IN3070/4070 – Logic – Autumn 2023

Lecture 6: Unification, Normal Forms

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27th September 2023



Department of Informatics



University of Oslo

Today's Plan

- Unification
- Normal Forms
- Negation Normal Form
- Conjunctive Normal Form
- Clausal Form
- Prenex Normal Forms
- Skolemization

Outline

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- To terms are unifiable if they have a unifier.

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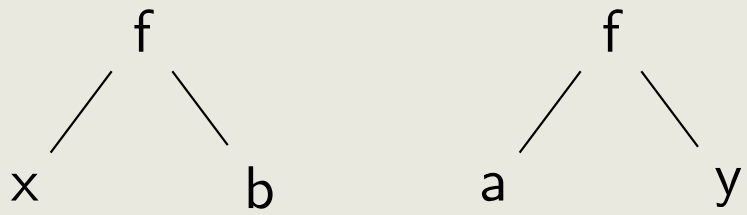
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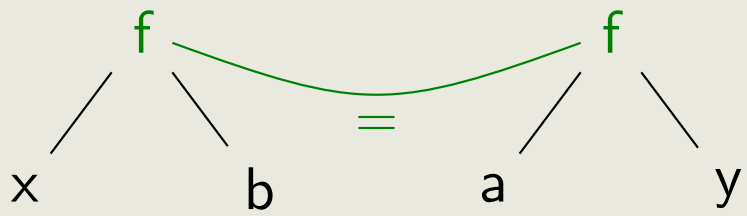
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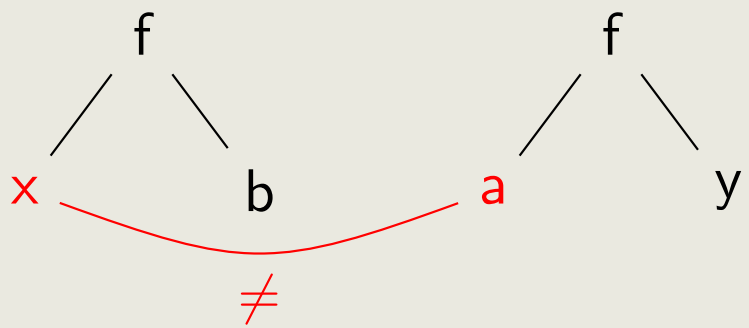
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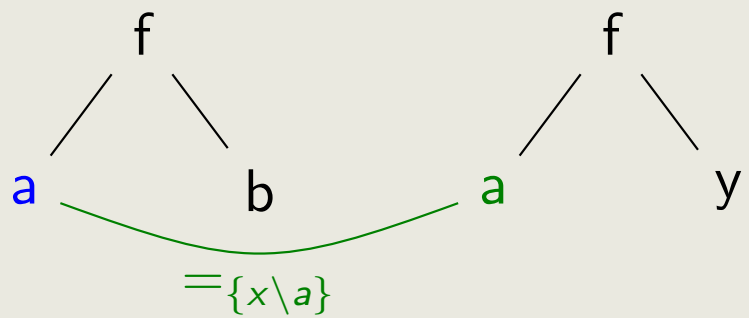
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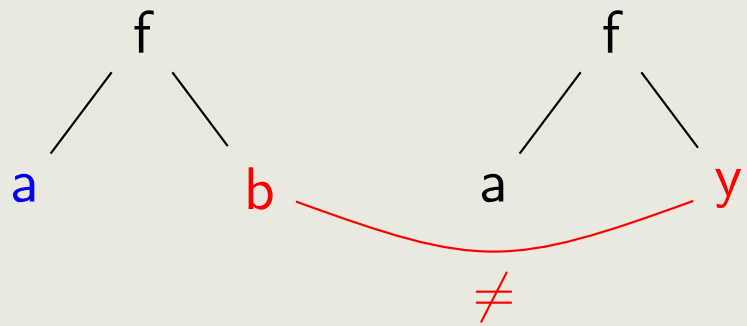
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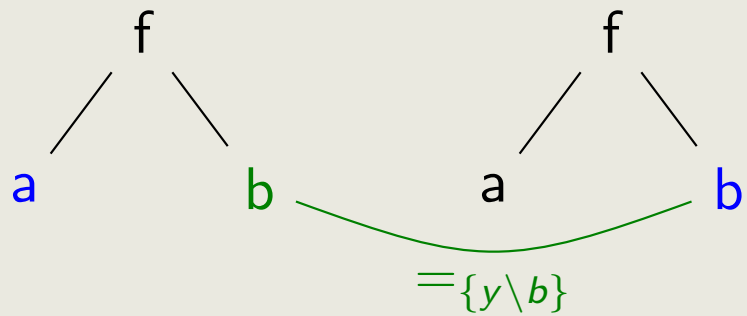
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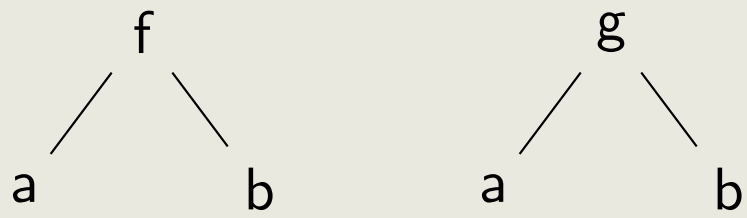
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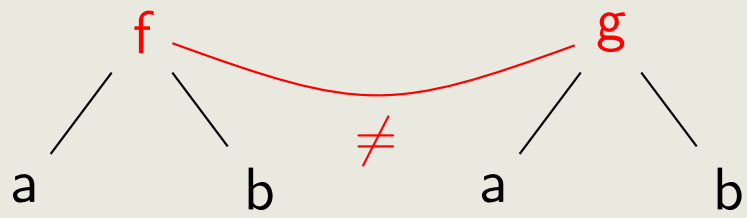
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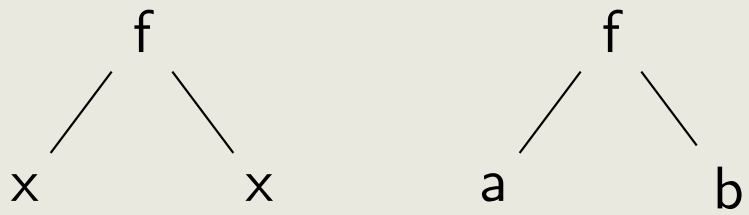
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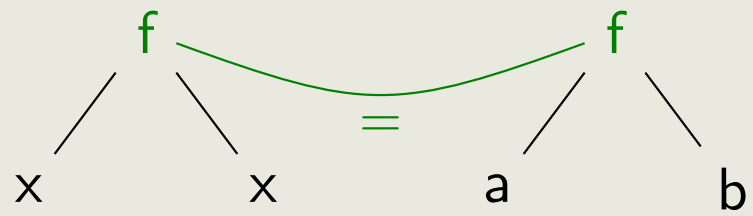
- The root symbols are different, and can *not* be unified!

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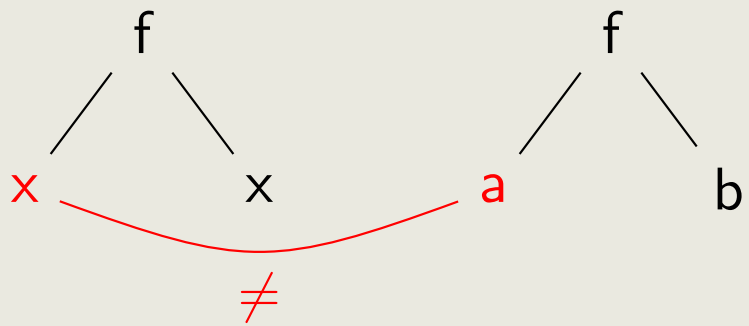


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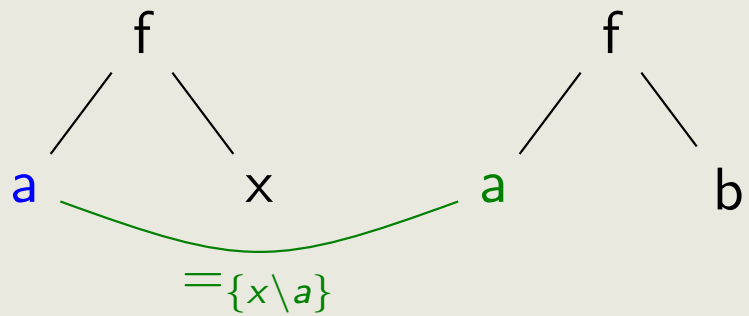
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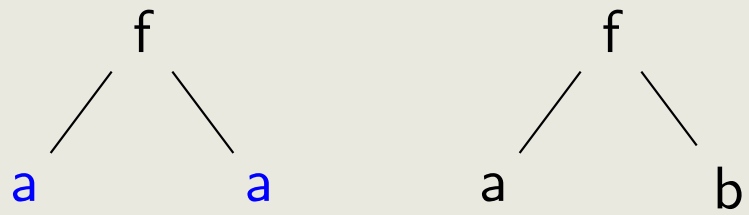
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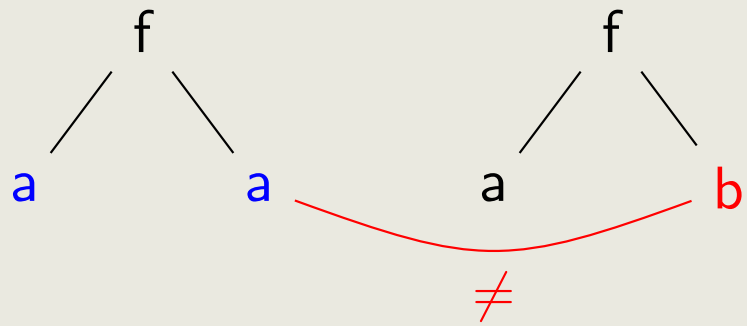
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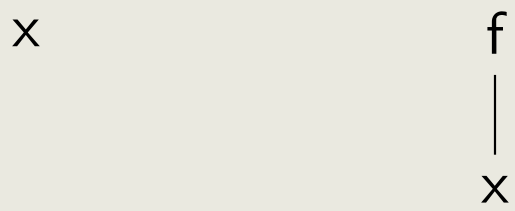
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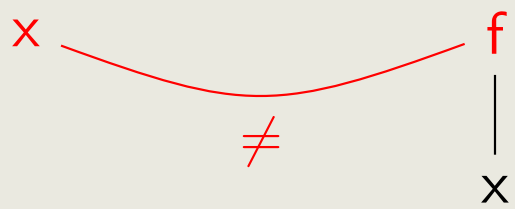
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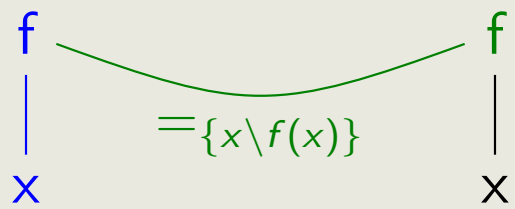


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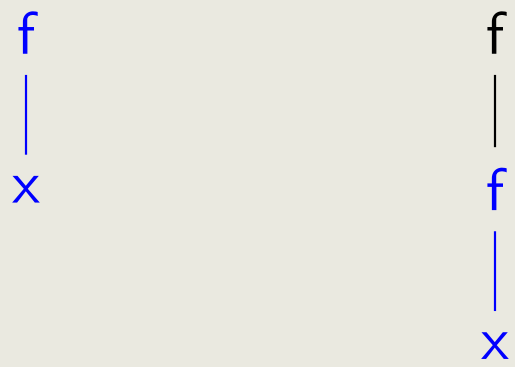
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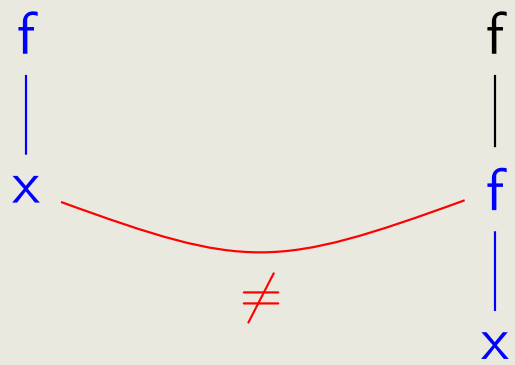
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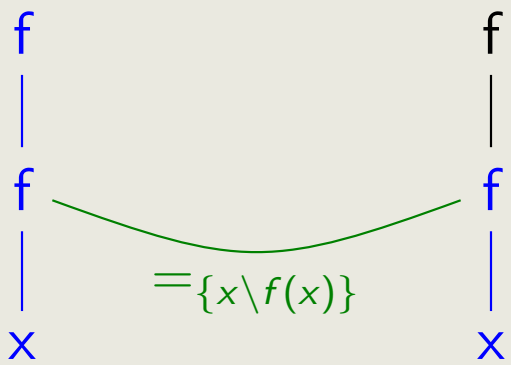
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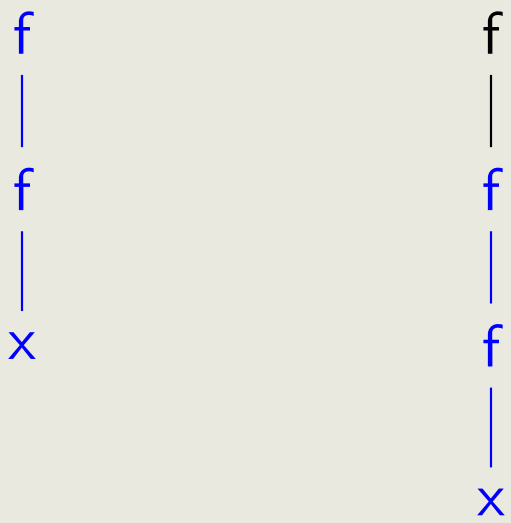
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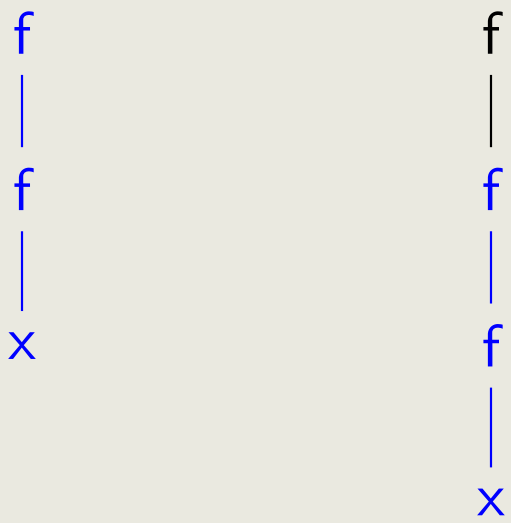
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- This continues indefinitely

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- Such a unifier is known as a most general unifier.

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- Exercise: show that $(\tau\sigma)(A) = \tau(\sigma(A))$ for all formlae A and all substitutions σ and τ.

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Yes, since $\{x\backslash f(y)\} = \{\}\{x\backslash f(y)\}$, where $\{\}$ is the identity substitution.

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Let τ be such that $\sigma_1 = \tau\sigma_2$.
$\sigma_1(s) = (\tau\sigma_2)(s) = \tau(\sigma_2(s)) = \tau(\sigma_2(t)) = (\tau\sigma_2)(t) = \sigma_1(t)$. □

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- $\sigma_3 = \{x\backslash z, y\backslash x, z\backslash y, u\backslash a\}$

Variable Renaming

- The previous example shows that two terms can have several most general unifiers.
- But these mgus are always equal up to variable renaming.

Definition 1.4 (Variable Renaming).

A subsitution η is a variable renaming if

1. $\eta(x)$ *is a variable for all* $x \in \mathcal{V}$, *and*
2. $\eta(x) \neq \eta(y)$ *for all* $x, y \in \mathcal{V}$ *with* $x \neq y$.

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Uniqueness "up to variable renaming"

Proposition 1.3.

If σ_1 and σ_2 are most general unifiers for two terms s and t, then there is a variable renaming η such that $\eta\sigma_1 = \sigma_2$.

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- We leave out the proof.

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- So every term is a subterm of itself, but not a strict subterm.

Numbered Term Trees

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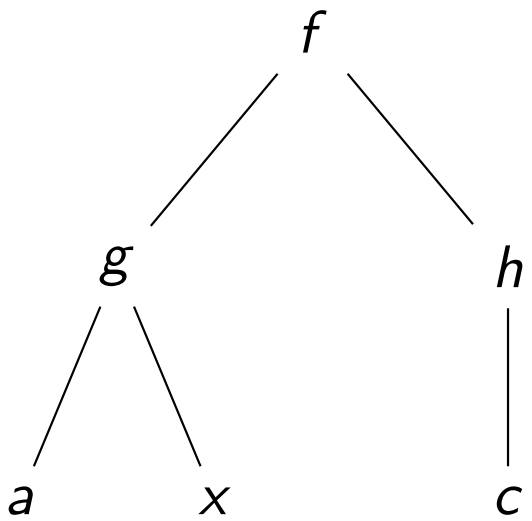
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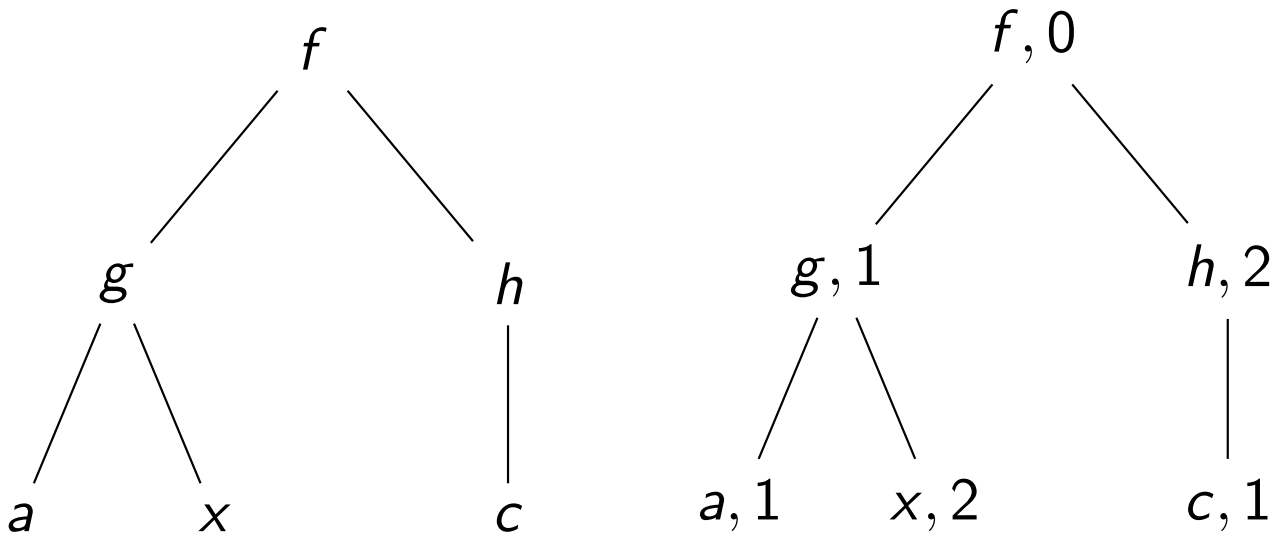
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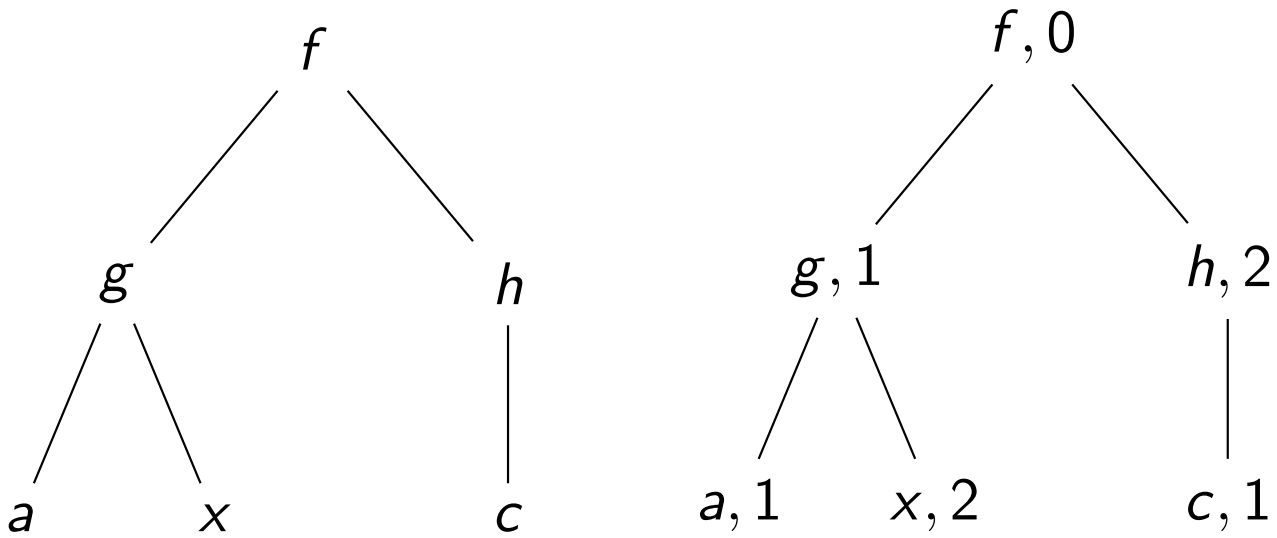
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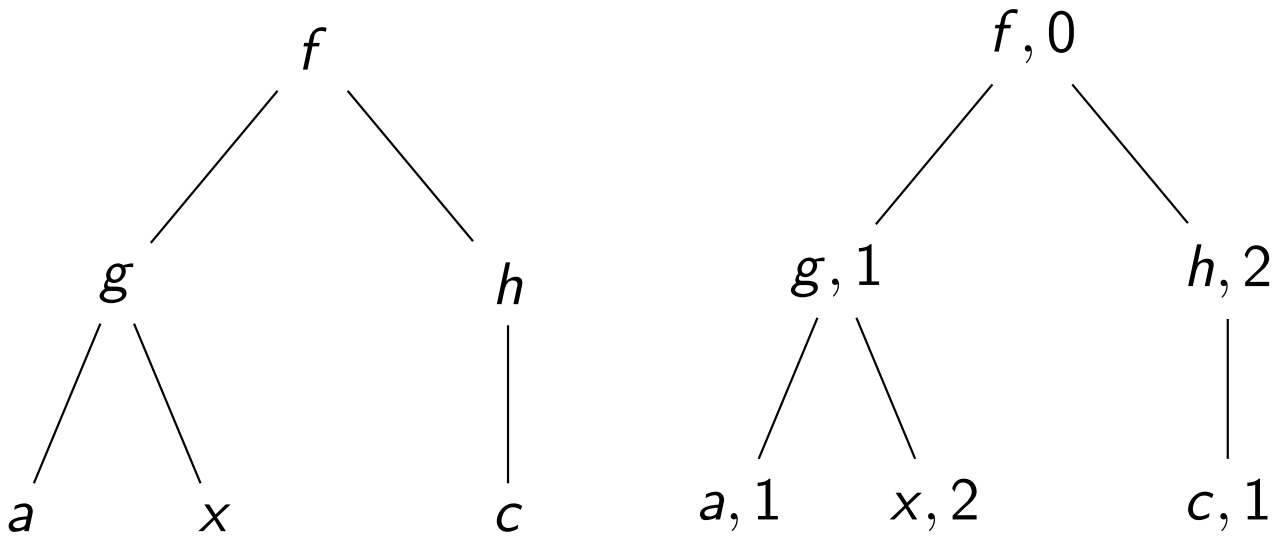
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- We write the root of the numbered term tree of t as root(t).

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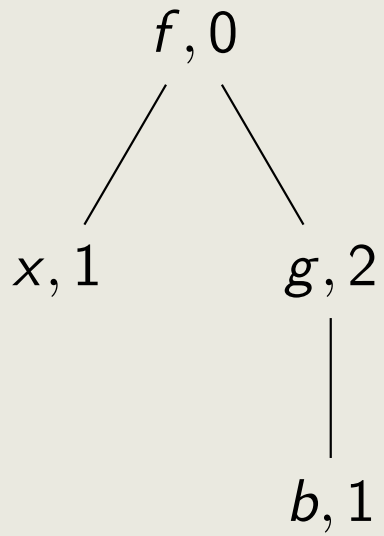
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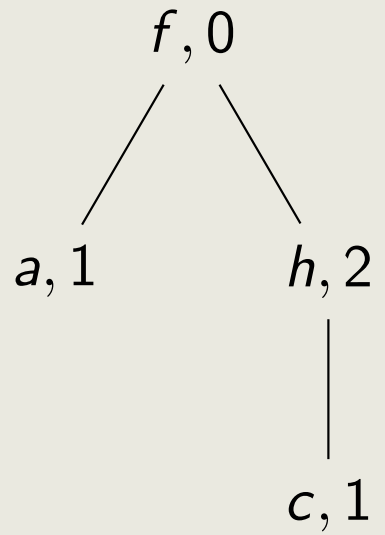
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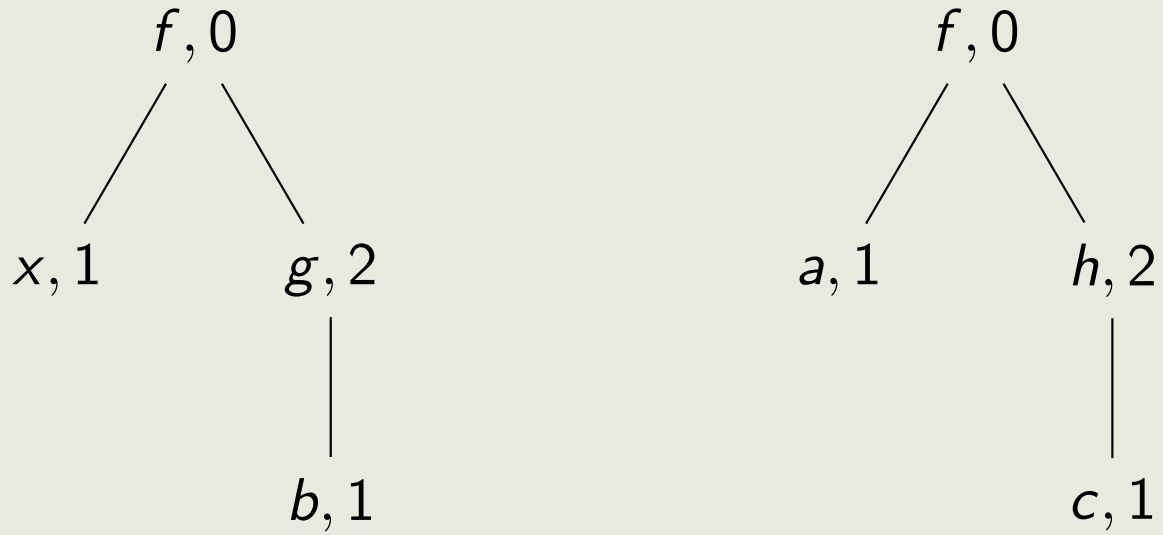
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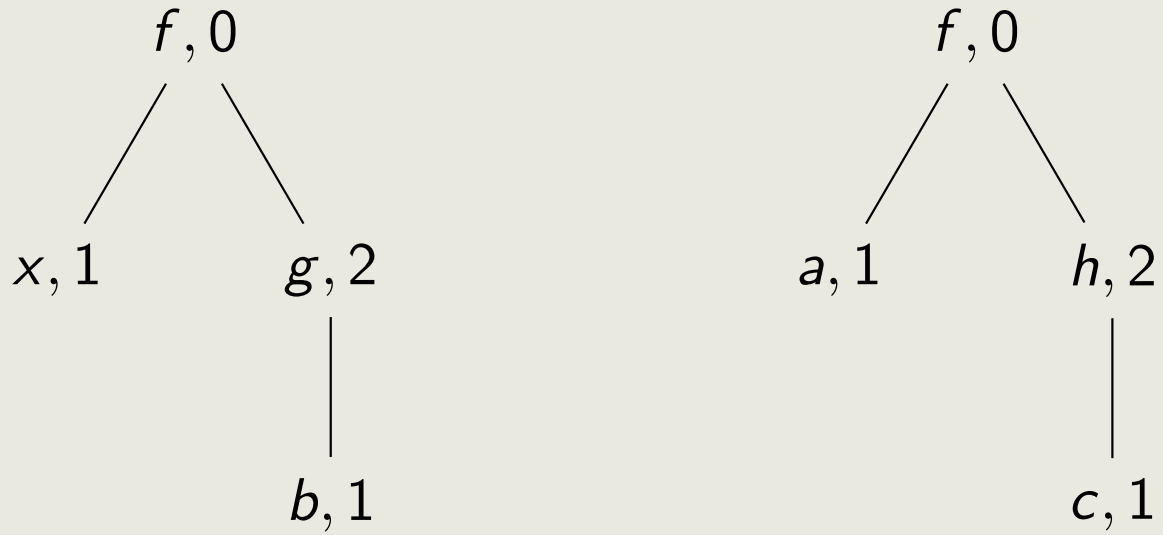
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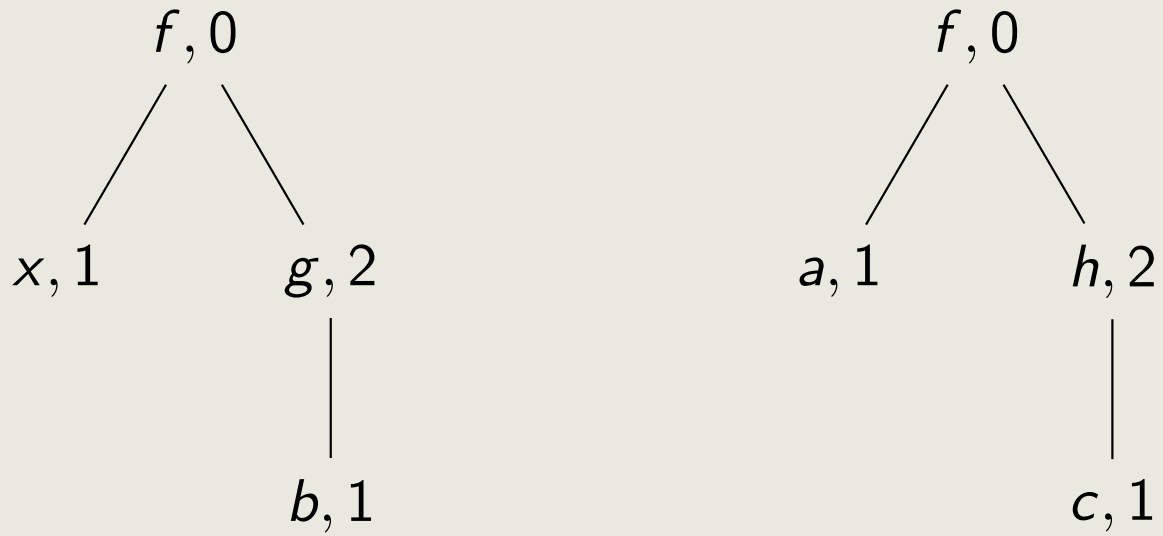
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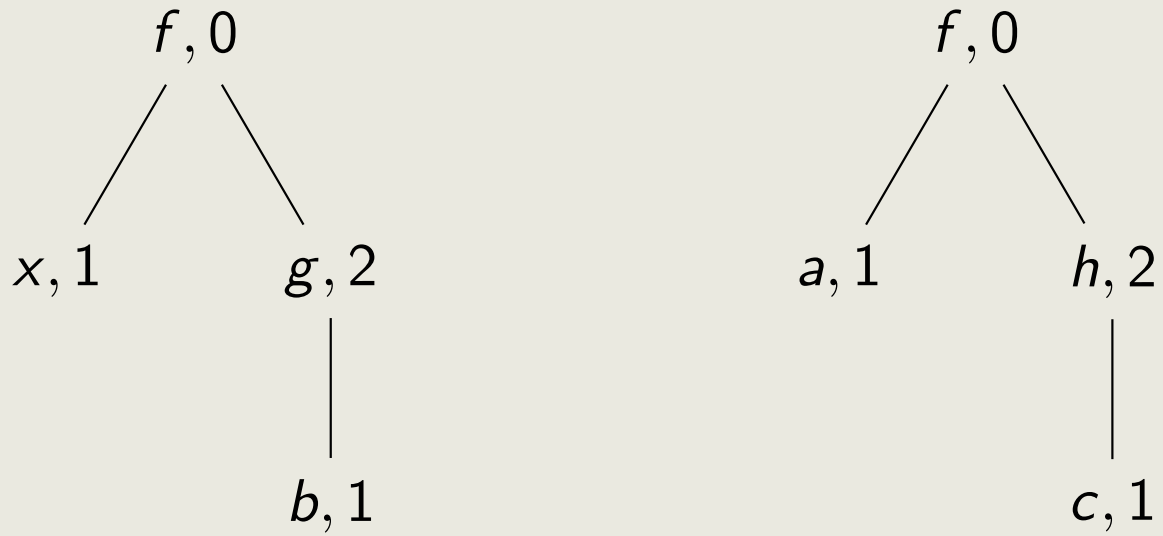
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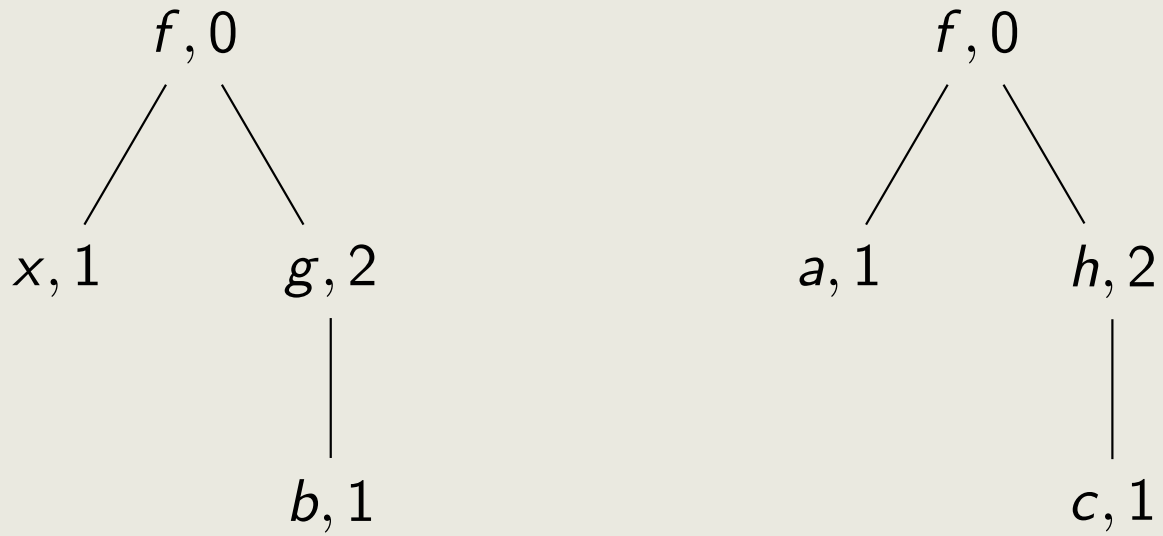
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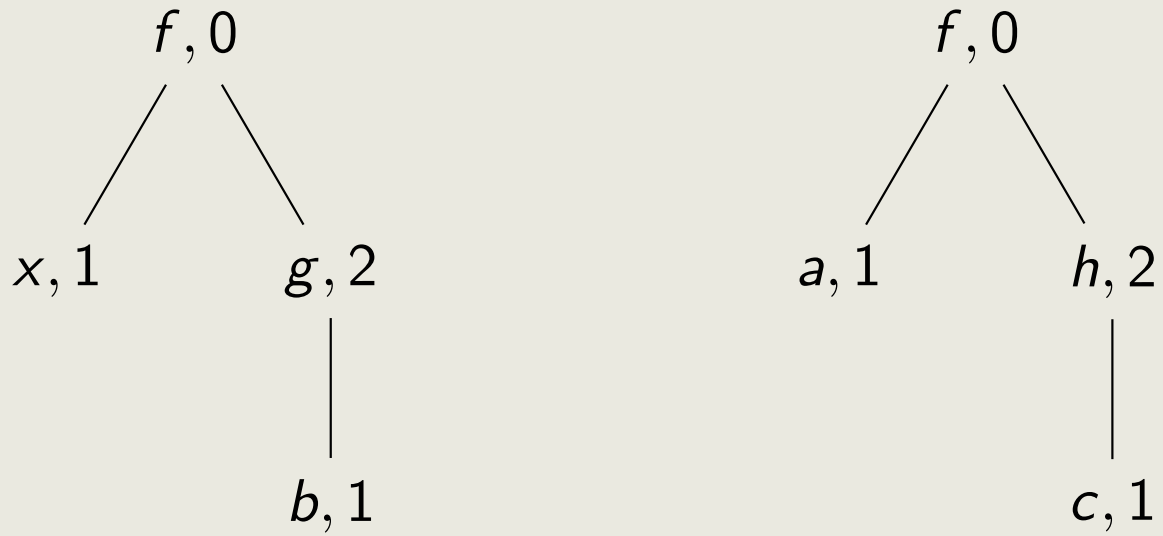
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Outline

- Unification
- Normal Forms
- Negation Normal Form
- Conjunctive Normal Form
- Clausal Form
- Prenex Normal Forms
- Skolemization

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- For every a, we can compute the (or a) N-normal form $f(a)$.

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- Then $\langle m/k, n/k \rangle$ is reduced and $\langle m/k, n/k \rangle \approx \langle m, n \rangle$
- So $\langle m/k, n/k \rangle$ is the "reduced normal form" of $\langle m, n \rangle$

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- Unification
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Theorem 3.1.

Every formula in first-order logic can be transformed into an equivalent formula in NNF.

Proof.

To convert an arbitrary formula to a formula in NNF, remove implications, and push negations inwards, preserving equivalence, using the following:

$$A \to B \equiv \neg A \lor B$$
$$\neg(A \land B) \equiv \neg A \lor \neg B$$
$$\neg(A \lor B) \equiv \neg A \land \neg B$$
$$\neg(\forall x\, A) \equiv \exists x\, \neg A$$
$$\neg(\exists x\, A) \equiv \forall x\, \neg A$$
$$\neg(\neg A) \equiv A$$

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- Soundness and completeness proofs also have fewer cases.

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(2) Use the distributive laws to move conjunctions inside disjunctions to the outside

$$A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$$

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$(p \vee \neg q) \wedge (\neg p \vee q)$ in clausal form: $\{\{p, \neg q\}, \{\neg p, q\}\}$

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- $\overline{L}$ then represents its *complement*.

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$\forall x \forall y((p(x,y) \vee \neg p(y,x)) \wedge (q(x,y) \vee \neg q(y,x)))$ *is in PCNF.*

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Prenex Conjunctive Normal Form

Definition 6.1 (Prenex Conjunctive Normal Form).

A formula is in prenex conjunctive normal form (PCNF) iff it is of the form:

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Clausal Form for First-order Formulae

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Note: *The universal quantifiers are implicit.*

Outline

- Unification
- Normal Forms
- Negation Normal Form
- Conjunctive Normal Form
- Clausal Form
- Prenex Normal Forms
- Skolemization

Skolem's Theorem

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Named after the Norwegian mathematician and logician Thoralf Albert Skolem (1887–1963).

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- The formula is now in PCNF.

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- If there are no universal quantifiers preceding $\exists x$, replace x by a new constant (0-ary function).
- These new function symbols are Skolem functions and the process of replacing existential quantifiers by functions is Skolemization.

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- *Are they equivalent? No!*
- *Are they equisatisfiable? Yes!*
- *The Skolemization of* $\forall x \exists y p(x, y)$ *is* $\forall x p(x, f(x))$*, and if one of them has a model, so does the other.*

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- For any choice of elements $d_1, \ldots, d_n$ from D, there is an element d_{n+1} in D such that

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- Let $f^{\iota'}(d_1, \ldots, d_n) = d_{n+1}$. This ensures that the claim holds.

Example

- Clause form of $\neg\exists x\,(p(x) \rightarrow \forall y\, p(y))$

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- First, transform to (equivalent) Prenex Normal Form

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Outlook

- We have seen the LK calculus for propositional and first-order logic
- Sound and complete, but not machine-oriented
- Machine-oriented calculi use:
  - Unification to find the right instantiations
  - Normal forms to simplify reasoning steps
- Free variable calculi
  - Similar to LK, but with unification
  - Often used with NNF or clause form
  - Not this year
- Resolution
  - Basis of many theorem provers, uses unification
  - Almost always on clause form