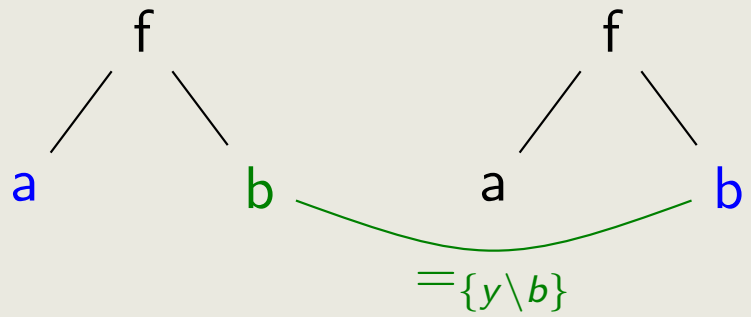

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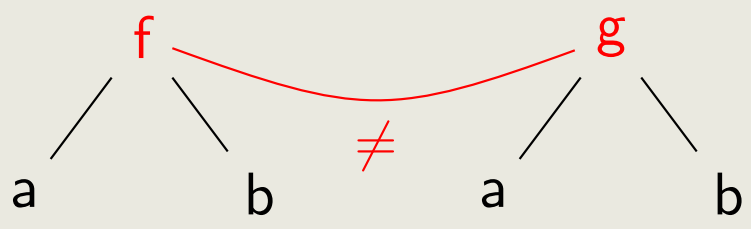

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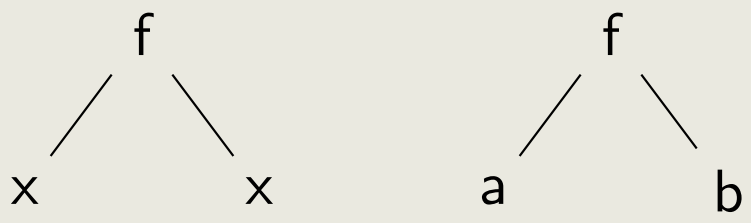

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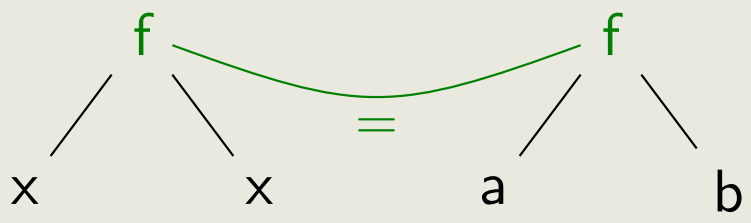

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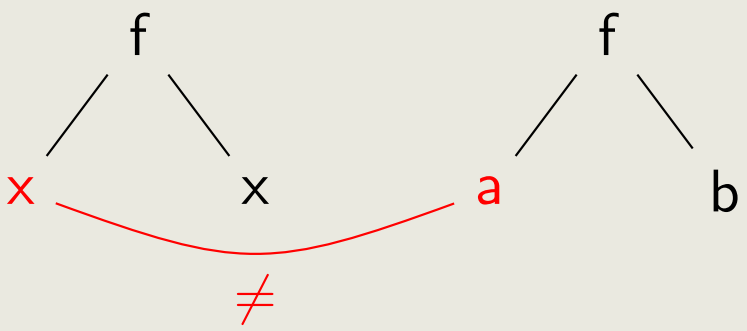

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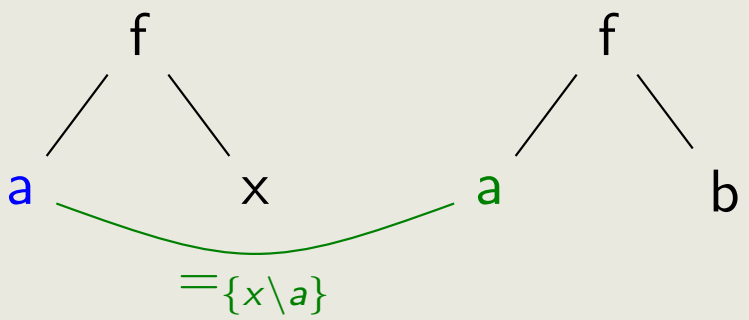

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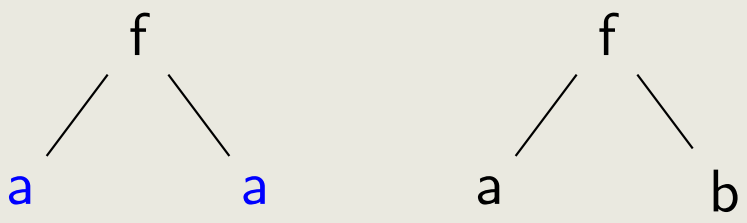

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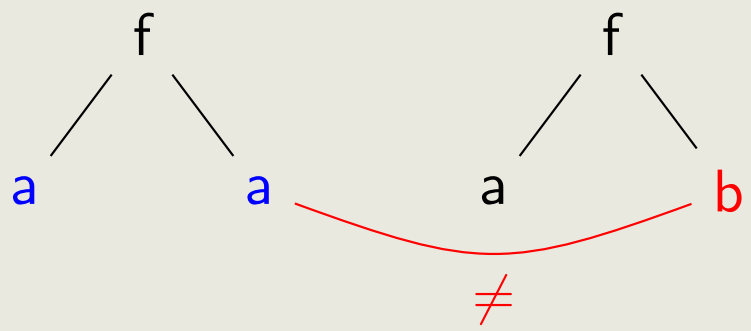

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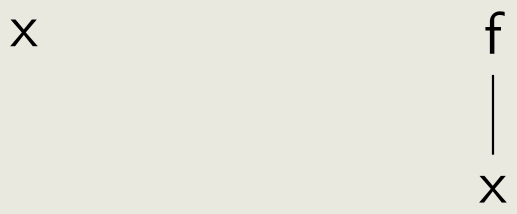

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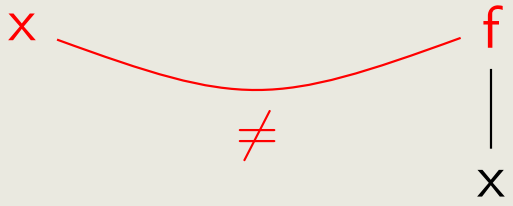

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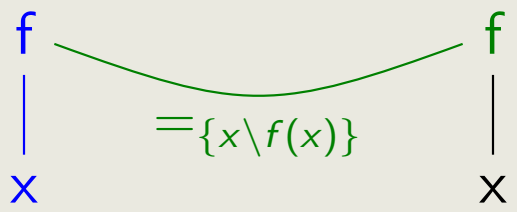

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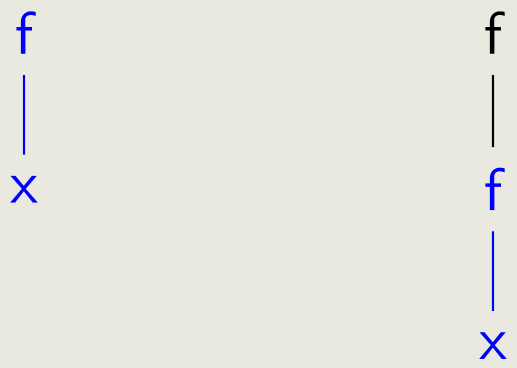

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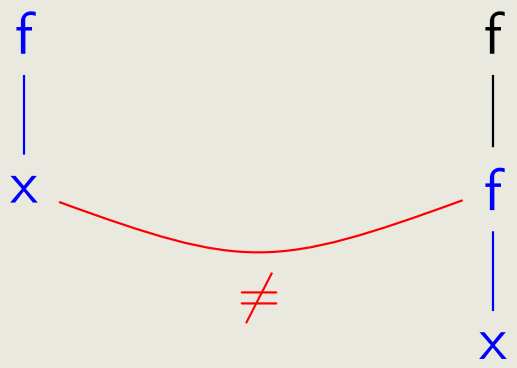

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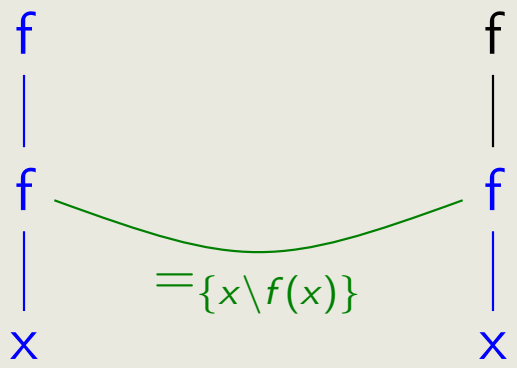

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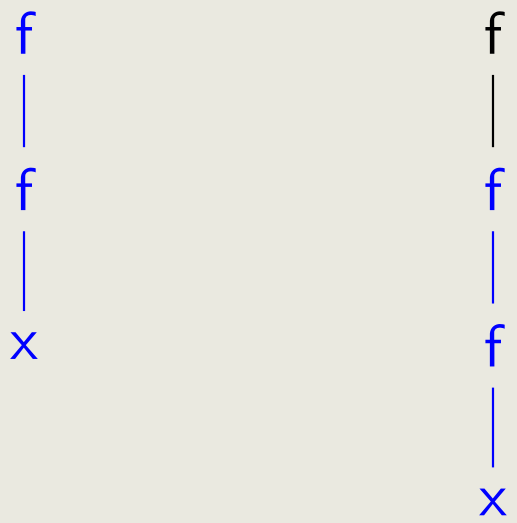

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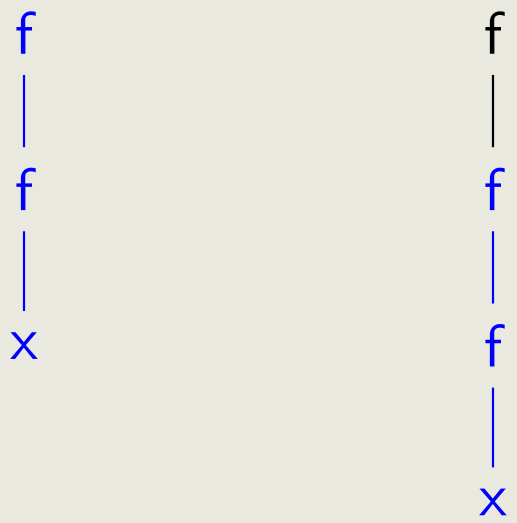

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- This continues indefinitely

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- Such a unifier is known as a most general unifier.

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- Exercise: show that $(\tau\sigma)(A) = \tau(\sigma(A))$ for all formlae A and all substitutions σ and τ.

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Proof.

Let τ be such that $\sigma_1 = \tau\sigma_2$.
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- So every term is a subterm of itself, but not a strict subterm.

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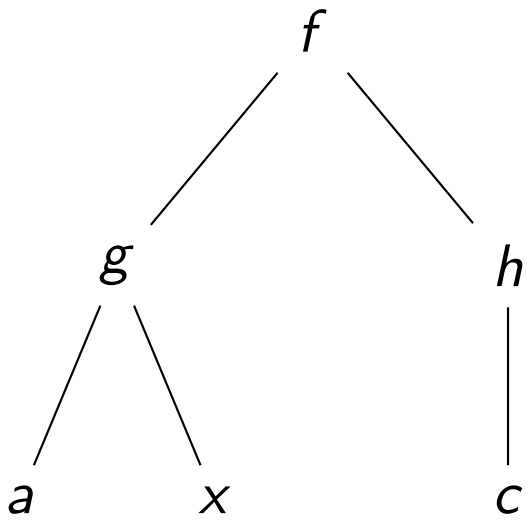

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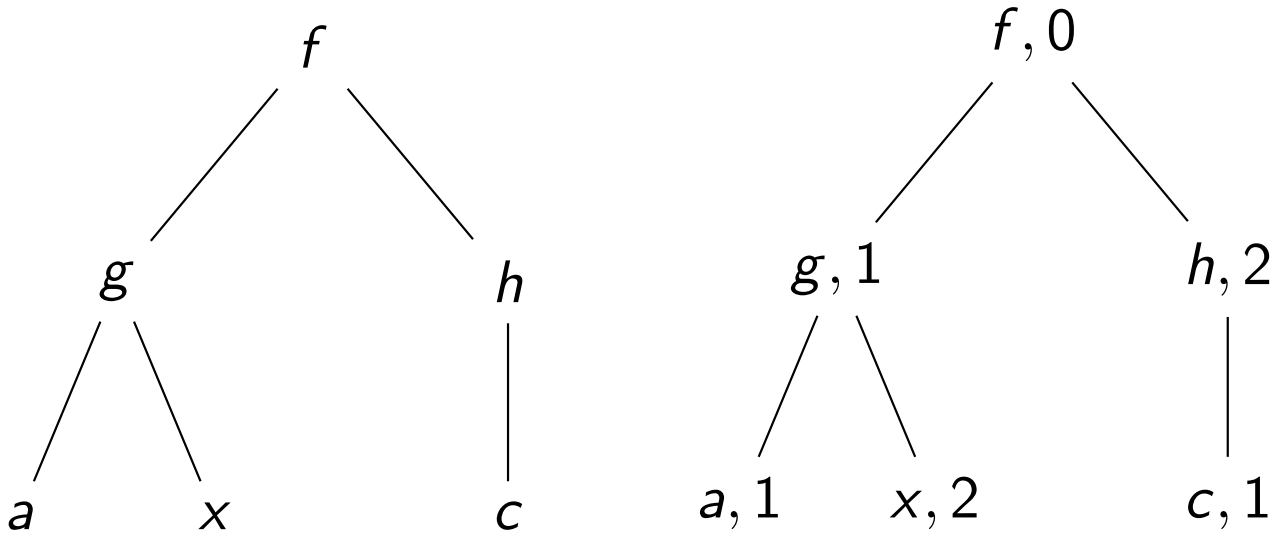

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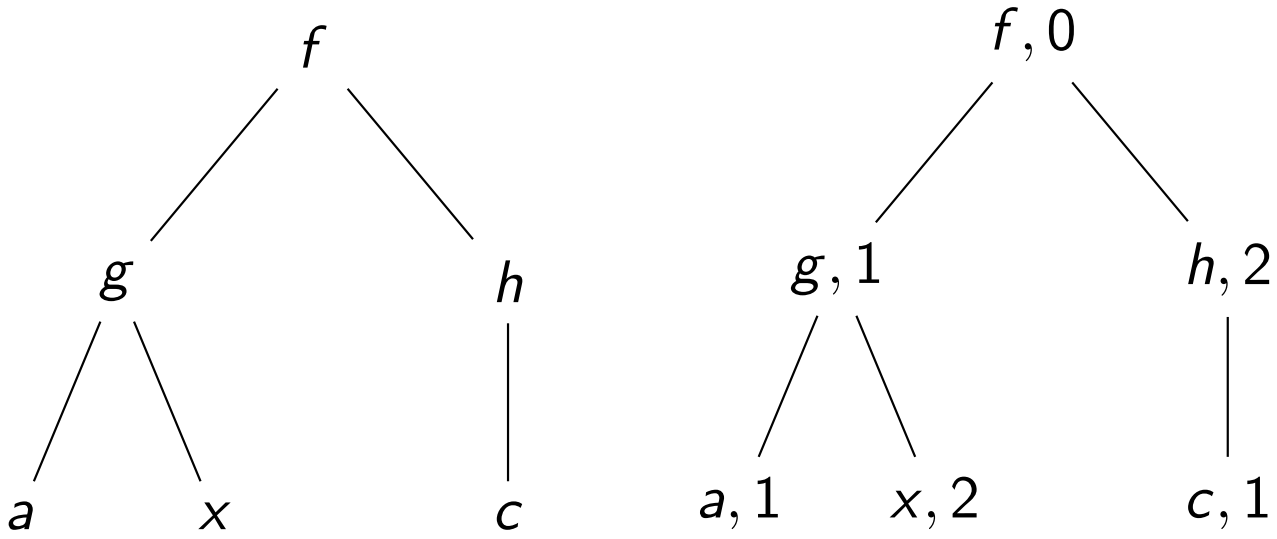

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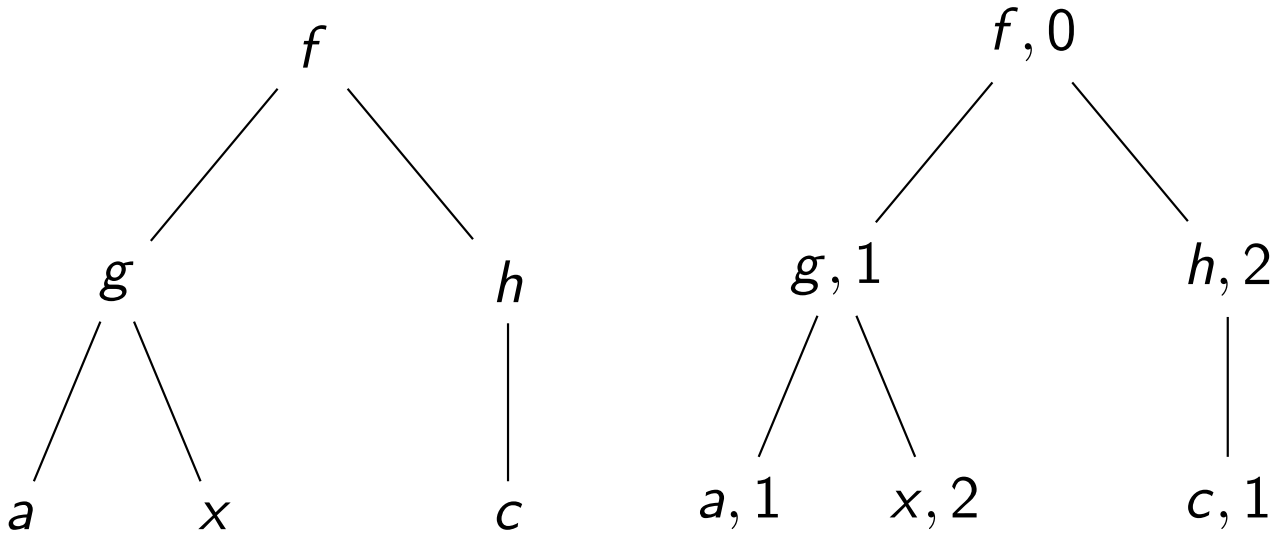

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- We write the root of the numbered term tree of t as $\mathrm{root}(t)$.

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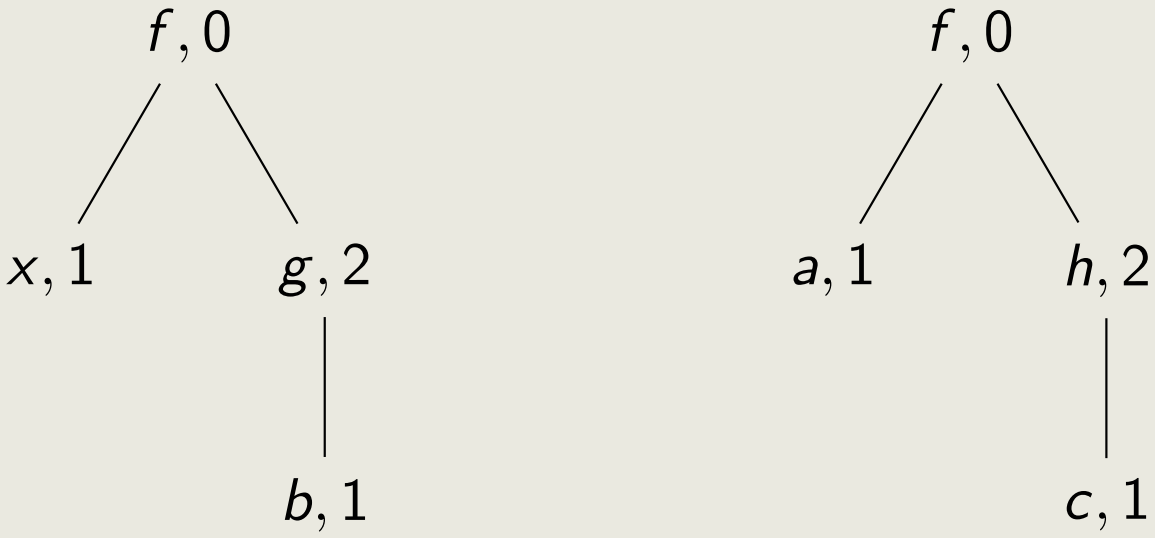

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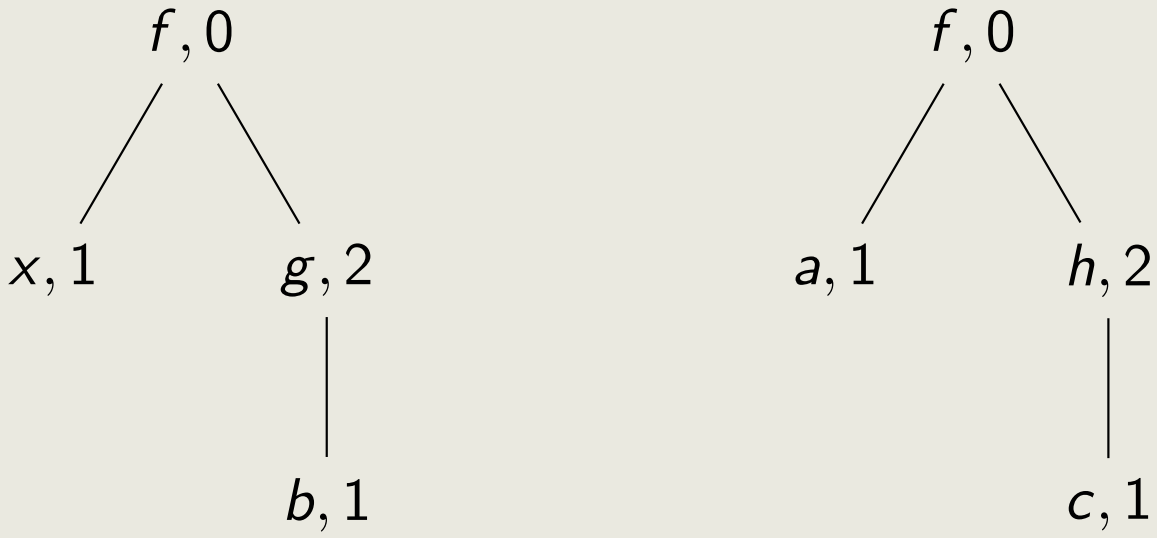

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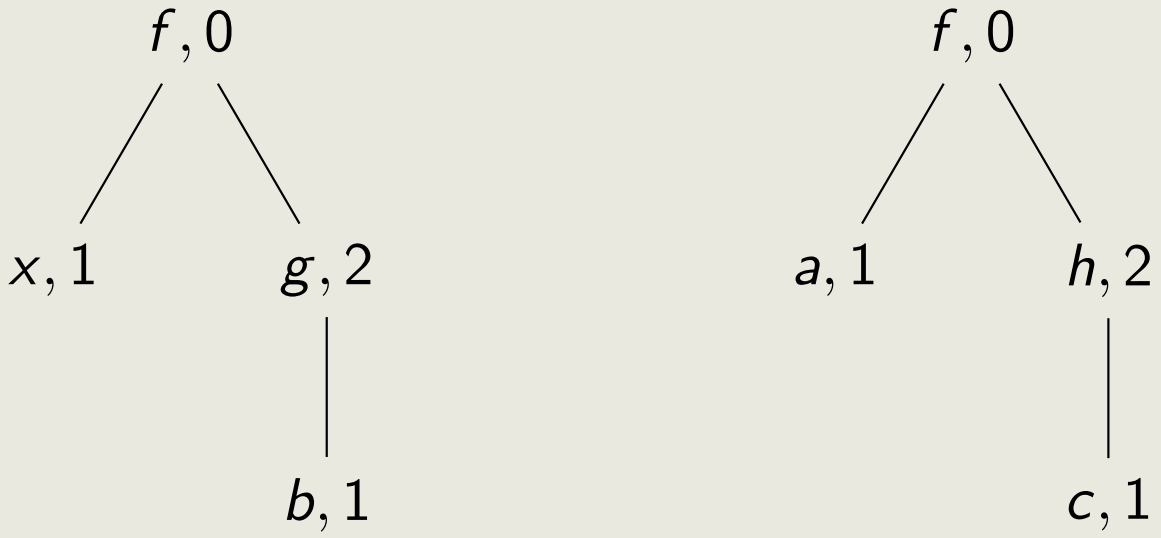

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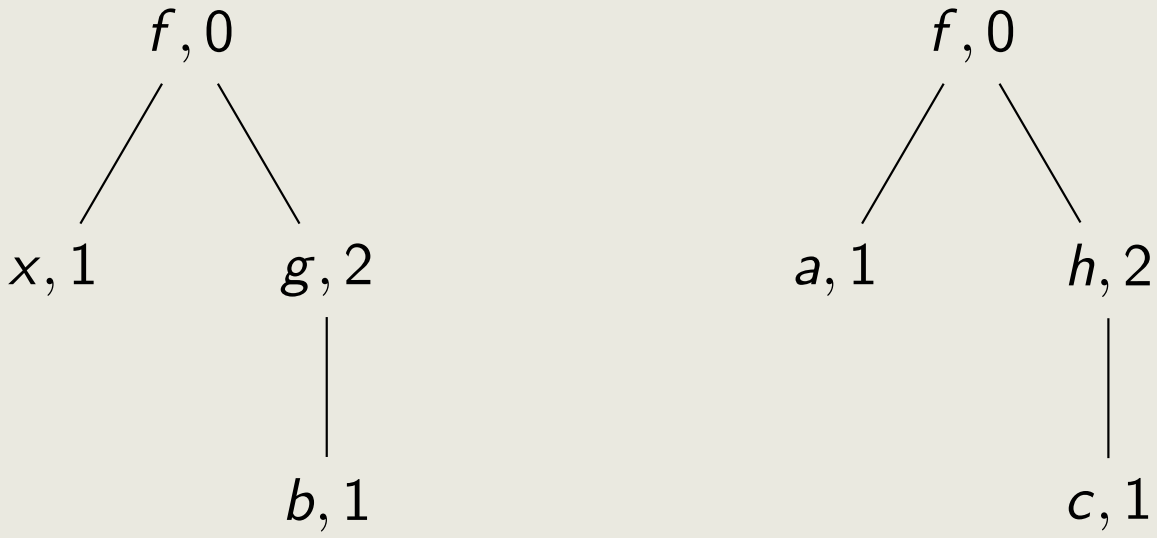

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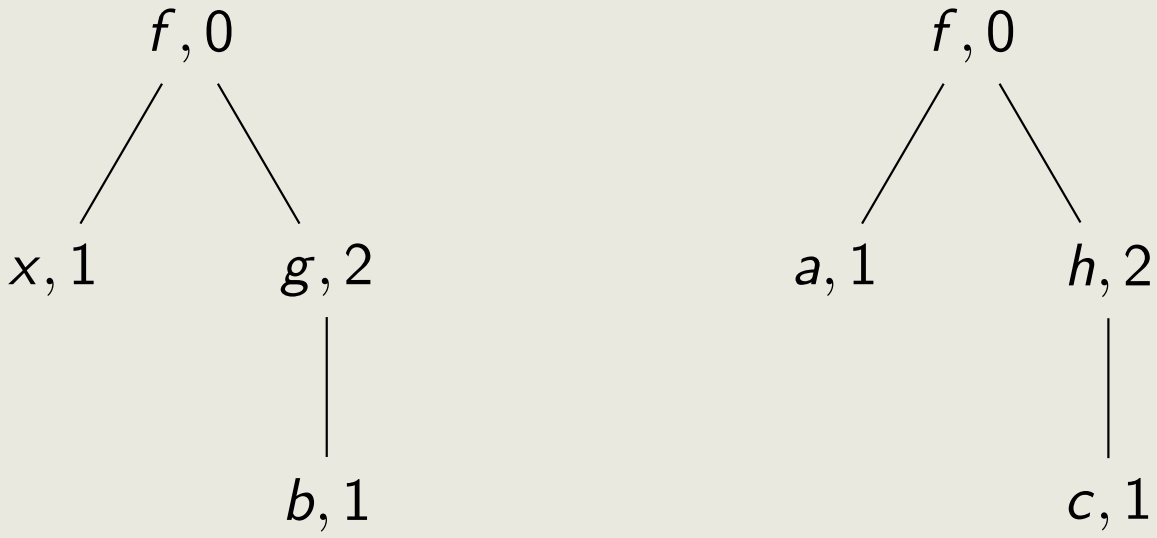

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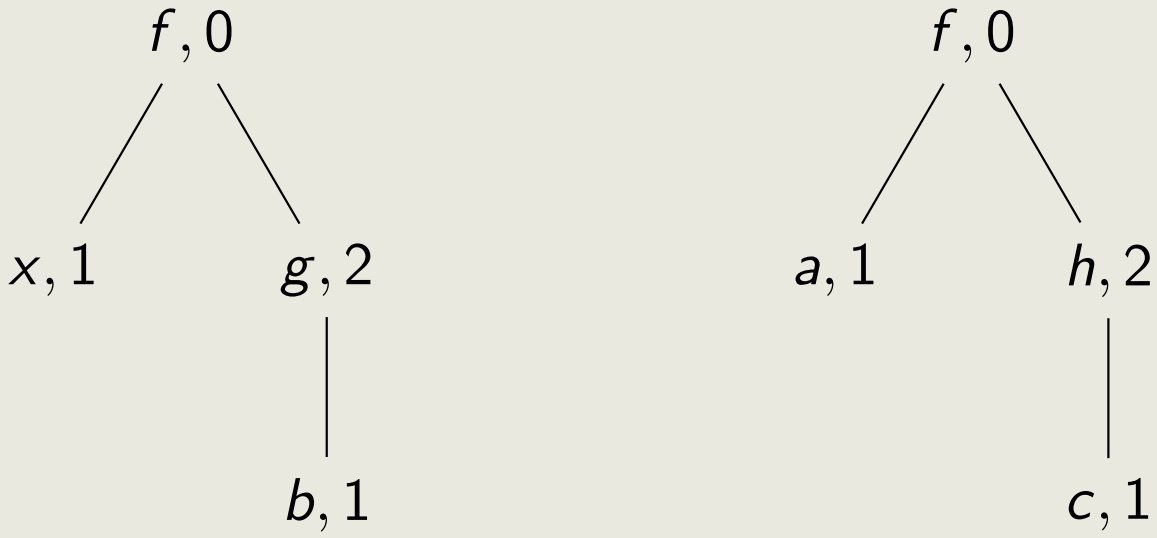

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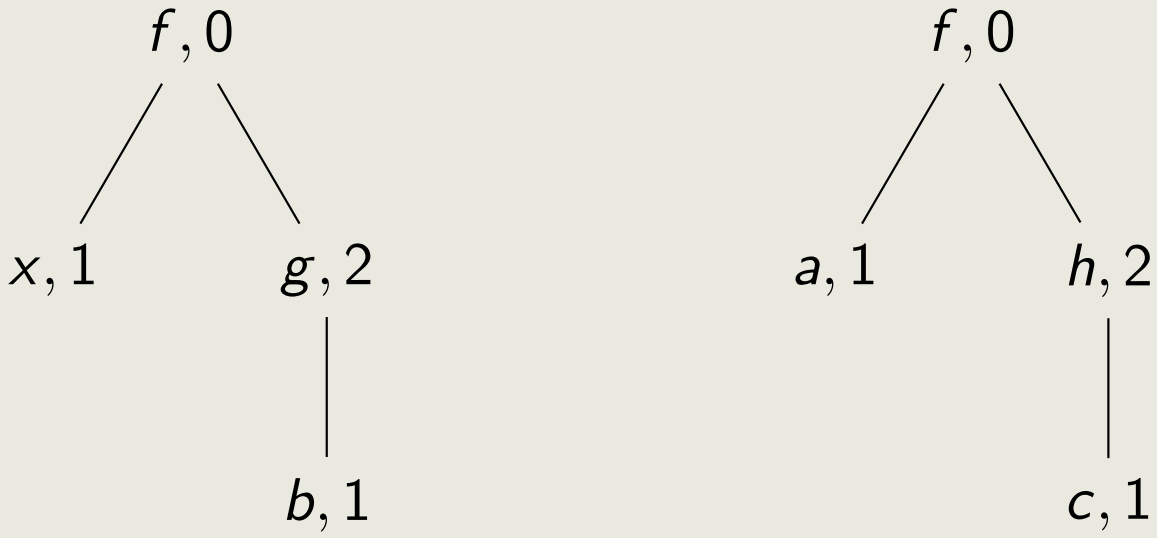

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Outline

- Unification
- Normal Forms
- Negation Normal Form
- Conjunctive Normal Form
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- Then $\langle m/k, n/k\rangle$ is reduced and $\langle m/k, n/k\rangle \approx \langle m, n\rangle$
- So $\langle m/k, n/k\rangle$ is the "reduced normal form" of $\langle m, n\rangle$

Outline

- Unification
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- Negation Normal Form
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- Clausal Form
- Prenex Normal Forms
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Theorem 3.1.

Every formula in first-order logic can be transformed into an equivalent formula in NNF.

Proof.

To convert an arbitrary formula to a formula in NNF, remove implications, and push negations inwards, preserving equivalence, using the following:

$$
\begin{aligned}
A \to B &\equiv \neg A \vee B \\
\neg(A \wedge B) &\equiv \neg A \vee \neg B \\
\neg(A \vee B) &\equiv \neg A \wedge \neg B \\
\neg(\forall x\, A) &\equiv \exists x\, \neg A \\
\neg(\exists x\, A) &\equiv \forall x\, \neg A \\
\neg(\neg A) &\equiv A
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- Soundness and completeness proofs also have fewer cases.

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(2) Use the distributive laws to move conjunctions inside disjunctions to the outside

$$A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$$

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$(p \vee \neg q) \wedge (\neg p \vee q)$ in clausal form: $\{\{p, \neg q\}, \{\neg p, q\}\}$

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- l^c then represents its *complement*.

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where the Q_i are quantifiers ($\forall/\exists$) and M is a quantifier-free formula in CNF. The sequence $Q_1 x_1 \cdots Q_n x_n$ is the prefix and M is the matrix.

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Note: *The universal quantifiers are implicit.*

Outline

- Unification
- Normal Forms
- Negation Normal Form
- Conjunctive Normal Form
- Clausal Form
- Prenex Normal Forms
- Skolemization

Skolem's Theorem

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Named after the Norwegian mathematician and logician Thoralf Albert Skolem (1887–1963).

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- Extract quantifiers from the matrix until all quantifiers appear in the prefix and the matrix is quantifier-free.

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- The formula is now in PCNF.

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- If there are no universal quantifiers preceding $\exists x$, replace x by a new constant (0-ary function).
- These new function symbols are Skolem functions and the process of replacing existential quantifiers by functions is Skolemization.

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- *Look at the formulas* $\forall x \exists y p(x, y)$ *and* $\forall x p(x, f(x))$.

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- *Are they equivalent? No!*
- *Are they equisatisfiable? Yes!*
- *The Skolemization of* $\forall x \exists y p(x, y)$ *is* $\forall x p(x, f(x))$*, and if one of them has a model, so does the other.*

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- For any choice of elements $d_1, \ldots, d_n$ from D, there is an element d_{n+1} in D such that

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- Let $f^{\iota'}(d_1, \ldots, d_n) = d_{n+1}$. This ensures that the claim holds.

Example

- Clause form of $\neg\exists x\,(p(x) \to \forall y\, p(y))$

Example

- Clause form of $\neg\exists x\,(p(x) \rightarrow \forall y\, p(y))$
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Outlook

- We have seen the LK calculus for propositional and first-order logic
- Sound and complete, but not machine-oriented
- Machine-oriented calculi use:
  - Unification to find the right instantiations
  - Normal forms to simplify reasoning steps
- Free variable calculi
  - Similar to LK, but with unification
  - Often used with NNF or clause form
  - Not this year
- Resolution
  - Basis of many theorem provers, uses unification
  - Almost always on clause form