IN3070/4070 – Logic – Autumn 2022

Lecture 7: Resolution

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Today's Plan

- Introduction
- Repetition: Negation Normal Form
- Conjunctive Normal Form
- Clausal Form
- Resolution
- Soundness of Resolution
- Completeness of Resolution

Oblig 1

- Published today, 4 October
- Has a question about first order resolution
  - Start on the rest, do that next week
- Deadline for first attempt in two weeks

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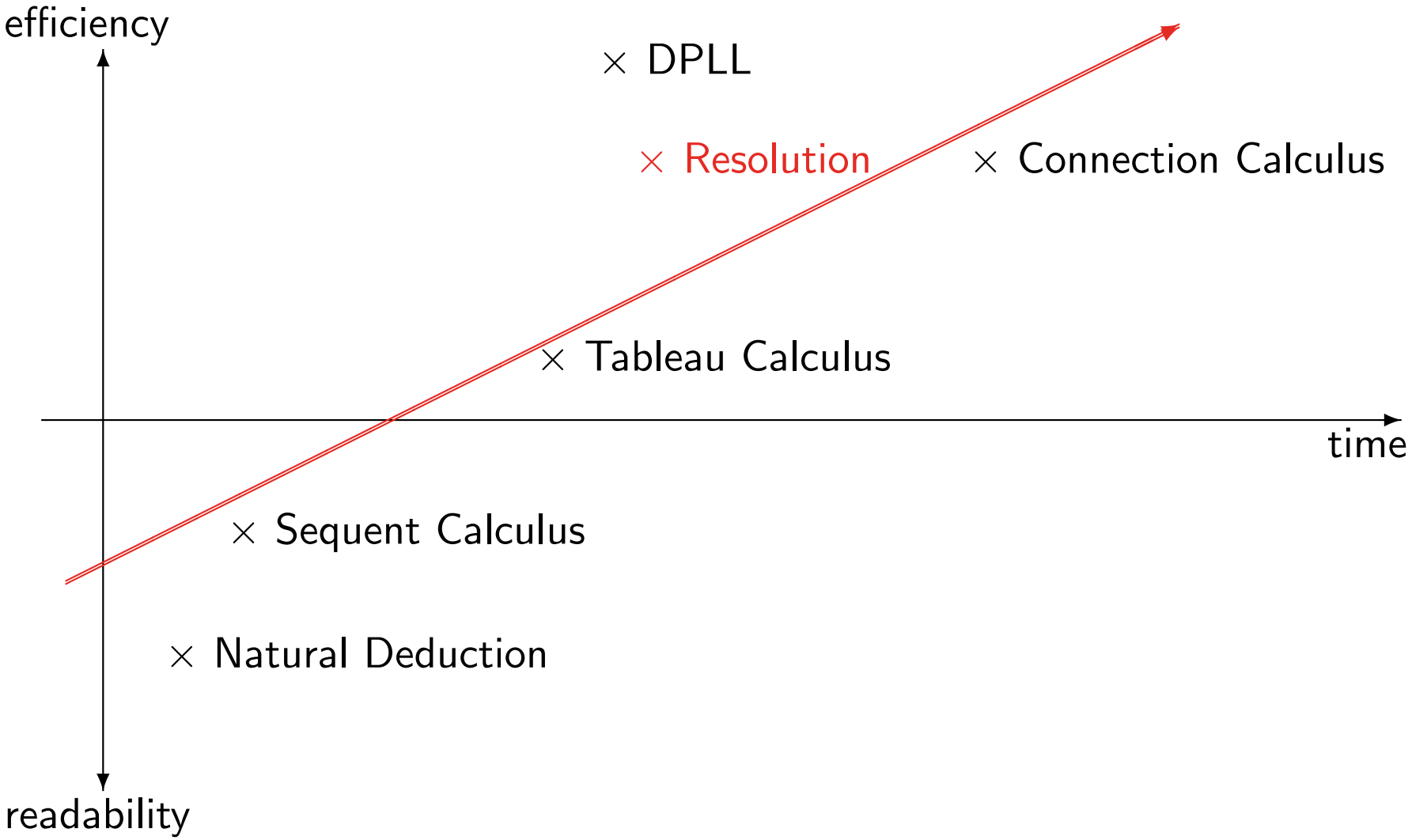
Proof Search Calculi



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- works for first-order formulae in clausal form (e.g. conjunctive or disjunctive normal form)
- consists of one (two for first-order) inference rules and one axiom
- is one of the most popular proof search calculi
- has been implemented in many automated theorem provers

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Theorem 2.1.

Every formula in first-order logic can be transformed into an equivalent formula in NNF.

Proof.

To convert an arbitrary formula to a formula in NNF, remove implications, and push negations inwards, preserving equivalence, using the following:

$$
\begin{aligned}
A \to B &\equiv \neg A \vee B \\
\neg(A \wedge B) &\equiv \neg A \vee \neg B \\
\neg(A \vee B) &\equiv \neg A \wedge \neg B \\
\neg(\forall x\, A) &\equiv \exists x\, \neg A \\
\neg(\exists x\, A) &\equiv \forall x\, \neg A \\
\neg(\neg A) &\equiv A
\end{aligned}
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To convert an arbitrary propositional formula to a formula in CNF perform the following steps, each of which preserves logical equivalence:

(1) Convert to negation normal form.

(2) Use the distributive laws to move conjunctions inside disjunctions to the outside

$$A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$$

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$(p \vee \neg q) \wedge (\neg p \vee q)$ in clausal form: $\{\{p, \neg q\}, \{\neg p, q\}\}$

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This follows from the previous theorem, where we transformed a formula to CNF. Each disjunction is then transformed to a clause (of literals), and the clausal form is the set of these clauses. □

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Let C_1, C_2 be clauses with $L \in C_1$ and $\overline{L} \in C_2$. The resolvent C' of C_1 and C_2 is $(C_1 \setminus \{L\}) \cup (C_2 \setminus \{\overline{L}\})$. C_1 and C_2 are the parents of C'.

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- the empty clause $\square$ is unsatisfiable
- goal: derive empty clause $\square$

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Definition 5.3 (Resolution Procedure).

Given a set of clauses S.

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- Resolve $\{p\}$ with $\{\neg p\}$

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- CNF: $p \wedge q \wedge \neg p$
- Clause set: $\{\{p\}, \{q\}, \{\neg p\}\}$
- Resolve $\{p\}$ with $\{\neg p\}$
- Resolvent: $\square$

Resolution Calculus – Example 2

- Prove validity of: $p \wedge (p \rightarrow q) \rightarrow q$

Resolution Calculus – Example 2

- Prove validity of: $p \wedge (p \rightarrow q) \rightarrow q$
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- Prove validity of: $p \wedge (p \rightarrow q) \rightarrow q$
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- Resolvent: $\{q\}$
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- Need to use resolution rule on new resolvents

Resolution Calculus – Example 3

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- Clauses:

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2. $\{p\}$

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- Order of resolution steps does not matter for completeness

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The resolution calculus has one axiom and one (inference) rule.

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- hence, a derivation in the resolution calculus has only one branch
- terminates, if all clauses $C_i \cup \{L\}, C_j \cup \{\overline{L}\}$ have been considered

Outline

- Introduction
- Repetition: Negation Normal Form
- Conjunctive Normal Form
- Clausal Form
- Resolution
- Soundness of Resolution
- Completeness of Resolution

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2. A set of clauses containing the empty clause is unsatisfiable

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$\mathcal{I} \models L$ $\mathcal{I} \models C_2$, and clauses are *disjunctions* of their literals, so $\mathcal{I}$ satisfies one of the literals in C_2, but not $\overline{L}$. So: $\mathcal{I} \models C_2 \setminus \{\overline{L}\}$.

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- $\mathcal{I} \models L$ $\mathcal{I} \models C_2$, and clauses are *disjunctions* of their literals, so $\mathcal{I}$ satisfies one of the literals in C_2, but not $\overline{L}$. So: $\mathcal{I} \models C_2 \setminus \{\overline{L}\}$.
- $\mathcal{I} \models \overline{L}$ By the same reasoning $\mathcal{I} \models C_1 \setminus \{L\}$.

So $\mathcal{I}$ satisfies at least one literal in either $C_1 \setminus \{L\}$ or $C_2 \setminus \{\overline{L}\}$.

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- $\mathcal{I} \models L$ $\mathcal{I} \models C_2$, and clauses are *disjunctions* of their literals, so $\mathcal{I}$ satisfies one of the literals in C_2, but not $\overline{L}$. So: $\mathcal{I} \models C_2 \setminus \{\overline{L}\}$.
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So $\mathcal{I}$ satisfies at least one literal in either $C_1 \setminus \{L\}$ or $C_2 \setminus \{\overline{L}\}$.
I.e. $\mathcal{I} \models (C_1 \setminus \{L\}) \cup (C_2 \setminus \{\overline{L}\})$

Resolution Preserves Satisfiability

Lemma 6.1.

If a set of clauses S is satisfiable, then the result of adding the resolvent of two clauses $C_1, C_2 \in S$ to S is also satisfiable.

Proof.

Let S be a set of clauses, and $C_1, C_2 \in S$ with $L \in C_1$ and $\overline{L} \in C_2$. Let $\mathcal{I}$ be an interpretation with $\mathcal{I} \models S$.
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But the empty clause $\mathcal{I}$ contains no literals, so that is a contradiction.

$\square$

Outline

- Introduction
- Repetition: Negation Normal Form
- Conjunctive Normal Form
- Clausal Form
- Resolution
- Soundness of Resolution
- Completeness of Resolution

Prove Completeness like for LK?

- Plan:

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- We will go through Robinson's original proof

Semantic Trees

The completeness proof uses the following concept:

Definition 7.1 (Semantic Trees).

A semantic tree is a binary tree where:

- *The root is labelled by the symbol $\bot$,*
- *Every node has either no children or two children,*
- *For every node that has children, there is some atom A such that one child is labeled with A and the other with $\neg A$*
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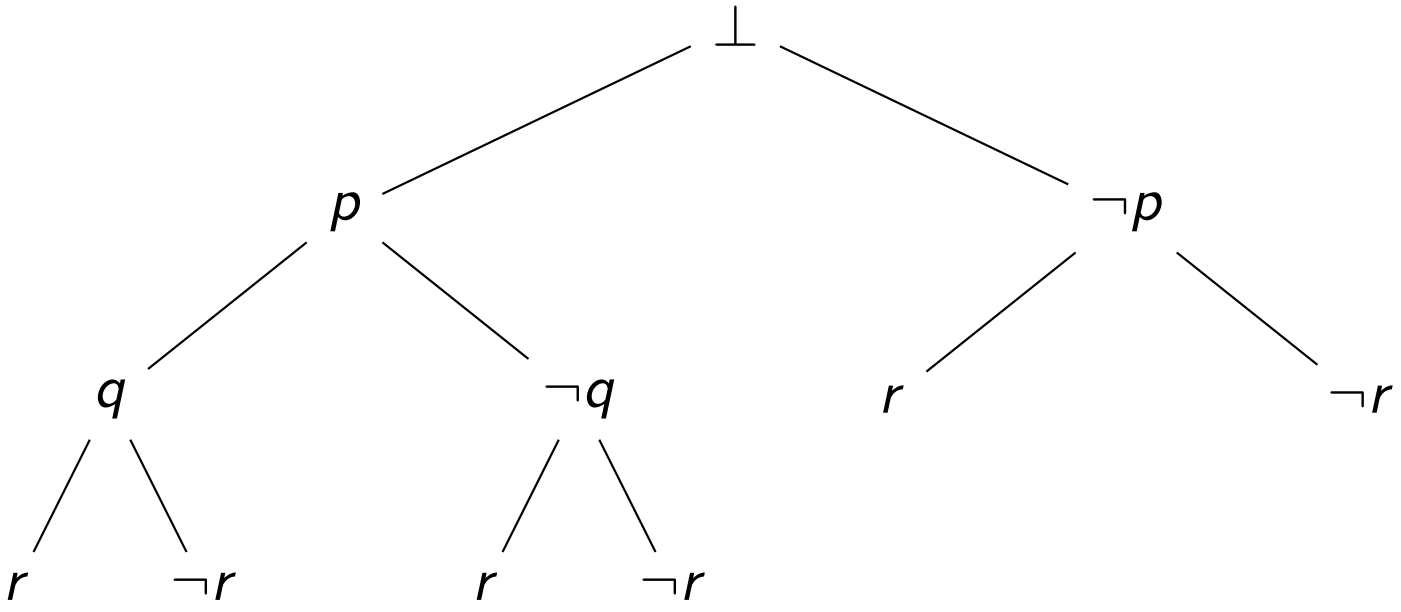
Not a data structure, just needed for the completeness proof

Semantic Trees — Example



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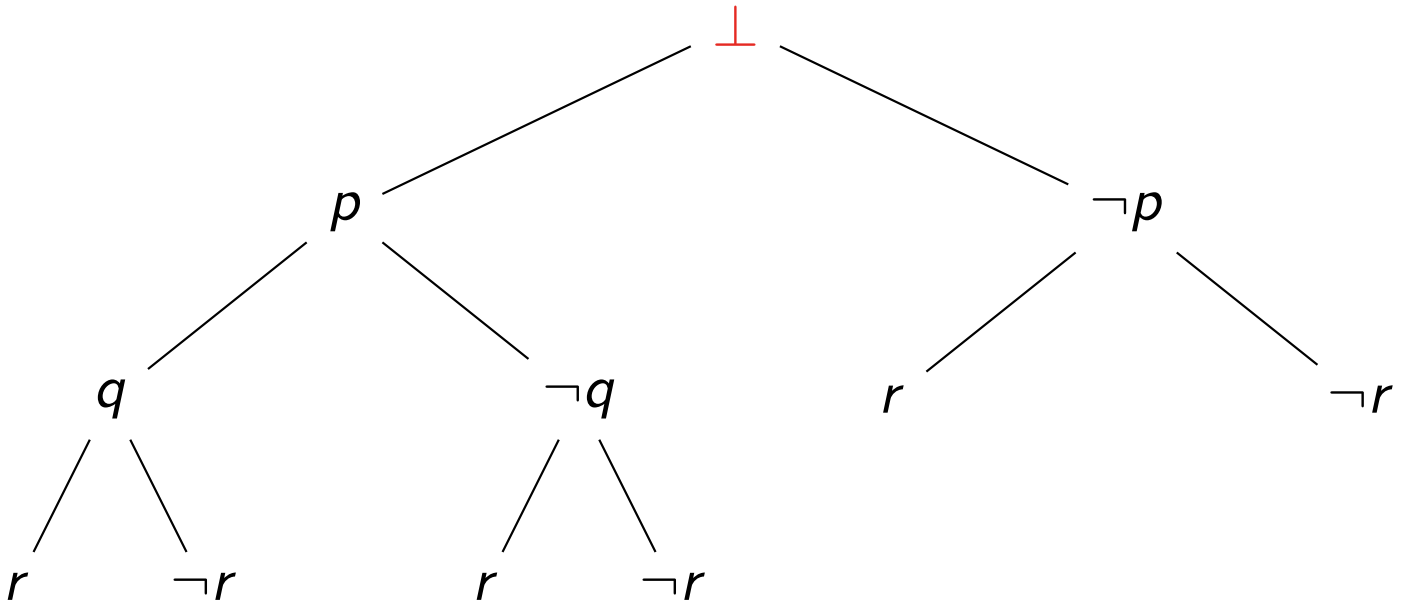

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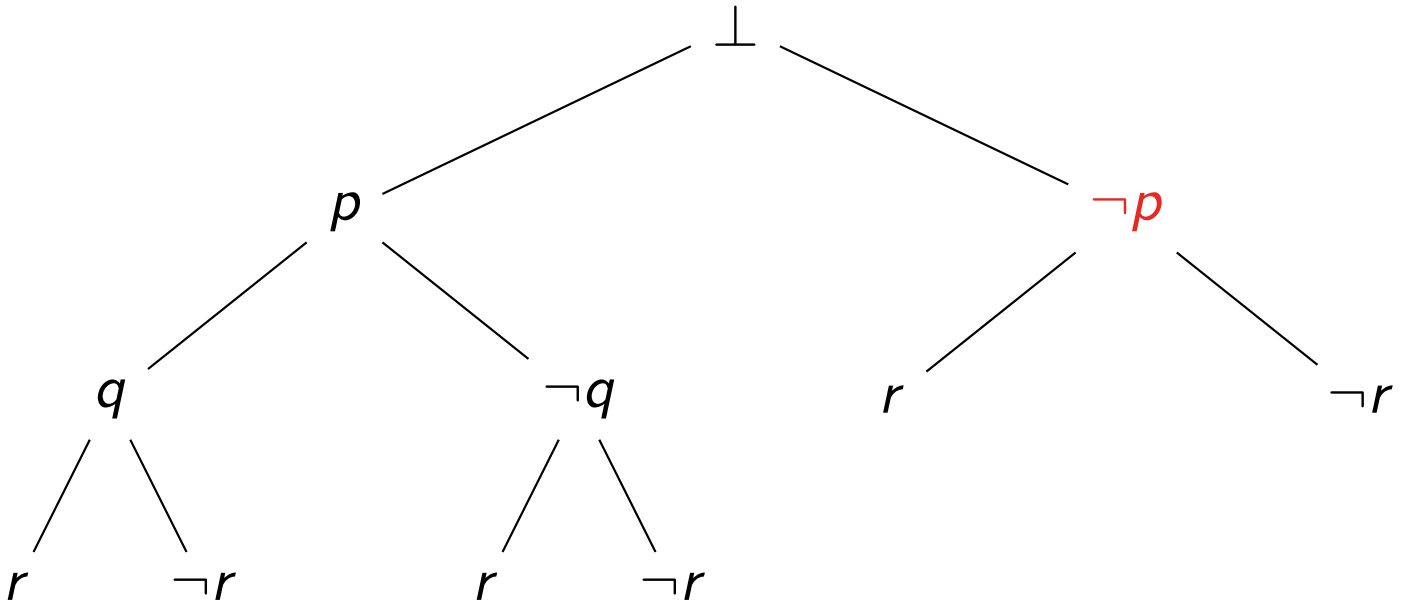

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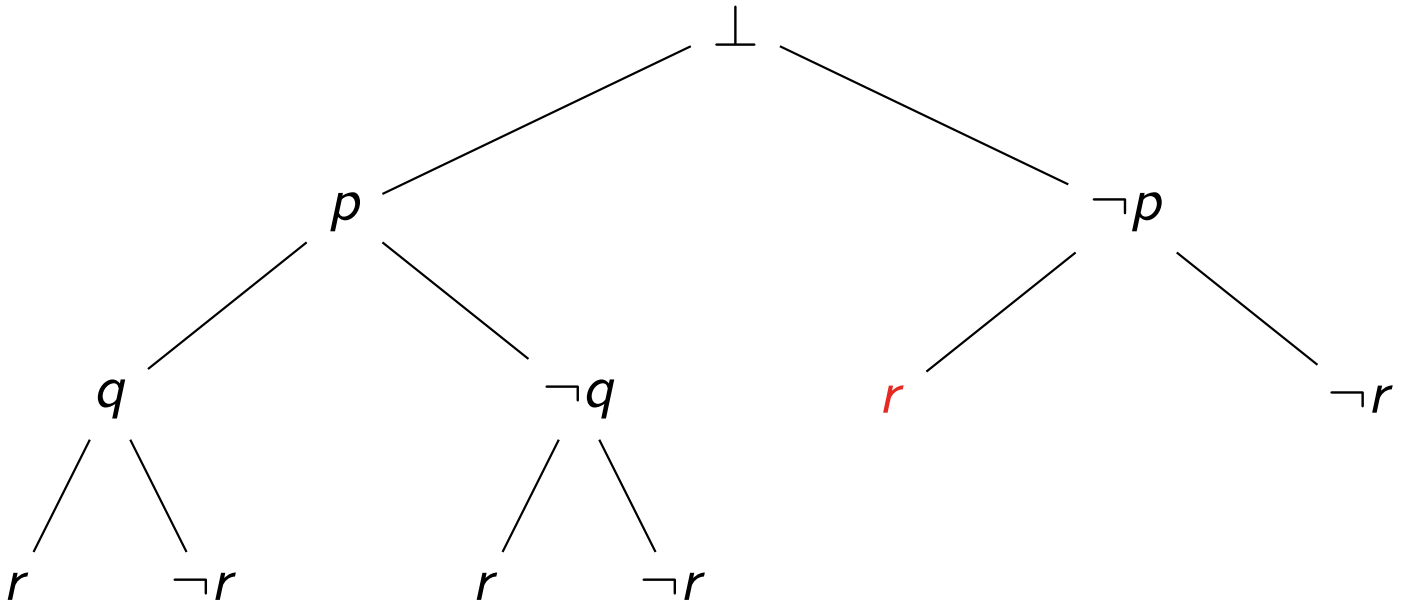

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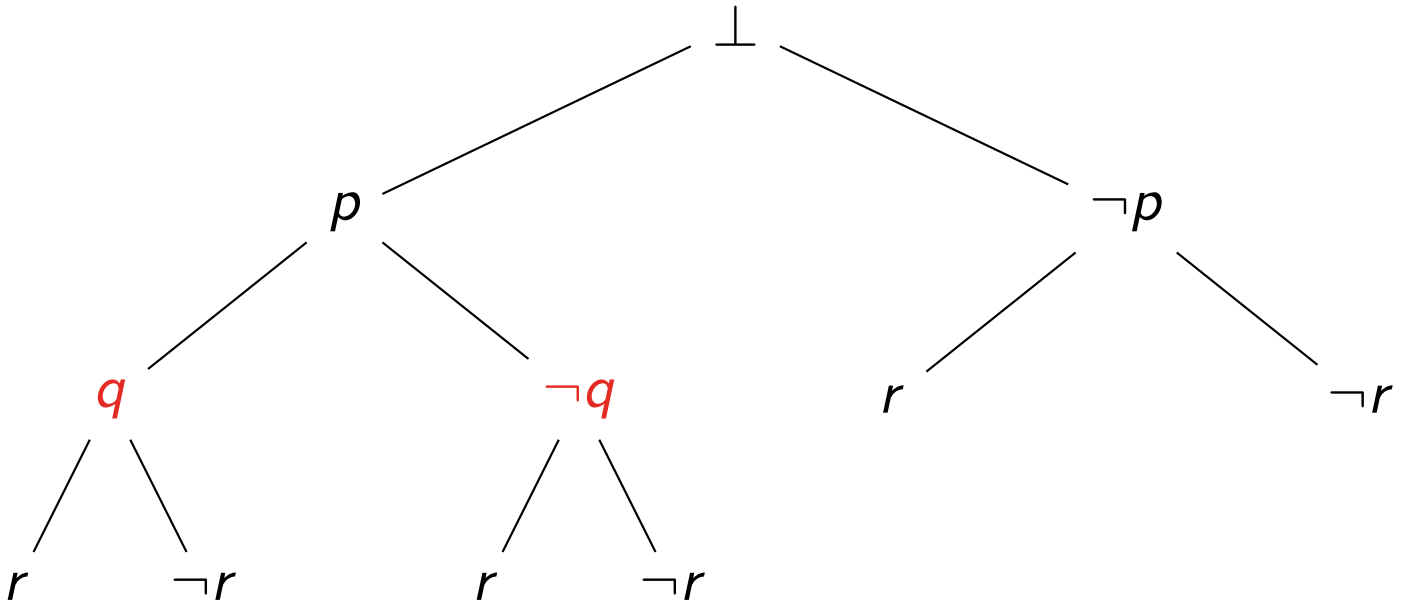
Semantic Trees — Example



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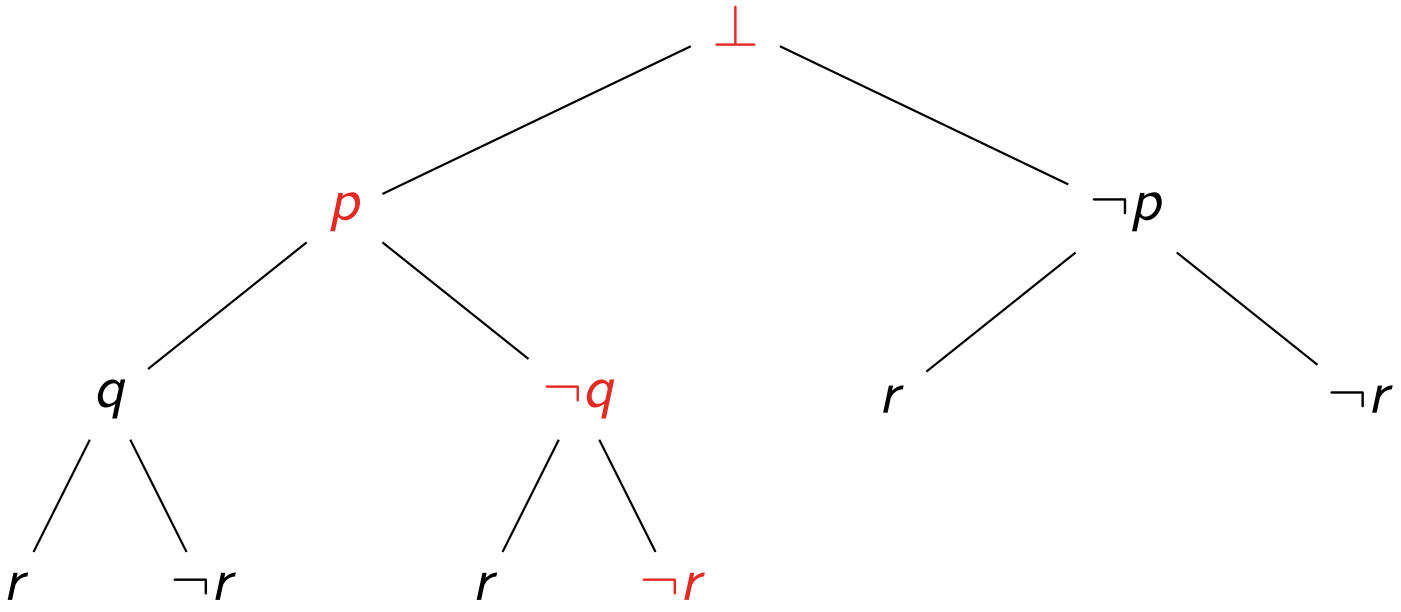

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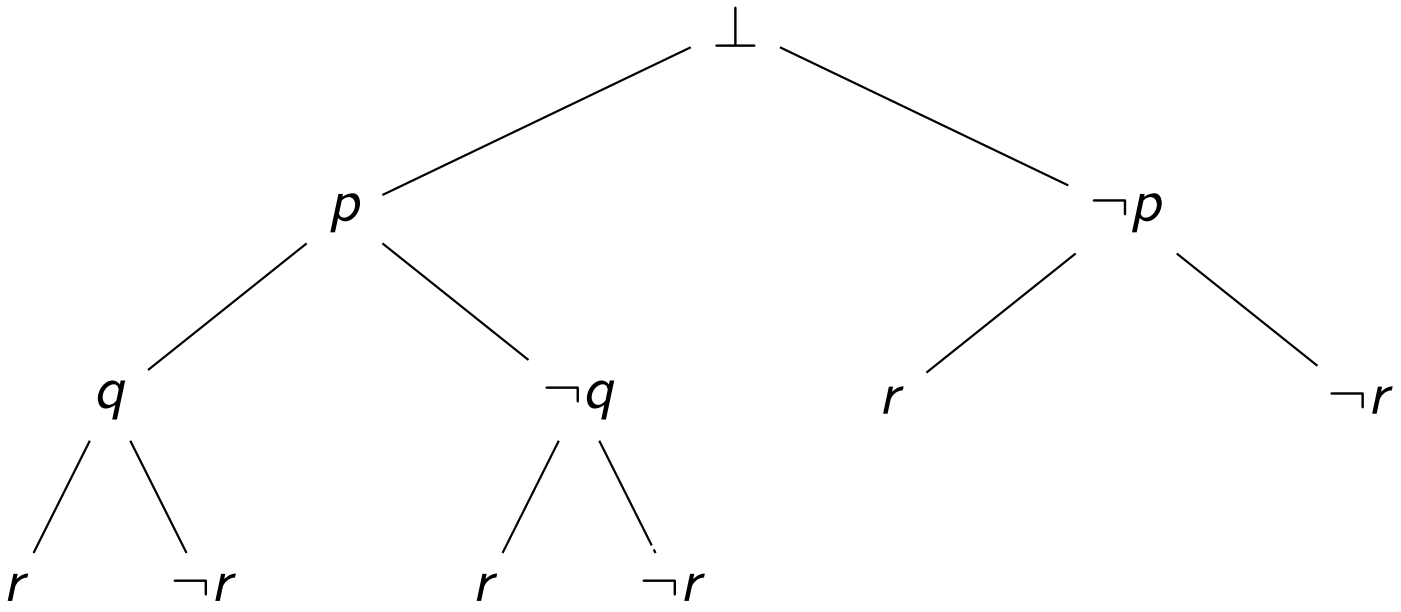
Partial Interpretations

The path to every node n in a semantic tree gives a 'partial interpretation' $\mathcal{I}_n$:



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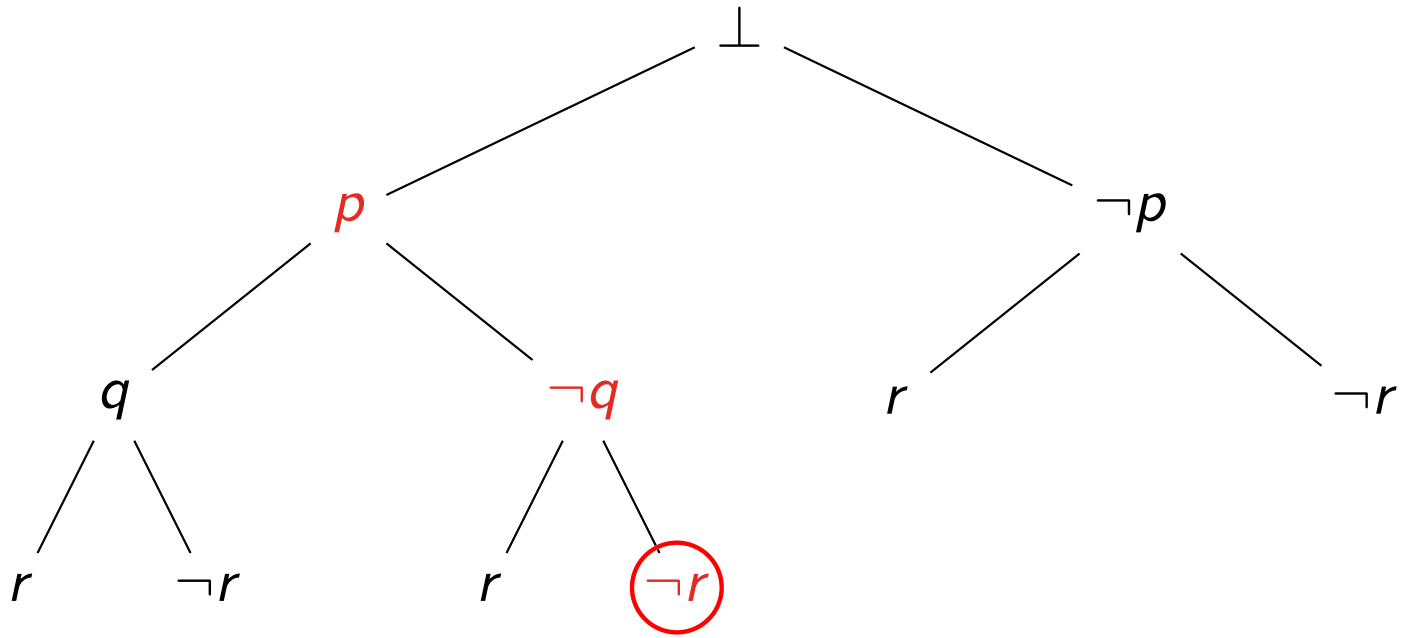

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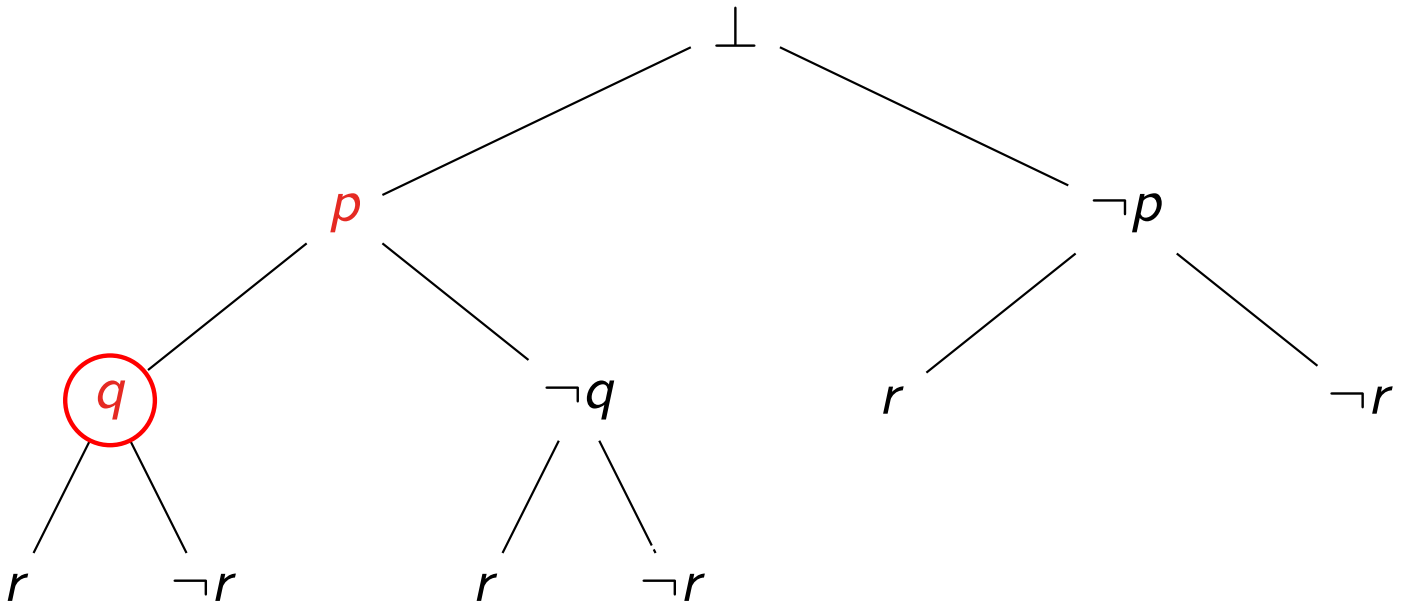

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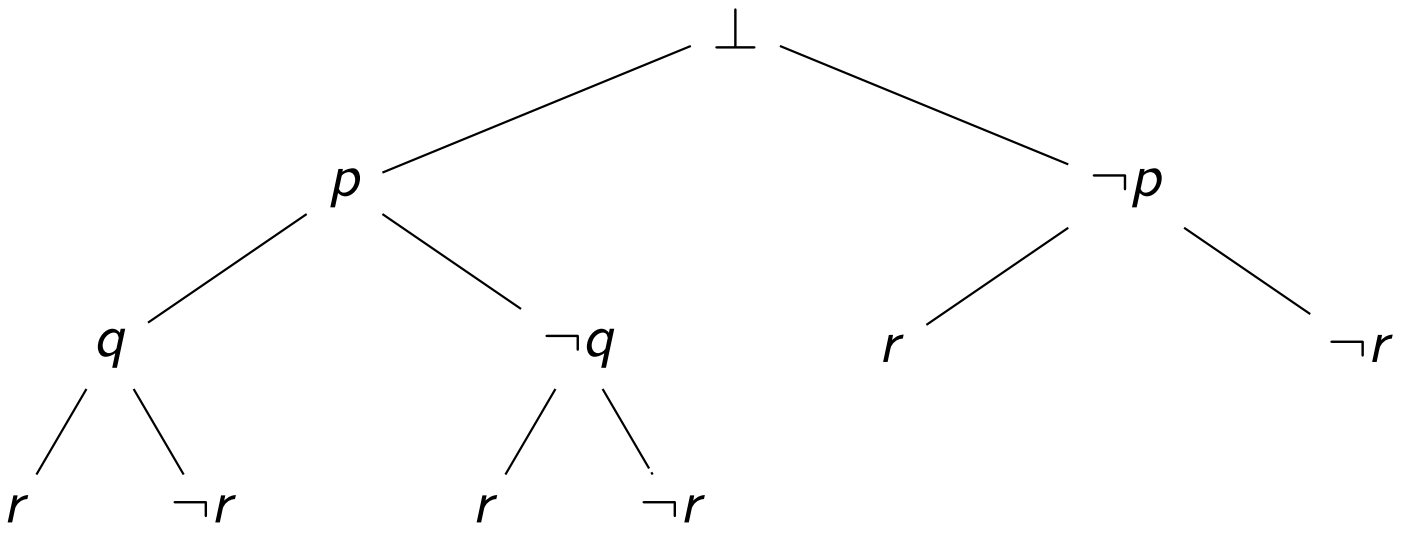
Failure Nodes – Motivation

Sometimes, such a 'partial interpretation' is enough to falsify a clause:



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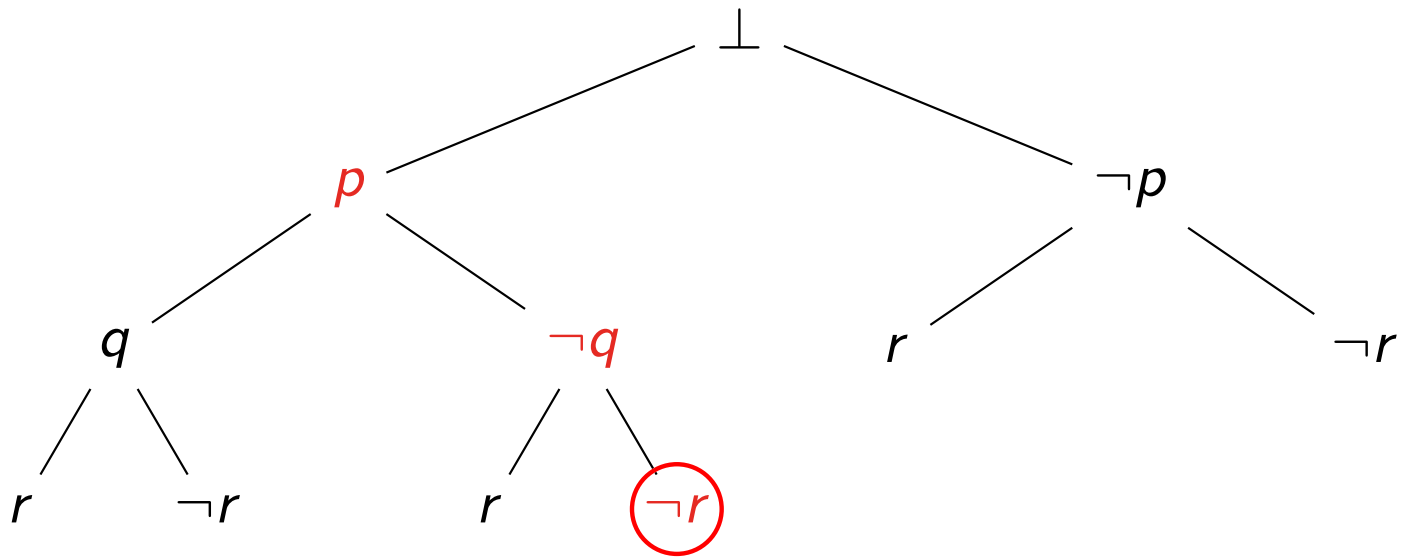

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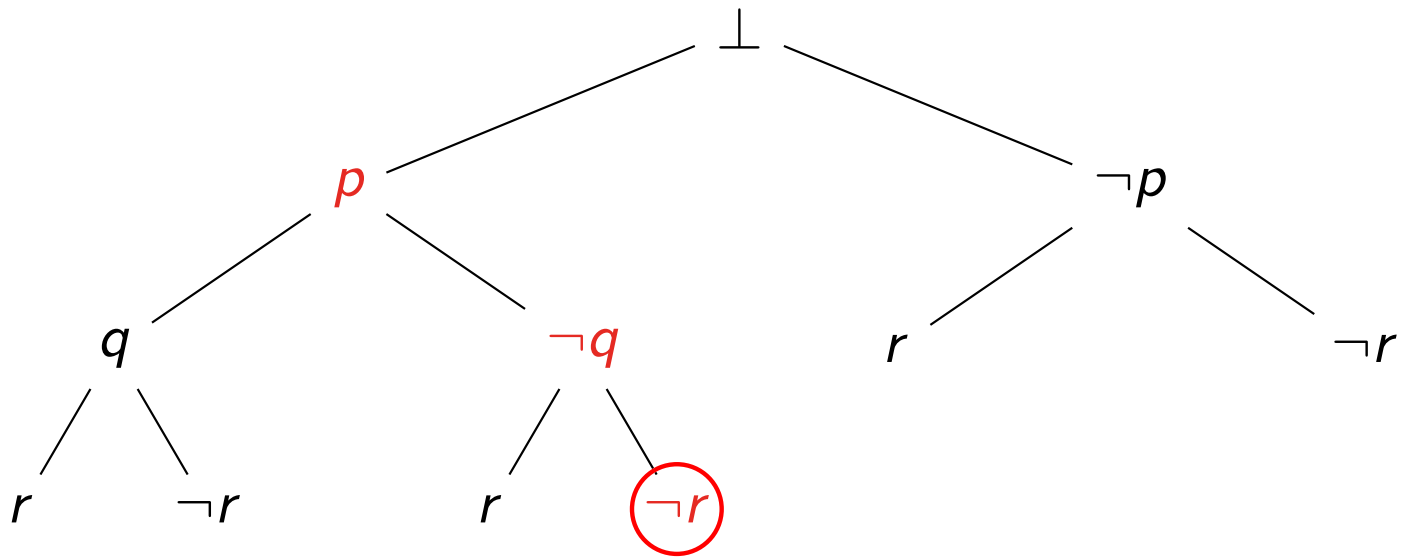

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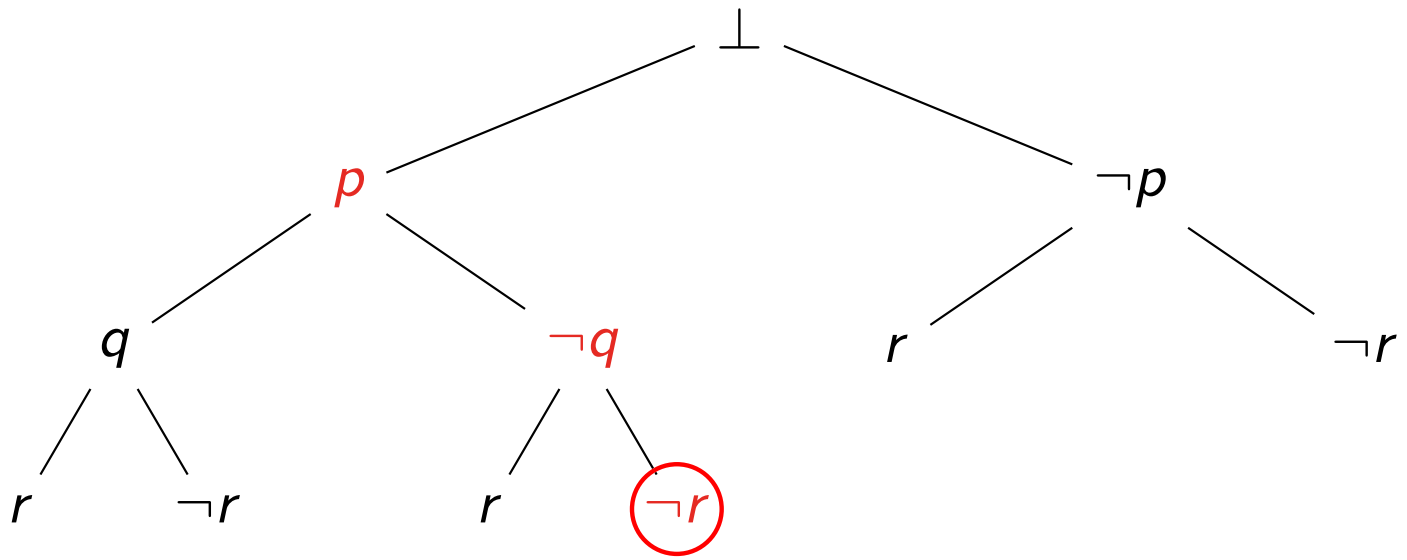

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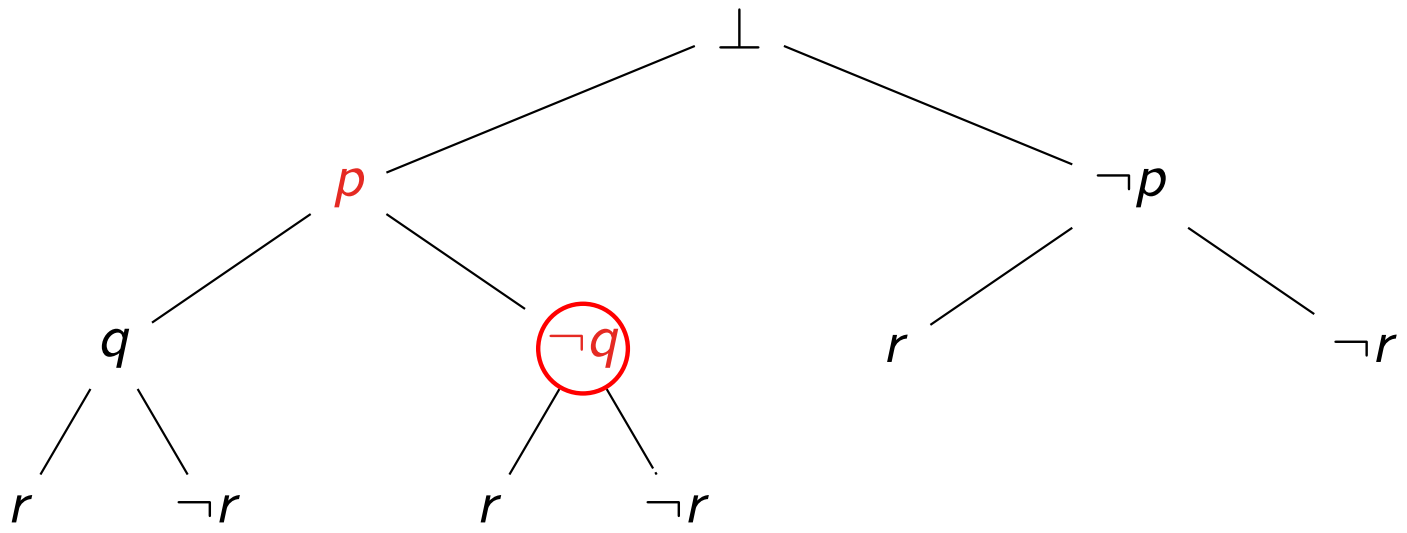

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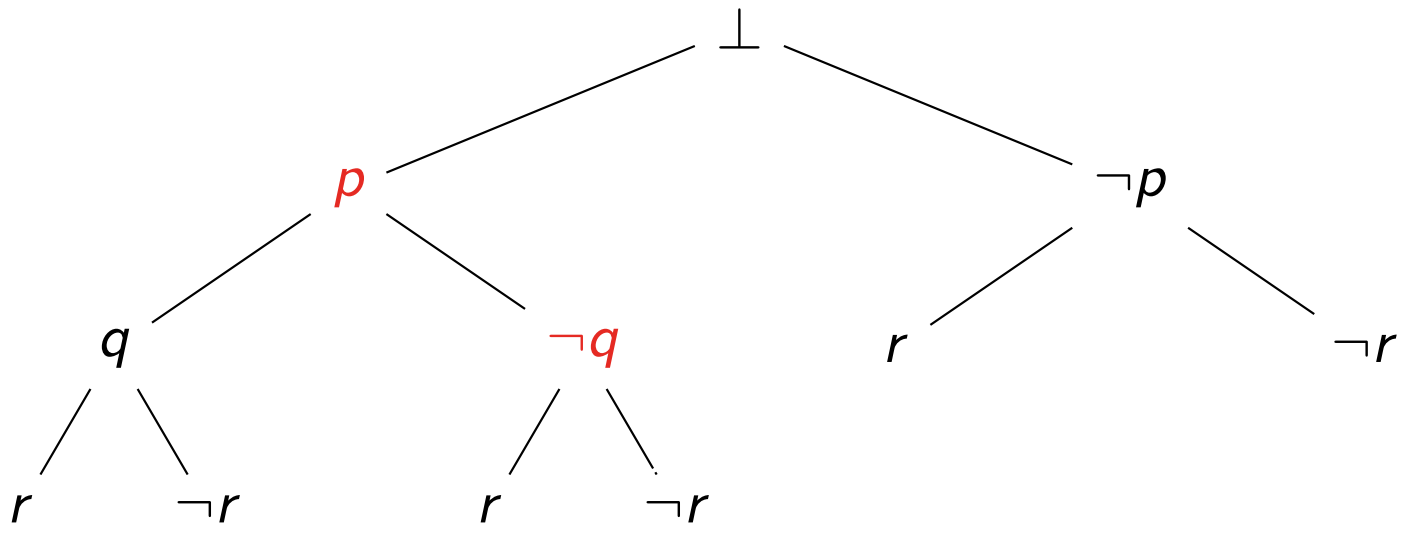

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- It remains false further down.

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Note: A has the root as a failure node iff $\square \in S$.

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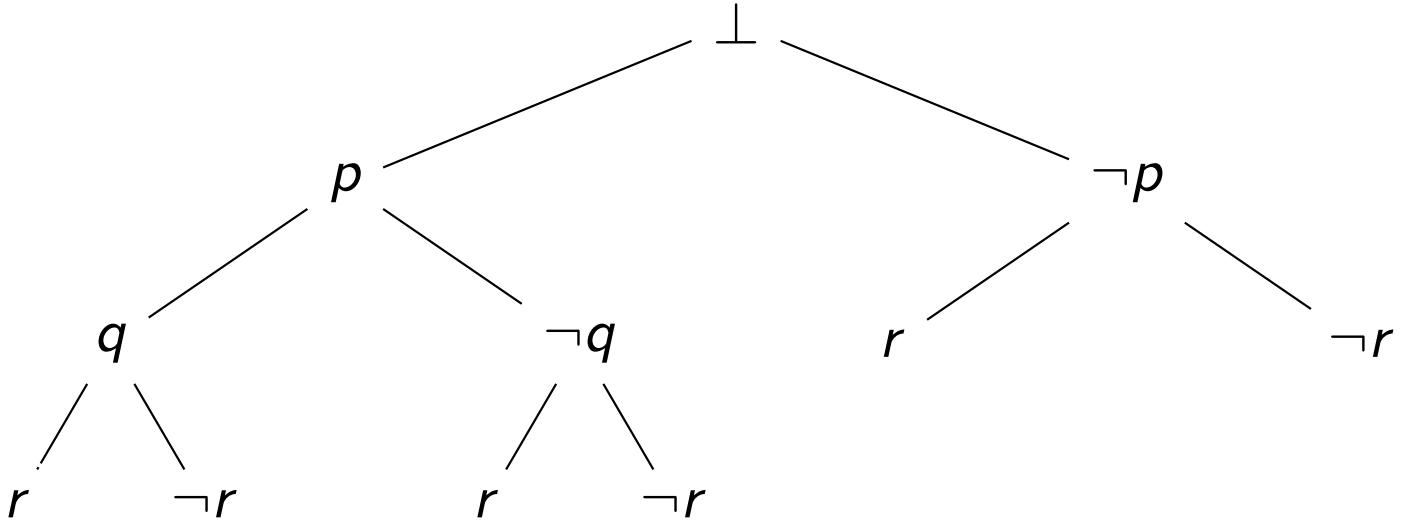

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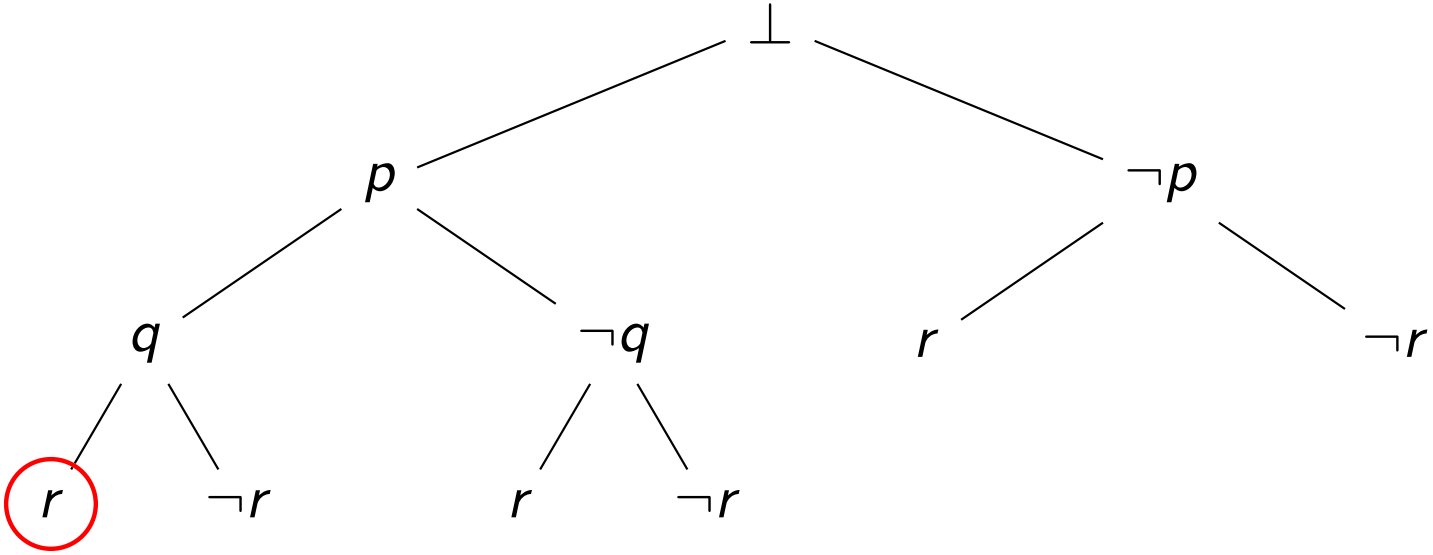

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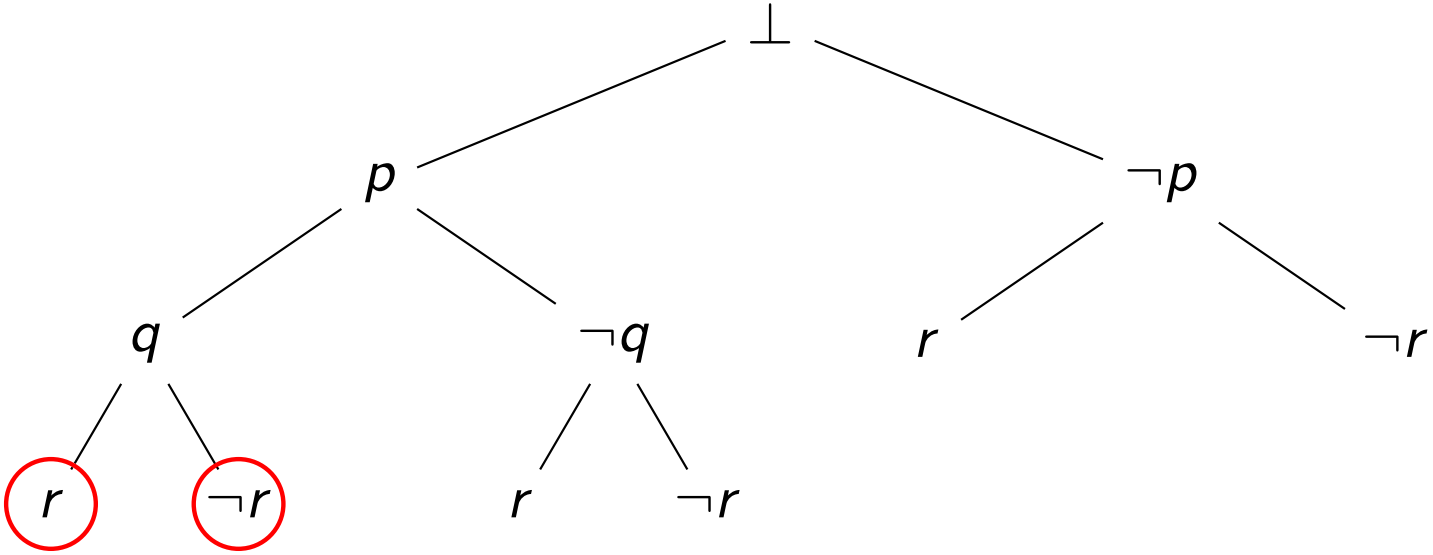

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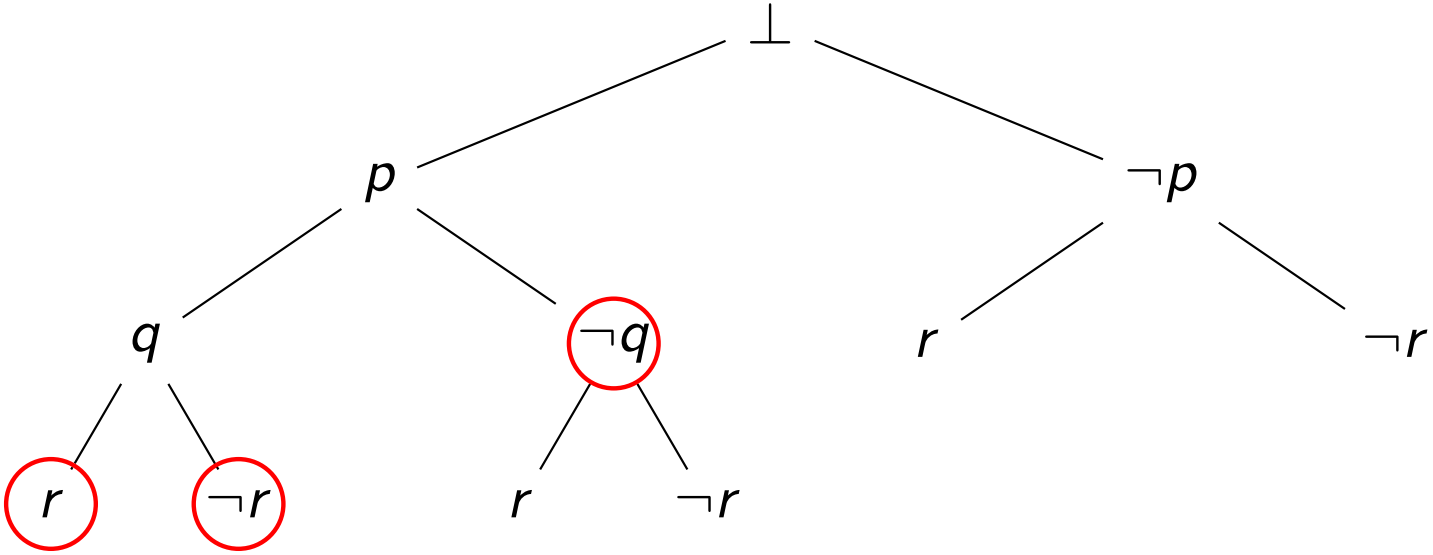

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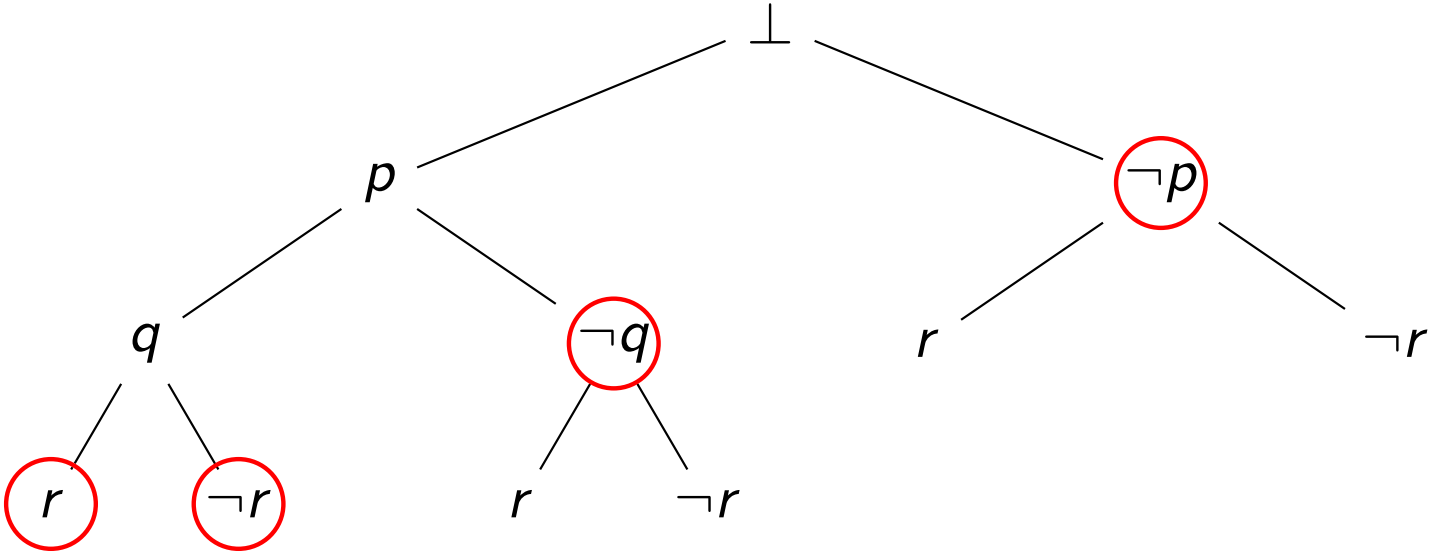

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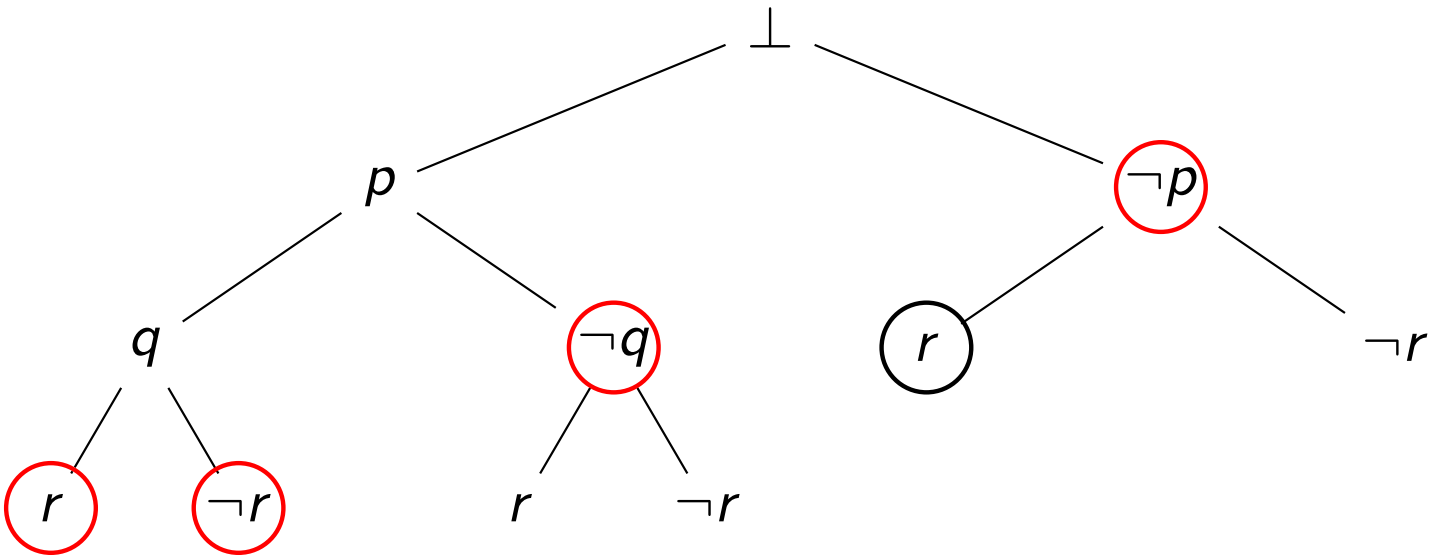
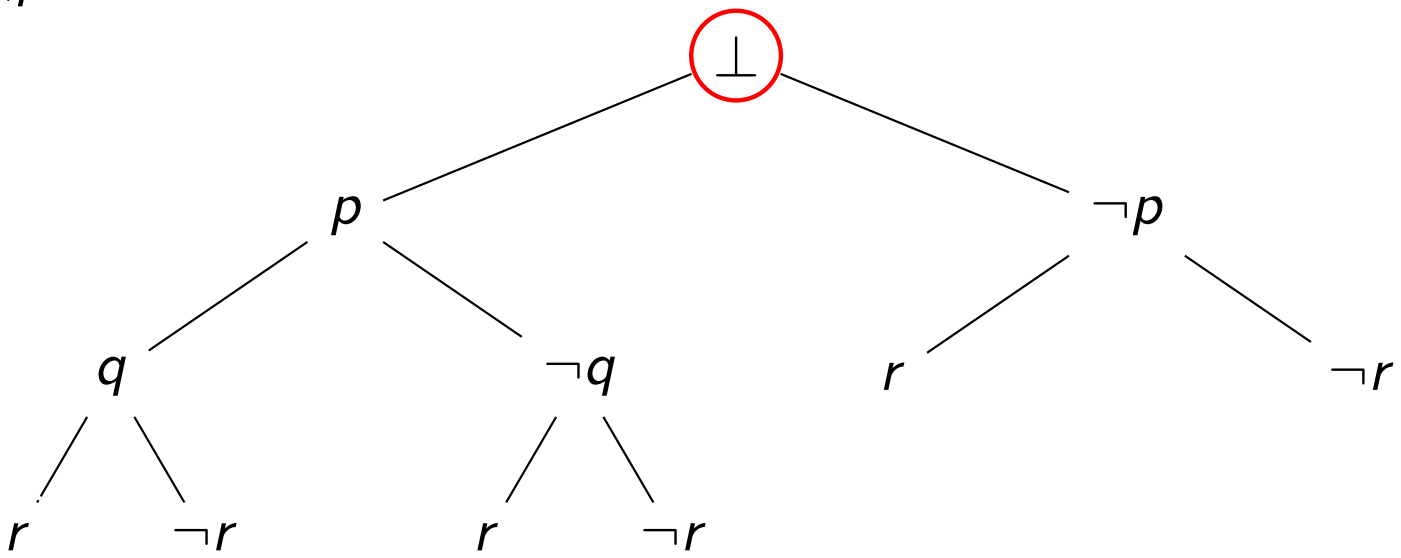


Not a failure node: parent node falsifies clause 4.

Failure Nodes – Example

1. $\neg p \lor \neg q \lor \neg r$
2. $\neg q \lor r$
3. $\neg p \lor q$
4. p
5. $p \lor \neg r$
6. $\square$



The empty clause is falsified by the root node

Closed Semantic Trees

Definition 7.4.

Given a semantic tree and a clause set S, a branch of the tree is closed if it contains a failure node.

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The semantic tree is closed if all branches contain failure nodes.

Closed Semantic Tree – Example

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2. $\neg q \vee r$
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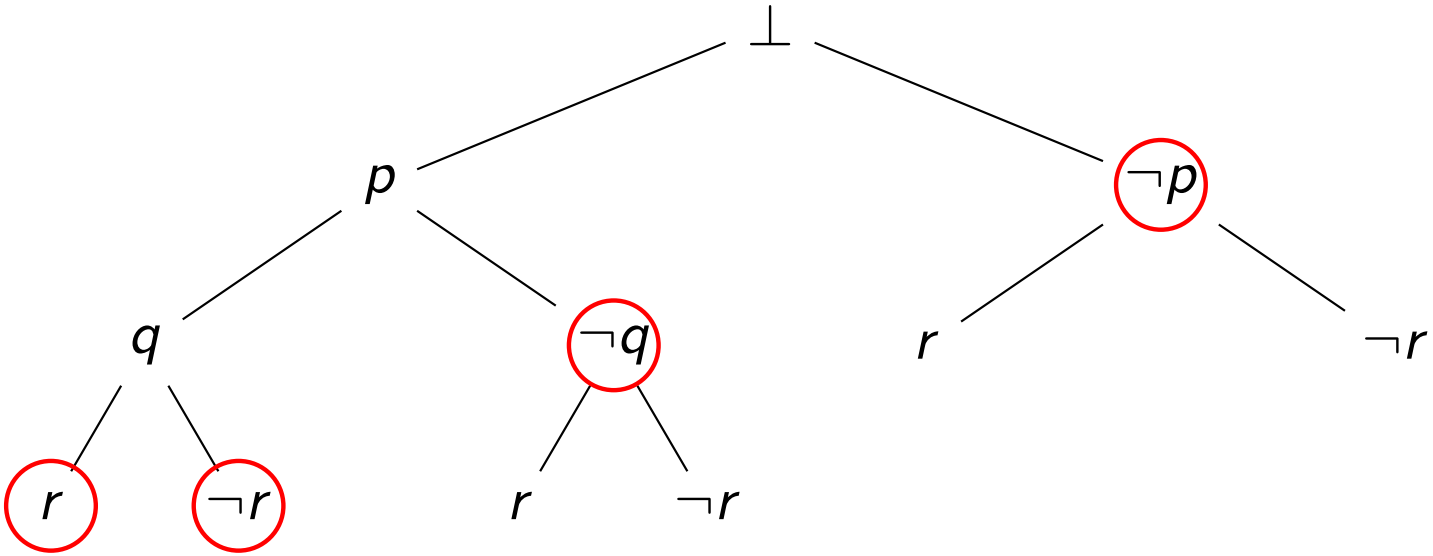
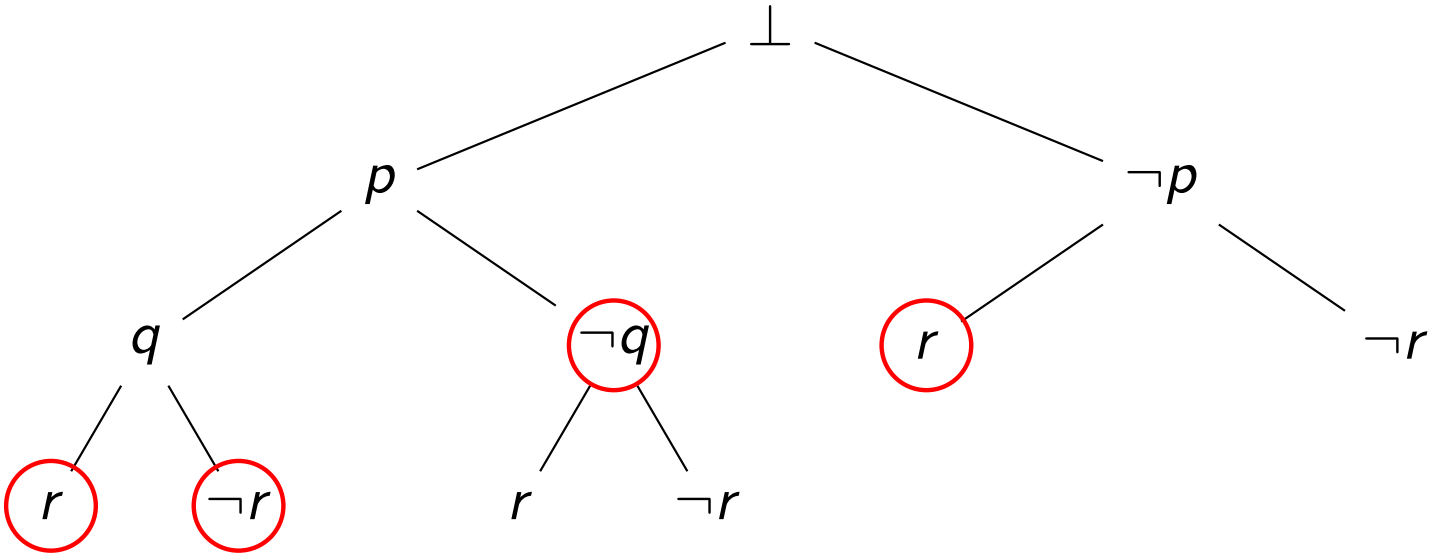


The semantic tree is closed for these 5 clauses.

Closed Semantic Tree – Example

1. $\neg p \vee \neg q \vee \neg r$
2. $\neg q \vee r$
3. $\neg p \vee q$

5. $p \vee \neg r$



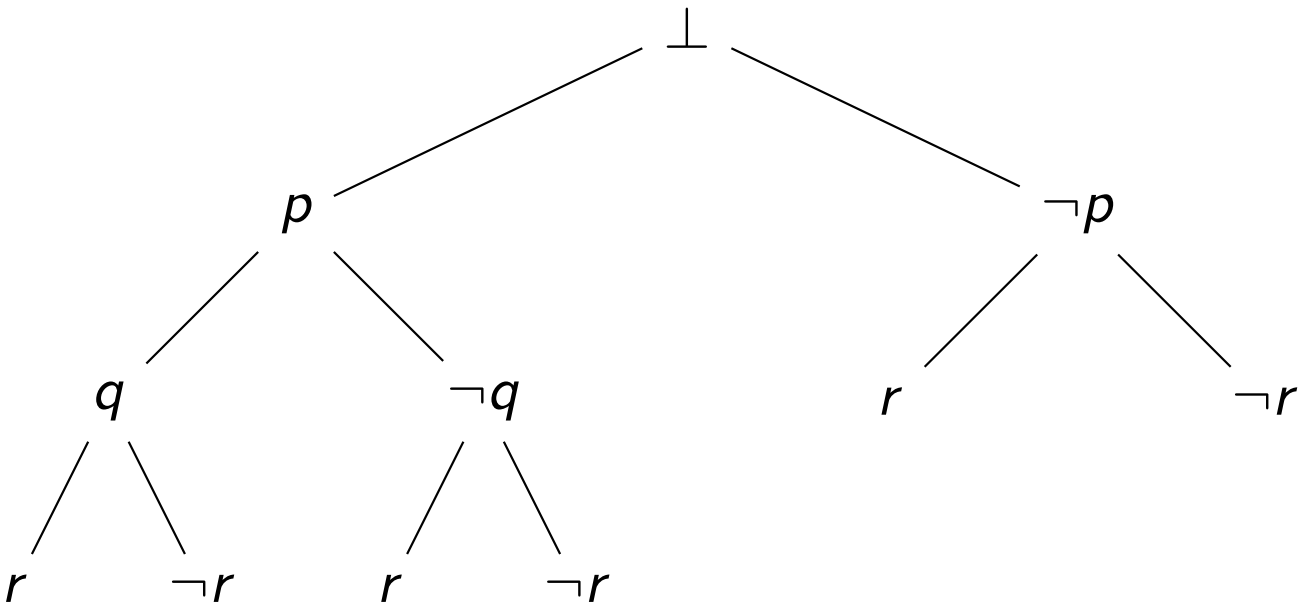
Without p, it is not closed.

Complete Semantic Trees

Definition 7.5.

A semantic tree is complete if for every atomic formula A and every branch (from root to leaf) either A or $\neg A$ occurs

Example: Complete Semantic Tree



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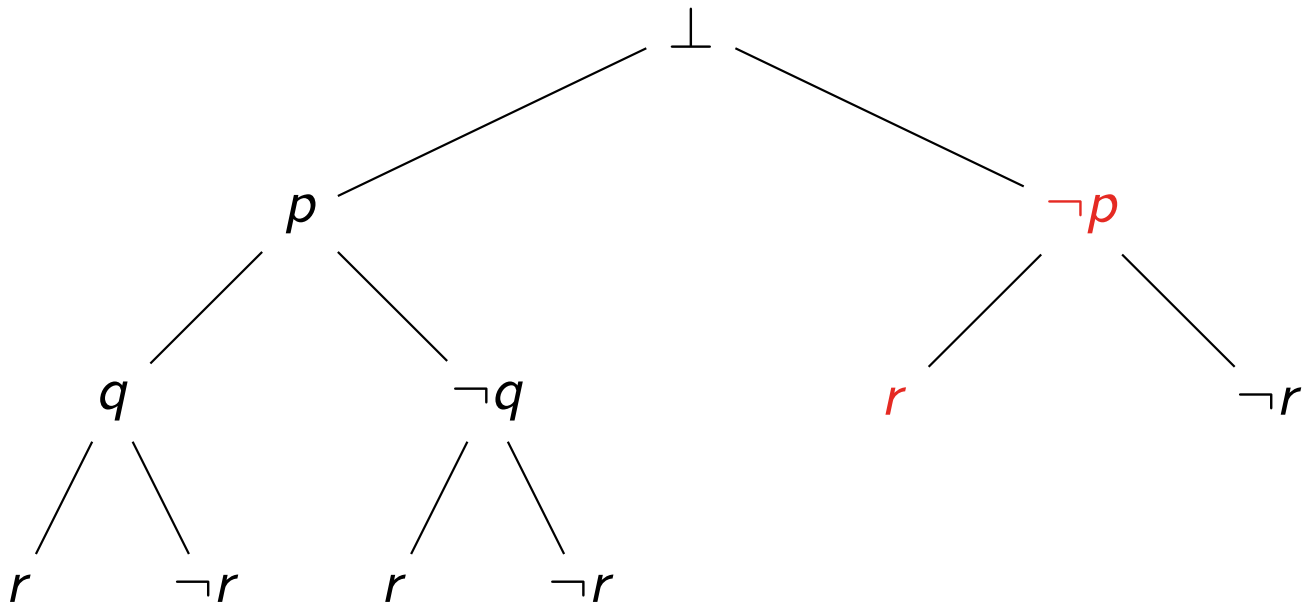
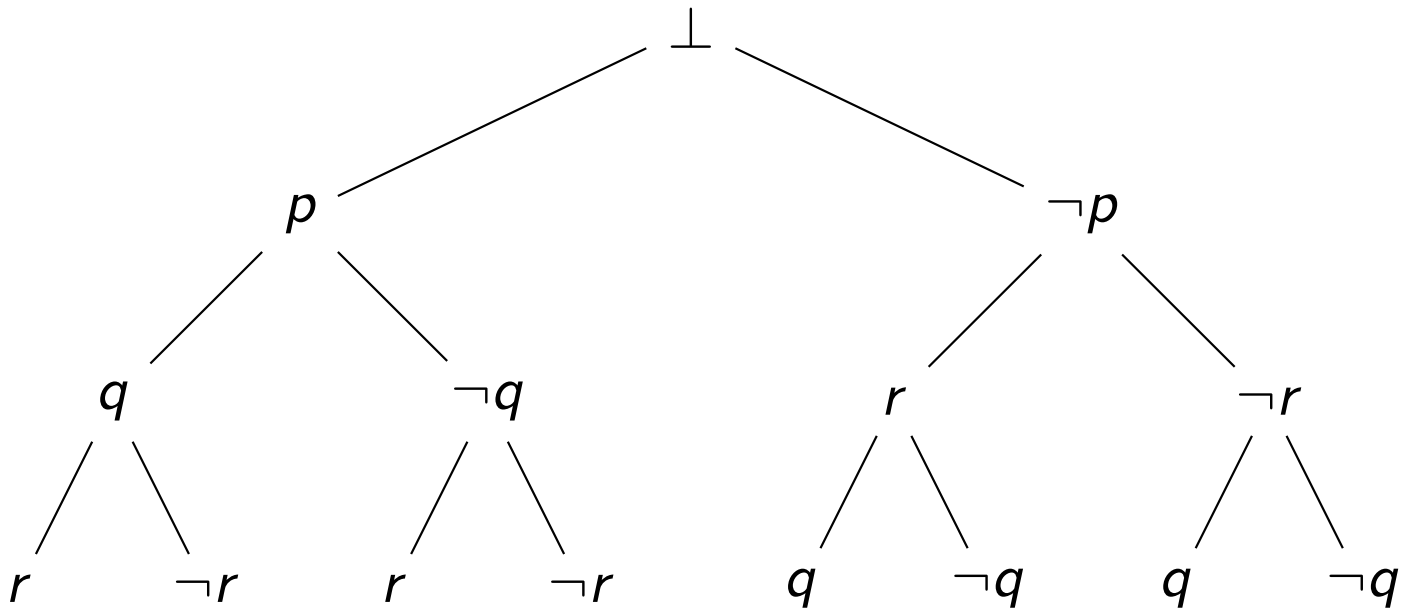


Not complete, since neither q nor $\neg q$ on branch

Example: Complete Semantic Tree



Complete for vocabulary $\{p, q, r\}$

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A complete semantic tree 'enumerates' all possible interpretations.

Unsatisfiable Clause Sets close Semantic Trees

Theorem 7.1.

A clause set is unsatisfiable iff there is a closed semantic tree for it.

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This holds for arbitrary interpretations $\mathcal{I}$, so S is unsatisfiable. $\square$

Resolution Steps from Closed Semantic Trees

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Resolution Steps from Closed Semantic Trees

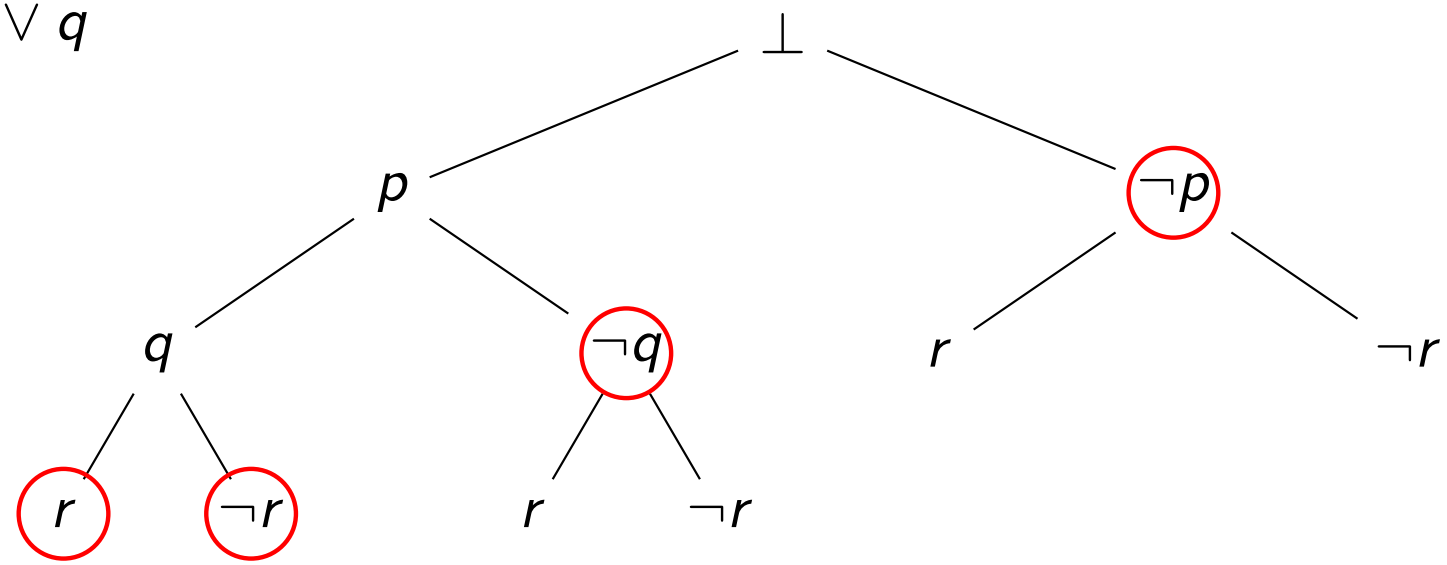
Lemma 7.2.

Let S be an unsatisfiable clause set, with a closed semantic tree, and $\square \notin S$. Then

- *a resolution step is possible from S,*
- *and the resulting clause set S' has a smaller closed semantic tree*

Idea of proof

1. $\neg p \vee \neg q \vee \neg r$
2. $\neg q \vee r$
3. $\neg p \vee q$
4. p



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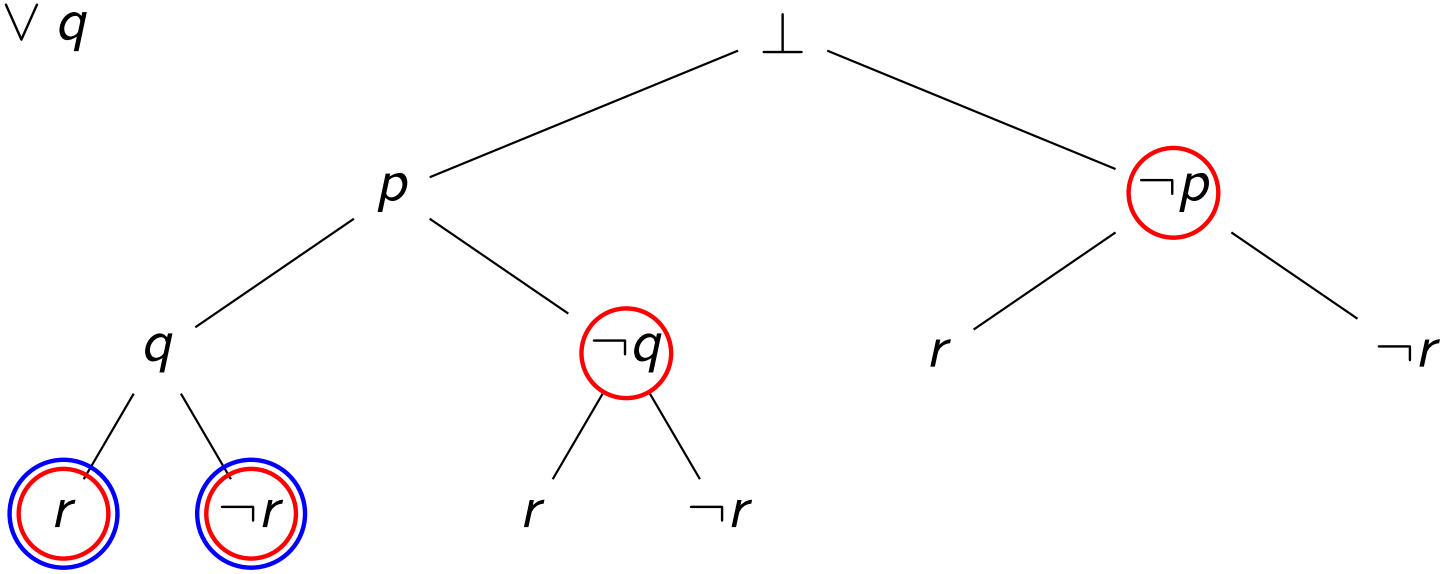

- There are two sibling failure nodes

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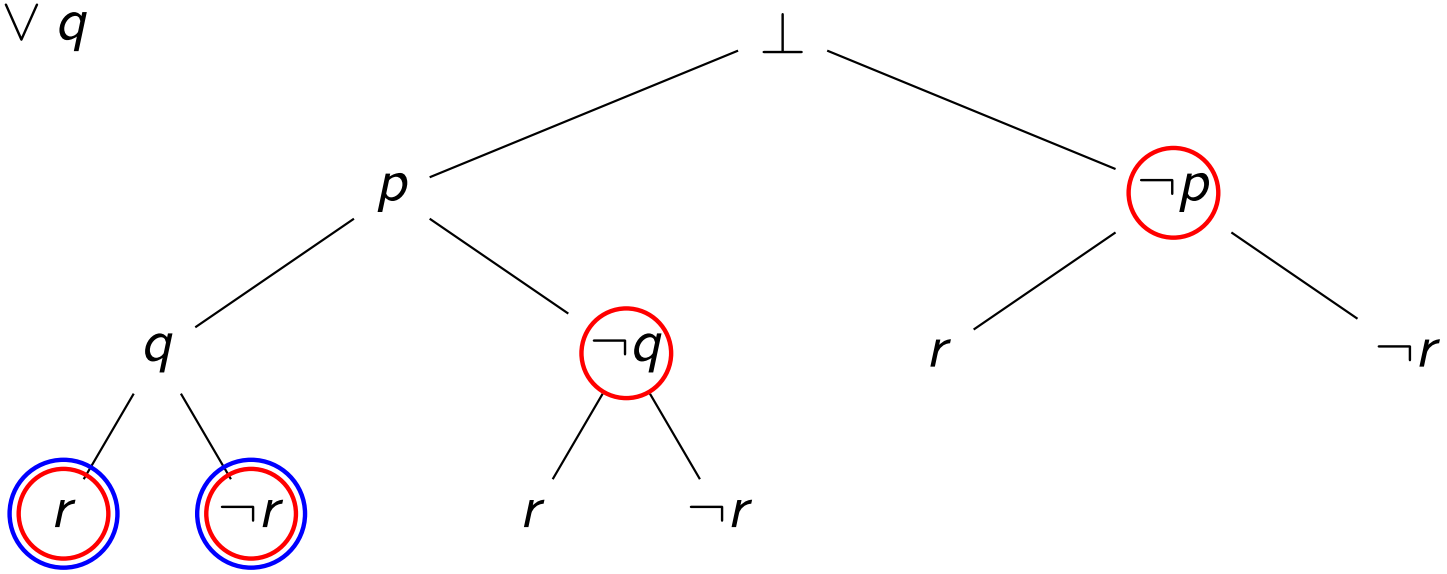
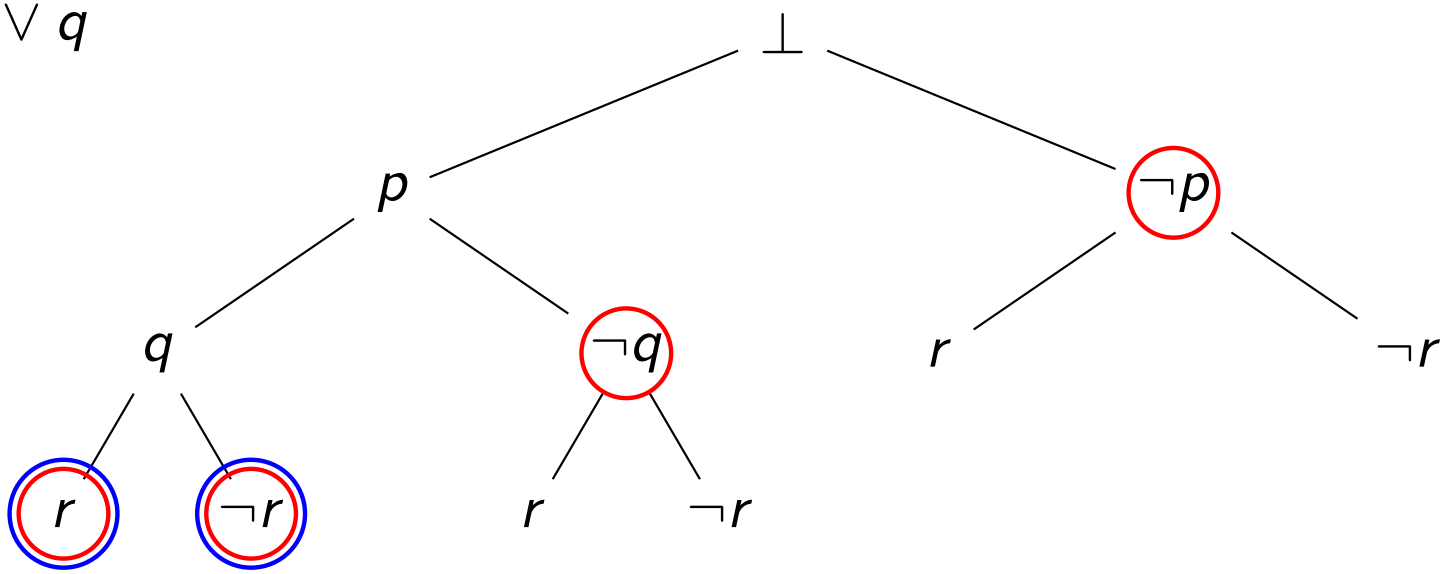


- There are two sibling failure nodes
- They falsify two clauses

Idea of proof

1. $\neg p \lor \neg q \lor \neg r$
2. $\neg q \lor r$
3. $\neg p \lor q$
4. p



- There are two sibling failure nodes
- They falsify two clauses with complementary literals

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4. p
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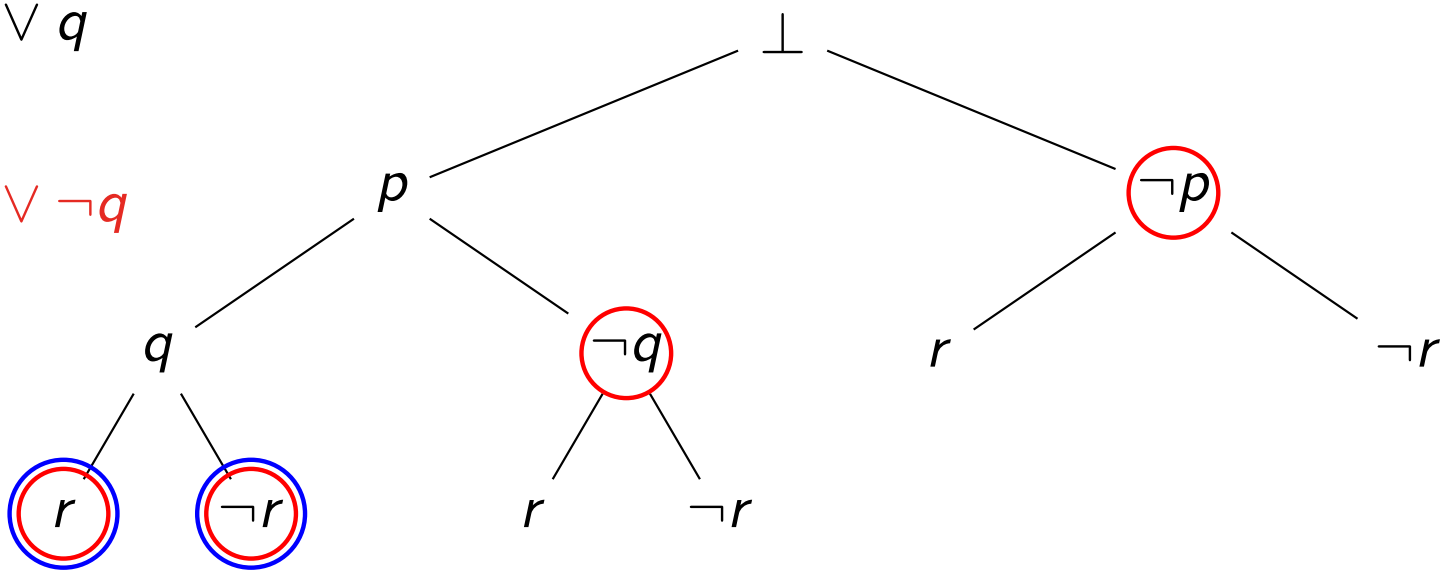

- There are two sibling failure nodes
- They falsify two clauses with complementary literals
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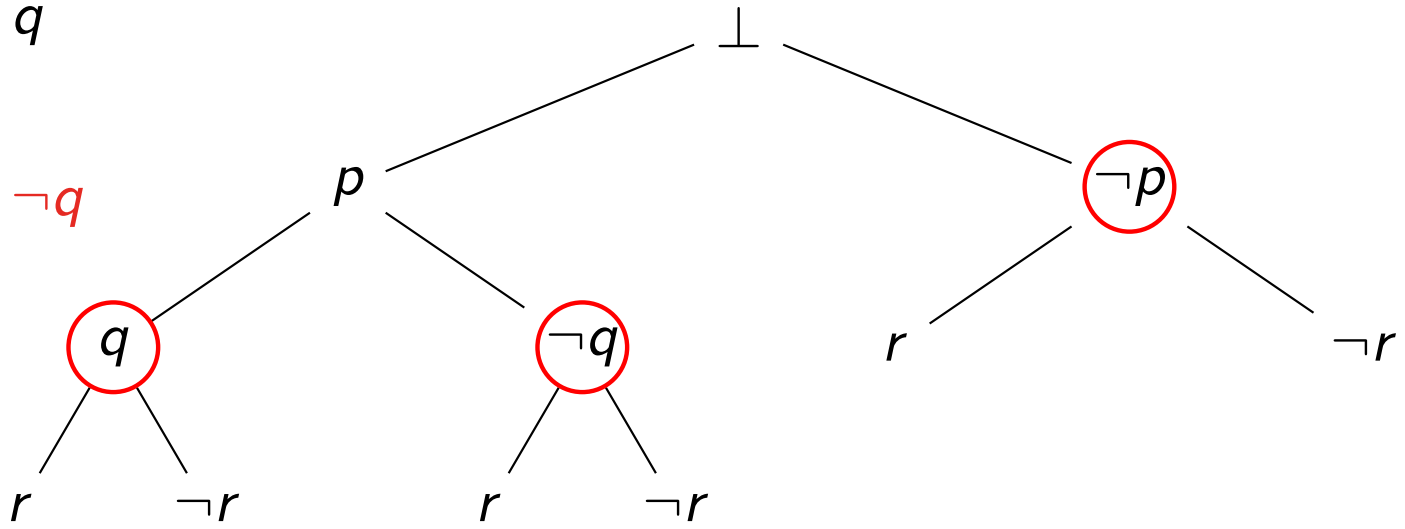
4. p
5. $\neg p \lor \neg q$



- There are two sibling failure nodes
- They falsify two clauses with complementary literals
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- Which is falsified by the parent node

There are two sibling failure nodes

- Let n_0 be the root.

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- ...
- This either finds sibling failure nodes...
- or it constructs a path in the tree without a failure node, but that is not possible.

Sibling Failure Nodes give Resolution Opportunities

- Let n_1 and n_2 be sibling failure nodes

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- Similarly $A \in C_2$

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- C_1 and C_2 can be resolved to $C := (C_1 \setminus \{\neg A\}) \cup (C_2 \setminus \{A\})$

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- Therefore $\neg A \in C_1$
- Similarly $A \in C_2$
- C_1 and C_2 can be resolved to $C := (C_1 \setminus \{\neg A\}) \cup (C_2 \setminus \{A\})$
- Every literal in C has its negation in N, so C is falisfied by n.
- Either n or a node between n and the root becomes a failure node.

Sibling Failure Nodes give Resolution Opportunities

- Let n_1 and n_2 be sibling failure nodes
  - falsifying C_1 and C_2,
  - labeled A and $\neg A$.
- The parent node n of n_1 and n_2 does not falsify C_1 and C_2.
- Let N be the set of literals on the nodes up to and including n.
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- This gives a closed semantic tree with ≥ 2 nodes less than before.

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Theorem 7.2.

If S is an unsatisfiable clause set, then there is a resolution derivation of the empty clause from S.

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$\square$