IN3070/4070 – Logic – Autumn 2023

Lecture 7: Resolution

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4th October 2023



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Today's Plan

- Introduction
- Repetition: Negation Normal Form
- Conjunctive Normal Form
- Clausal Form
- Resolution
- Soundness of Resolution
- Completeness of Resolution

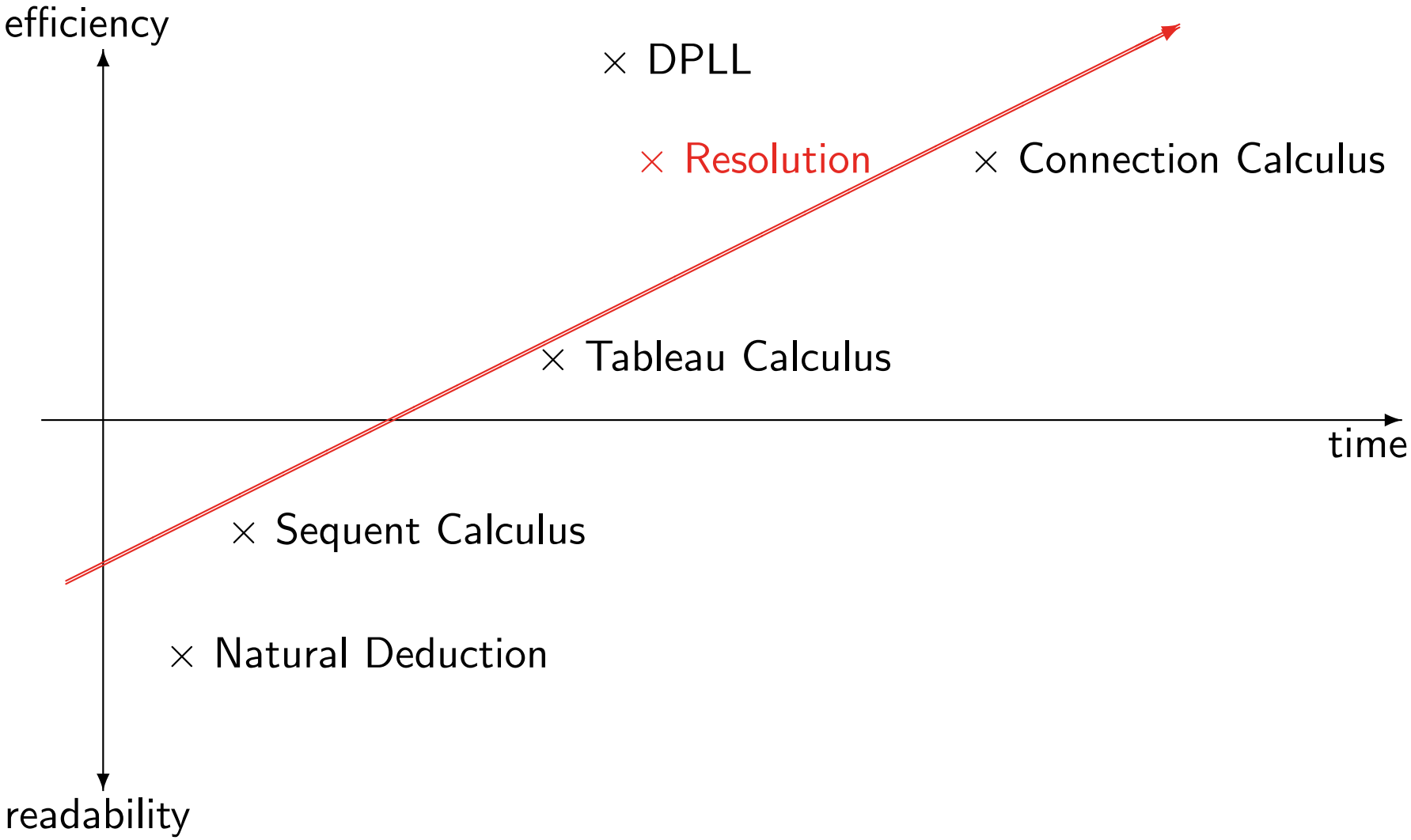
Oblig 2

- Published today, 4 October
- Has a question about first order resolution
  - Start on the rest, do that next week
- Deadline for first attempt in two weeks

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Proof Search Calculi



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- works for first-order formulae in clausal form (e.g. conjunctive or disjunctive normal form)
- consists of one (two for first-order) inference rules and one axiom
- is one of the most popular proof search calculi
- has been implemented in many automated theorem provers

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Theorem 2.1.

Every formula in first-order logic can be transformed into an equivalent formula in NNF.

Proof.

To convert an arbitrary formula to a formula in NNF, remove implications, and push negations inwards, preserving equivalence, using the following:

$$A \to B \equiv \neg A \vee B$$
$$\neg(A \wedge B) \equiv \neg A \vee \neg B$$
$$\neg(A \vee B) \equiv \neg A \wedge \neg B$$
$$\neg(\forall x\, A) \equiv \exists x\, \neg A$$
$$\neg(\exists x\, A) \equiv \forall x\, \neg A$$
$$\neg(\neg A) \equiv A$$

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To convert an arbitrary propositional formula to a formula in CNF perform the following steps, each of which preserves logical equivalence:

(1) Convert to negation normal form.

(2) Use the distributive laws to move conjunctions inside disjunctions to the outside

$$A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$$

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$(p \vee \neg q) \wedge (\neg p \vee q)$ in clausal form: $\{\{p, \neg q\}, \{\neg p, q\}\}$

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This follows from the previous theorem, where we transformed a formula to CNF. Each disjunction is then transformed to a clause (of literals), and the clausal form is the set of these clauses. □

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The Resolution Rule

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Definition 5.2 (Resolution Rule).

Let C_1, C_2 be clauses with $L \in C_1$ and $\overline{L} \in C_2$. The resolvent C' of C_1 and C_2 is $(C_1 \setminus \{L\}) \cup (C_2 \setminus \{\overline{L}\})$. C_1 and C_2 are the parents of C'.

The Resolution Rule – Example

Example: Let $C_1 = \{a, b, \neg c\}$ and $C_2 = \{b, c, \neg e\}$.

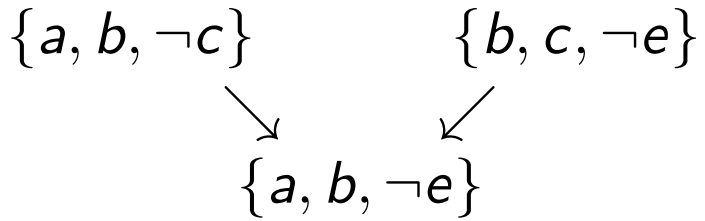
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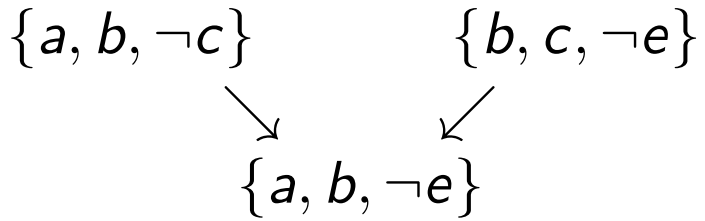
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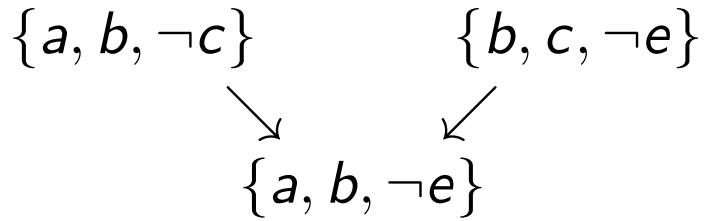
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Observations:

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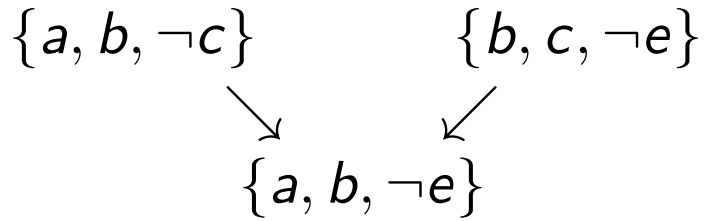
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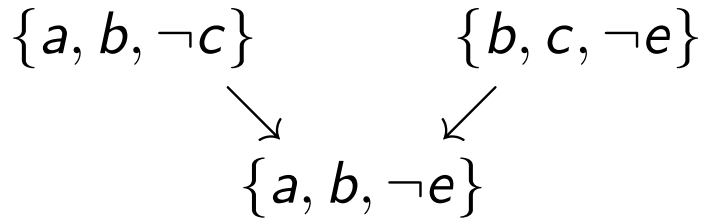
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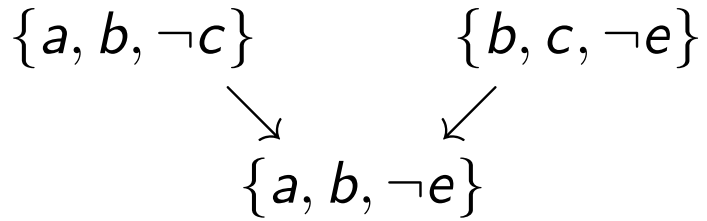
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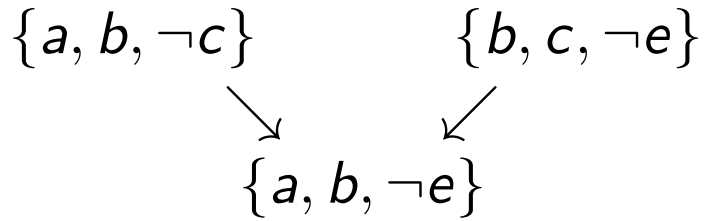
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- the empty clause $\square$ is unsatisfiable
- goal: derive empty clause $\square$

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Definition 5.3 (Resolution Procedure).

Given a set of clauses S.

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- Prove validity of: $(p \wedge q) \rightarrow p$
- Show unsatisfiability of: $\neg((p \wedge q) \rightarrow p)$

Resolution Calculus – Example 1

- Prove validity of: $(p \wedge q) \rightarrow p$
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Resolution Calculus – Example 1

- Prove validity of: $(p \wedge q) \rightarrow p$
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- Prove validity of: $(p \wedge q) \rightarrow p$
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- CNF: $p \wedge q \wedge \neg p$
- Clause set: $\{\{p\}, \{q\}, \{\neg p\}\}$
- Resolve $\{p\}$ with $\{\neg p\}$
- Resolvent: $\square$

Resolution Calculus – Example 2

- Prove validity of: $p \wedge (p \rightarrow q) \rightarrow q$

Resolution Calculus – Example 2

- Prove validity of: $p \wedge (p \rightarrow q) \rightarrow q$
- Show unsatisfiability of: $\neg(p \wedge (p \rightarrow q) \rightarrow q)$

Resolution Calculus – Example 2

- Prove validity of: $p \wedge (p \rightarrow q) \rightarrow q$
- Show unsatisfiability of: $\neg(p \wedge (p \rightarrow q) \rightarrow q)$
- Equivalent to: $p \wedge (p \rightarrow q) \wedge \neg q$

Resolution Calculus – Example 2

- Prove validity of: $p \wedge (p \rightarrow q) \rightarrow q$
- Show unsatisfiability of: $\neg(p \wedge (p \rightarrow q) \rightarrow q)$
- Equivalent to: $p \wedge (p \rightarrow q) \wedge \neg q$
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- Prove validity of: $p \wedge (p \rightarrow q) \rightarrow q$
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- Equivalent to: $p \wedge (p \rightarrow q) \wedge \neg q$
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- Prove validity of: $p \wedge (p \rightarrow q) \rightarrow q$
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- Clause set: $\{\{p\}, \{\neg p, q\}, \{\neg q\}\}$
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- Clause set: $\{\{p\}, \{\neg p, q\}, \{\neg q\}\}$
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- Resolvent: $\{q\}$
- New clause set: $\{\{p\}, \{\neg p, q\}, \{\neg q\}, \{q\}\}$

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- Prove validity of: $p \wedge (p \rightarrow q) \rightarrow q$
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- New clause set: $\{\{p\}, \{\neg p, q\}, \{\neg q\}, \{q\}\}$
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- Need to use resolution rule on new resolvents

Resolution Calculus – Example 3

- Prove validity of: $(p \rightarrow (q \rightarrow r)) \rightarrow (p \wedge q \rightarrow r)$

Resolution Calculus – Example 3

- Prove validity of: $(p \to (q \to r)) \to (p \wedge q \to r)$
- Show unsatisfiability of: $\neg((p \to (q \to r)) \to (p \wedge q \to r))$

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7. $\square$ — resolvent of 4. and 6.

Resolution Calculus – Example 4

- Prove validity of: $(p \to q) \to ((p \to r) \to (p \to (q \wedge r)))$

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- Order of resolution steps does not matter for completeness

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Definition 5.4 (Resolution Calculus).

The resolution calculus has one axiom and one (inference) rule.

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- in contrast to natural deduction or the sequent calculus, the resolution calculus has no rule with more than one premise
- hence, a derivation in the resolution calculus has only one branch
- terminates, if all clauses $C_i \cup \{L\}, C_j \cup \{\overline{L}\}$ have been considered

Outline

- Introduction
- Repetition: Negation Normal Form
- Conjunctive Normal Form
- Clausal Form
- Resolution
- Soundness of Resolution
- Completeness of Resolution

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2. A set of clauses containing the empty clause is unsatisfiable

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But the empty clause $\mathcal{I}$ contains no literals, so that is a contradiction.

$\square$

Outline

- Introduction
- Repetition: Negation Normal Form
- Conjunctive Normal Form
- Clausal Form
- Resolution
- Soundness of Resolution
- Completeness of Resolution

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- We will go through Robinson's original proof

Semantic Trees

The completeness proof uses the following concept:

Definition 7.1 (Semantic Trees).

A semantic tree is a binary tree where:

- *The root is labelled by the symbol $\bot$,*
- *Every node has either no children or two children,*
- *For every node that has children, there is some atom A such that one child is labeled with A and the other with $\neg A$*
- *There are not two complementary literals A and $\neg A$ on any path starting at the root.*

Semantic Trees

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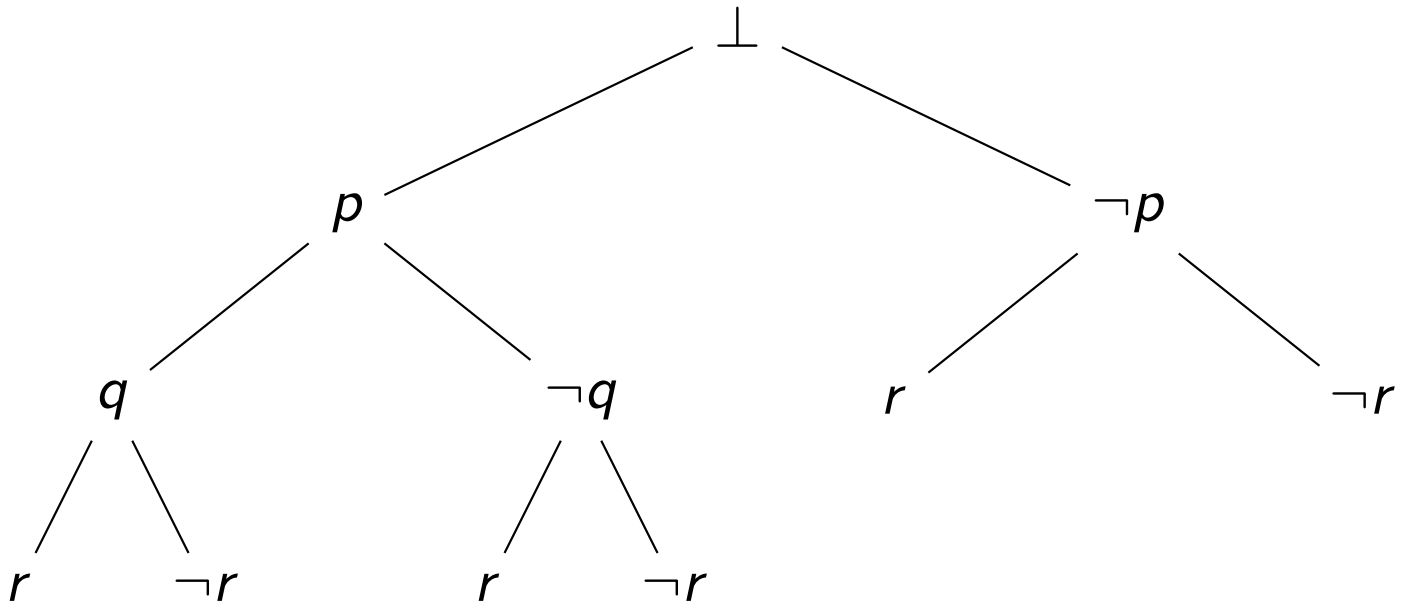
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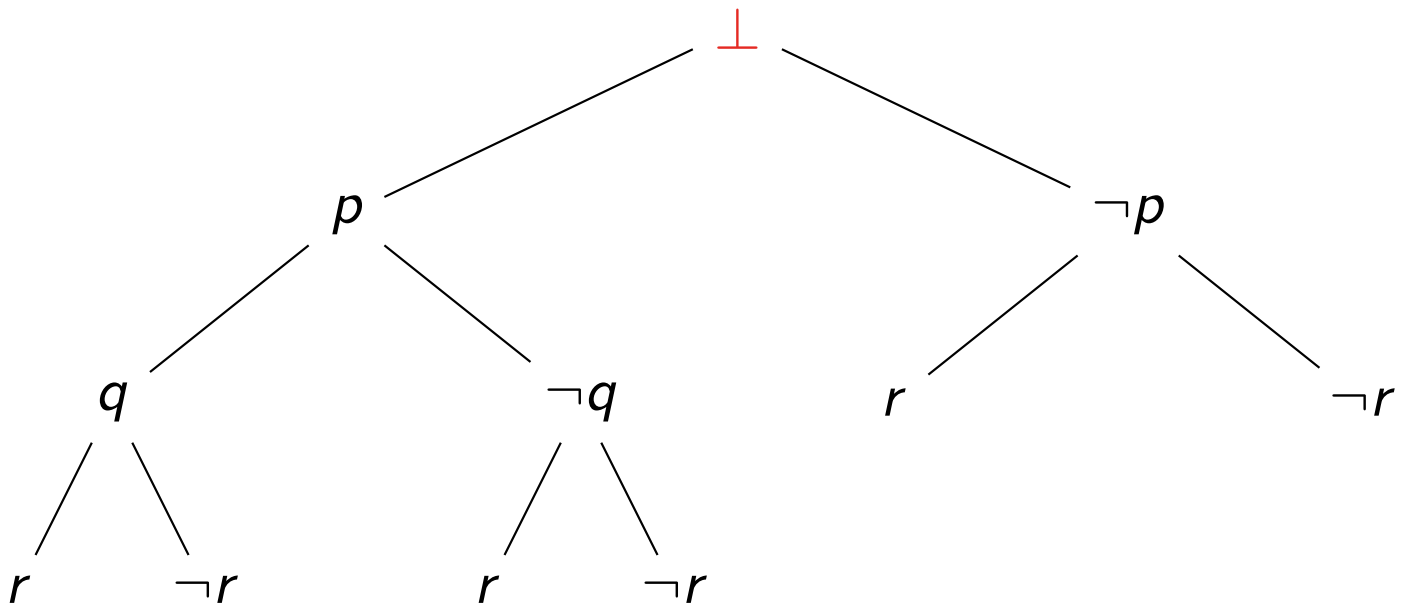
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Not a data structure, just needed for the completeness proof

Semantic Trees — Example

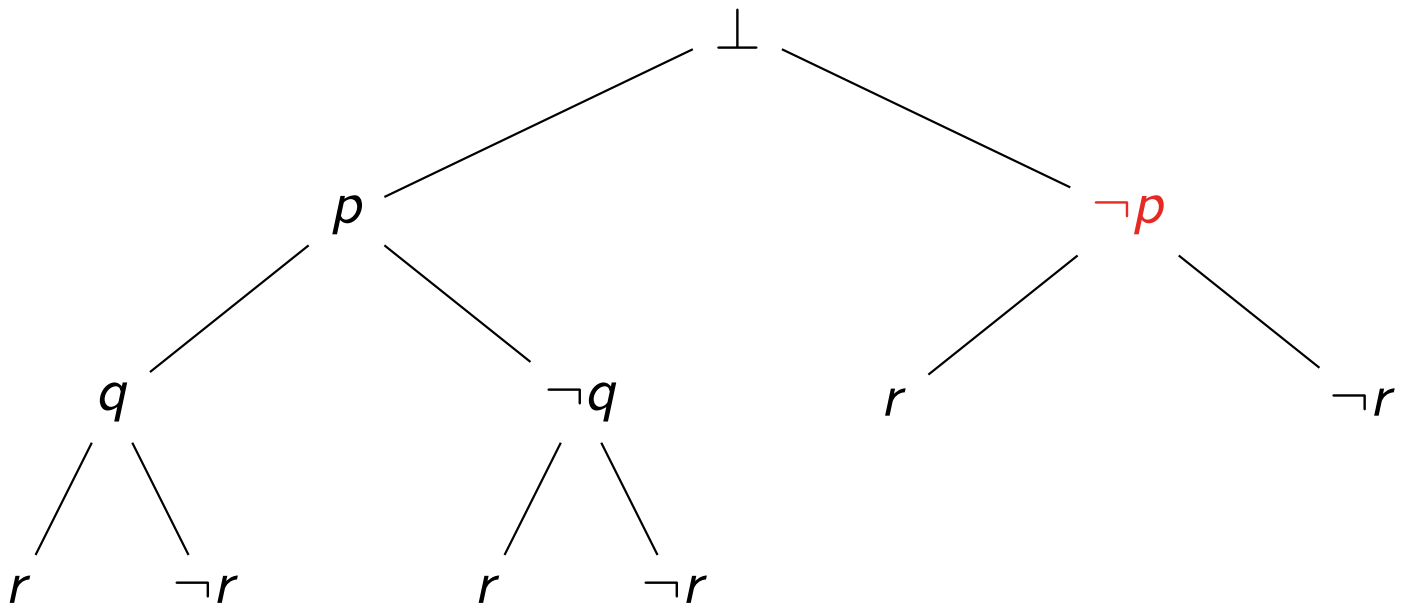


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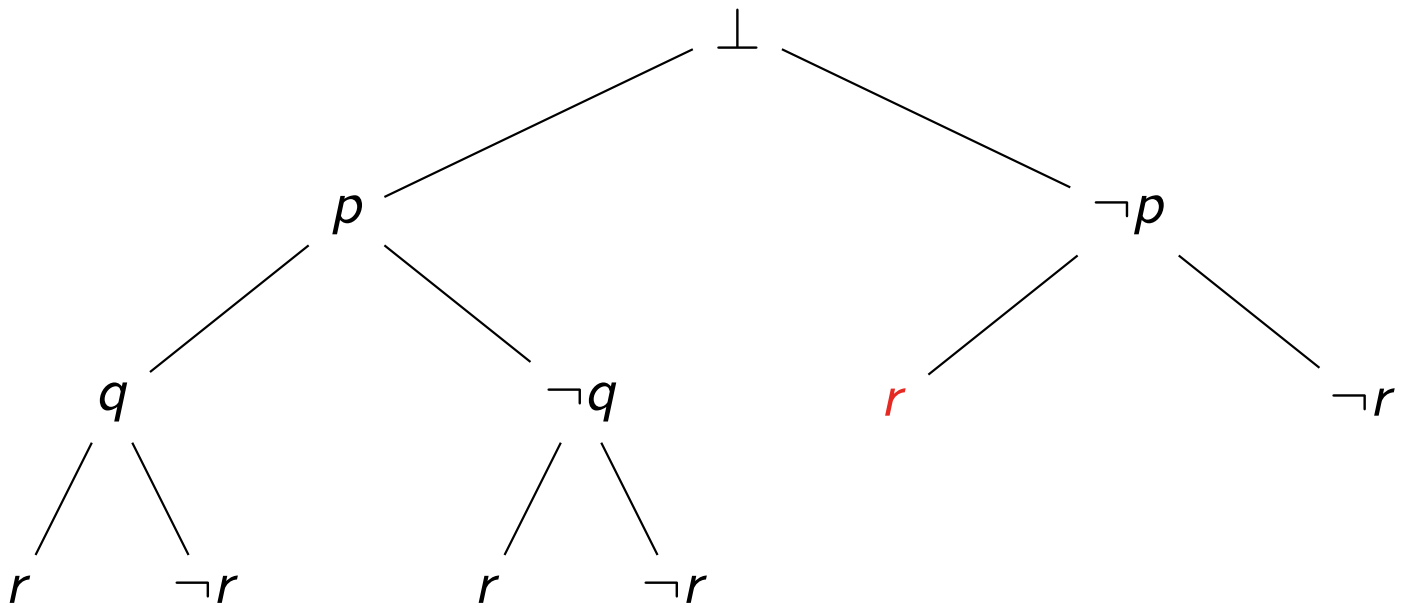
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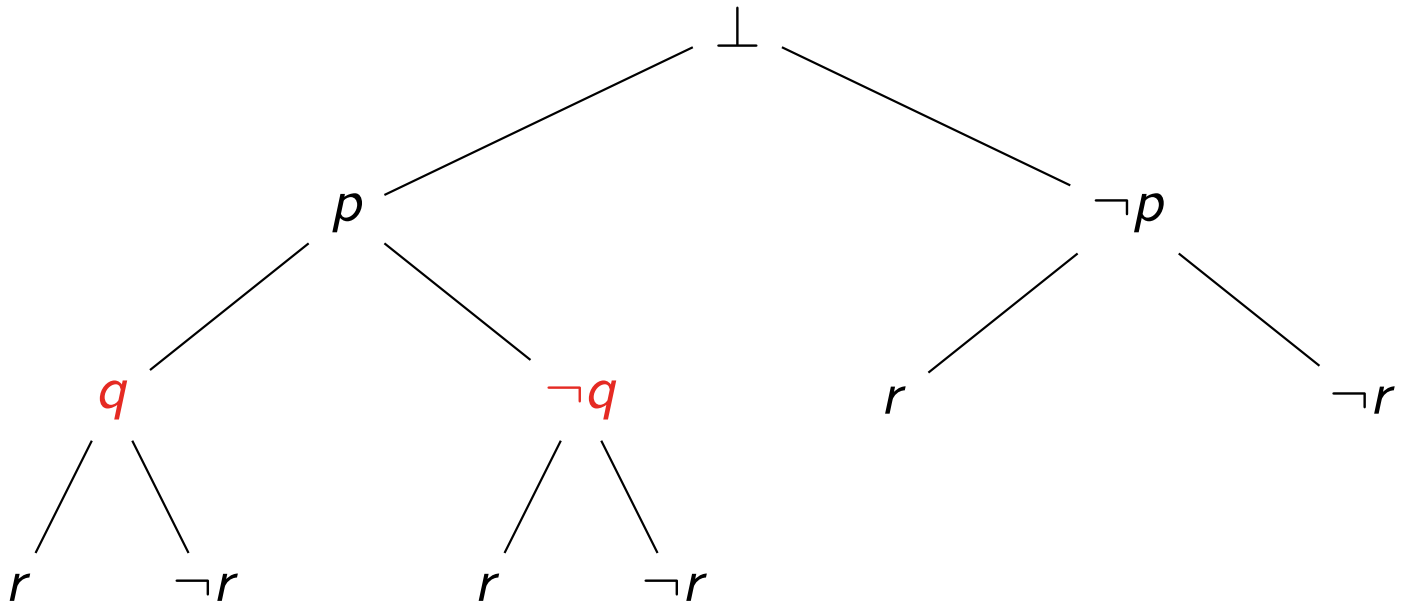
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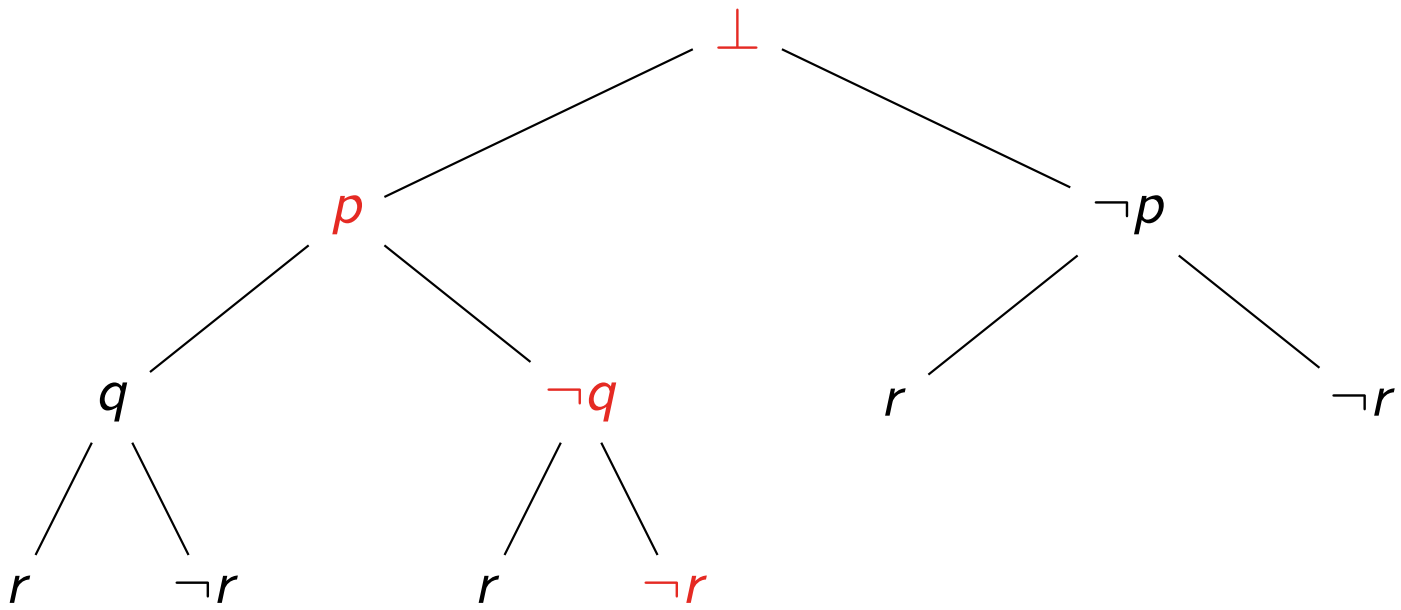
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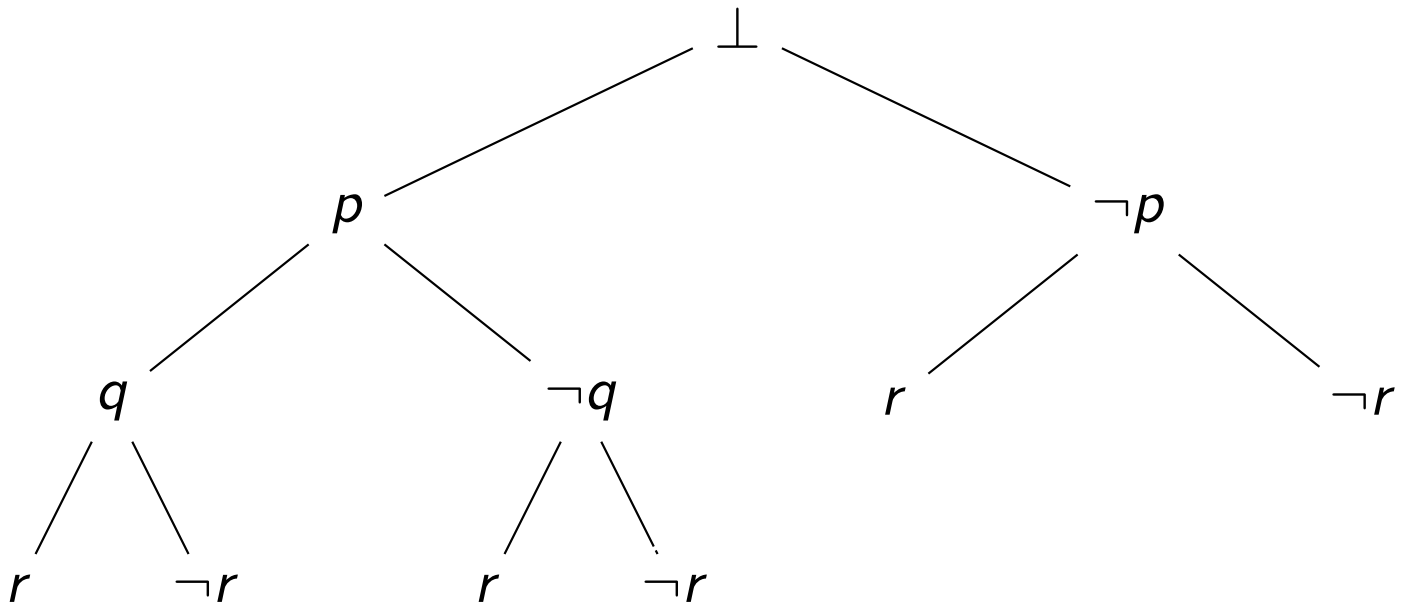
Semantic Trees — Example



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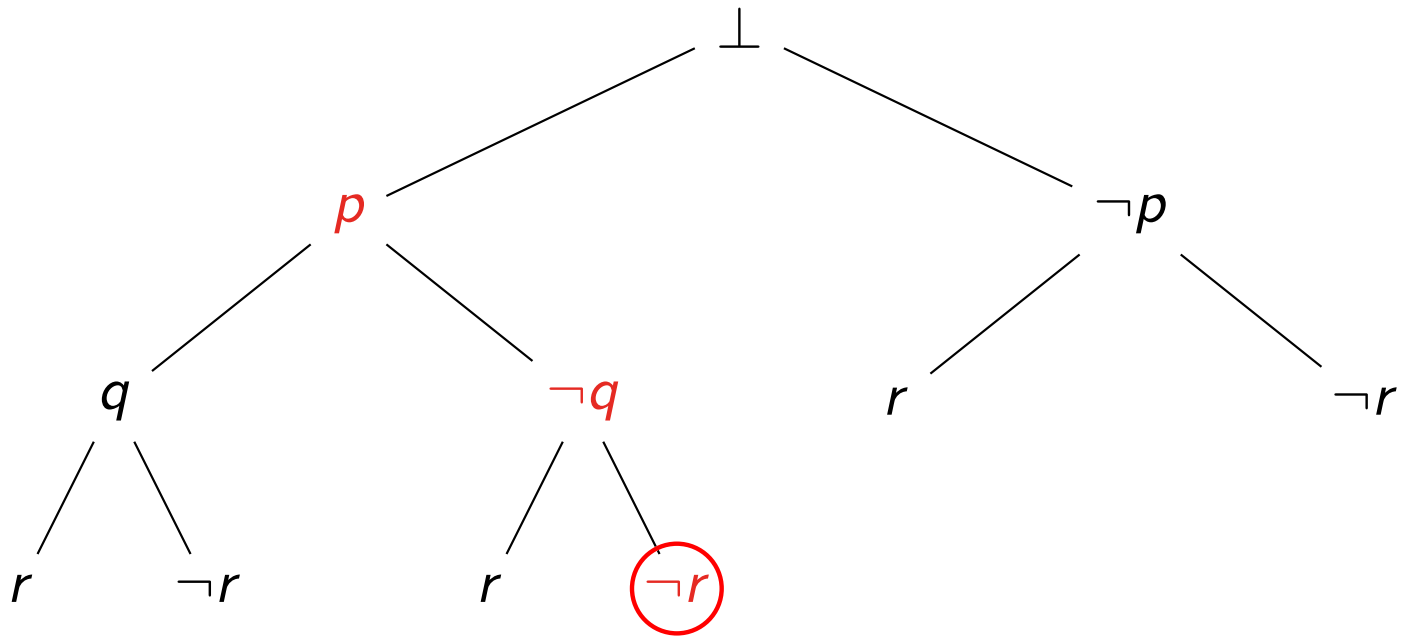
Partial Interpretations

The path to every node n in a semantic tree gives a 'partial interpretation' $\mathcal{I}_n$:



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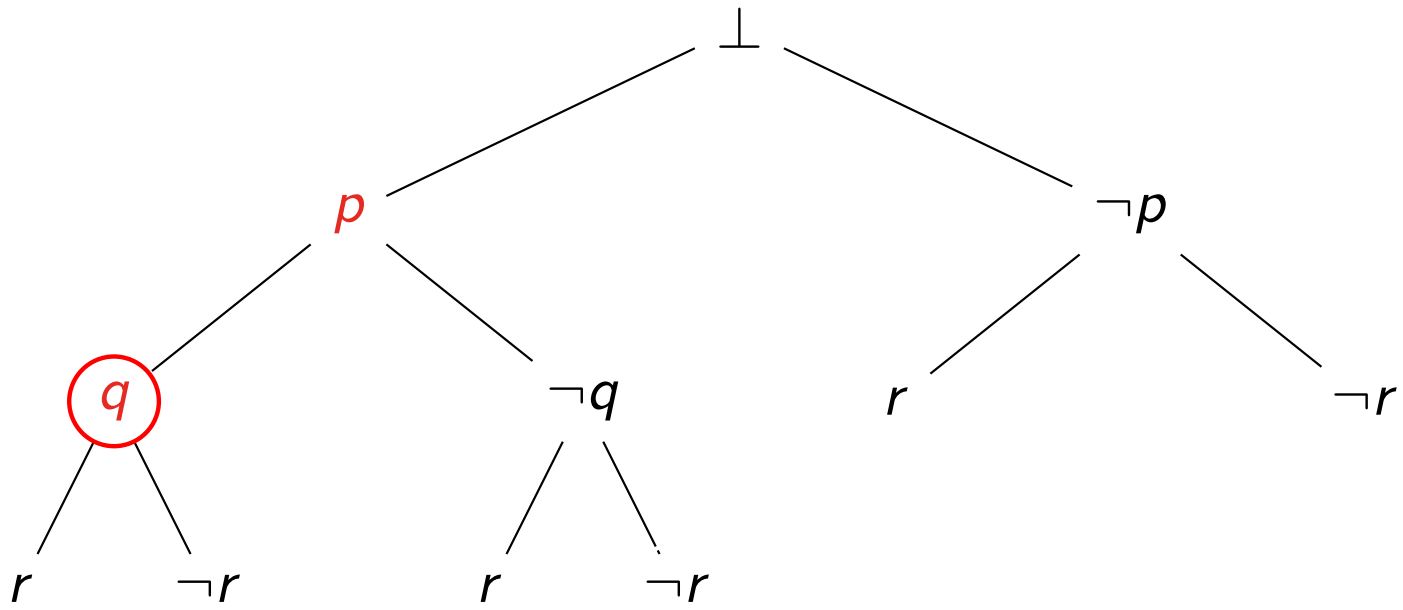
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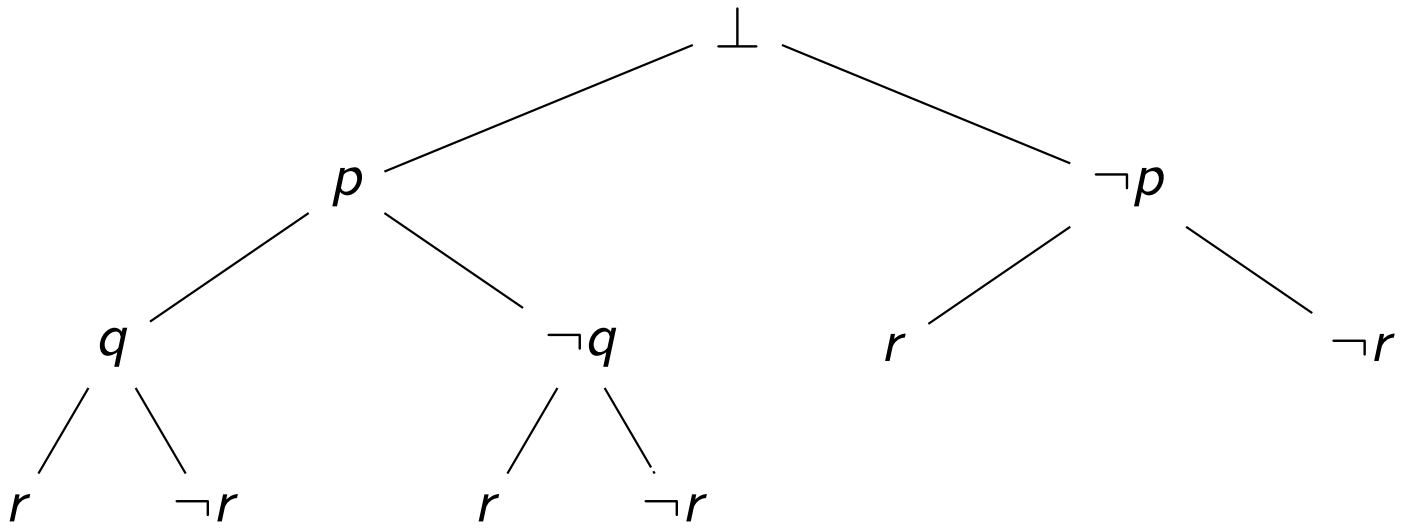
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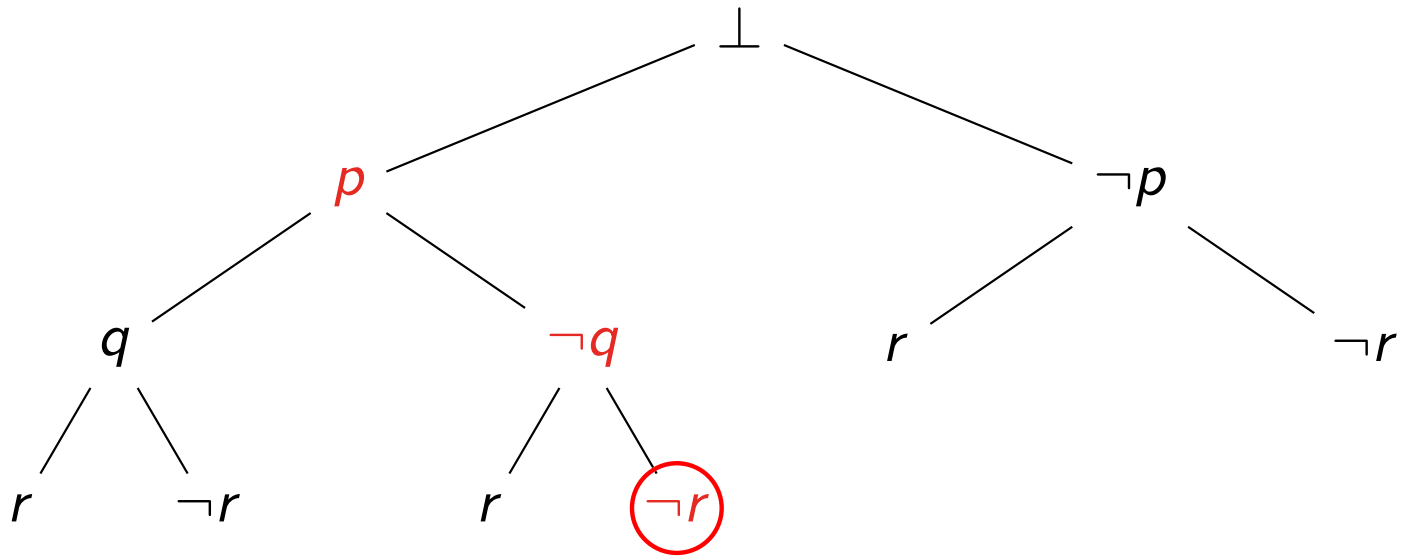
Failure Nodes – Motivation

Sometimes, such a ‘partial interpretation’ is enough to falsify a clause:



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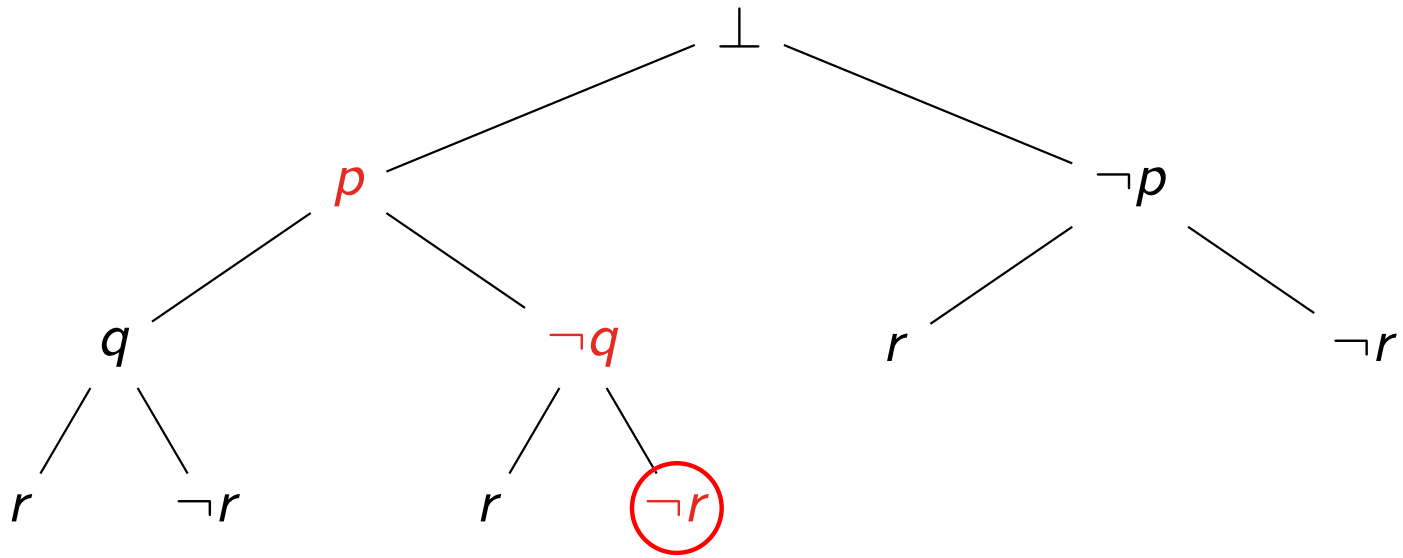
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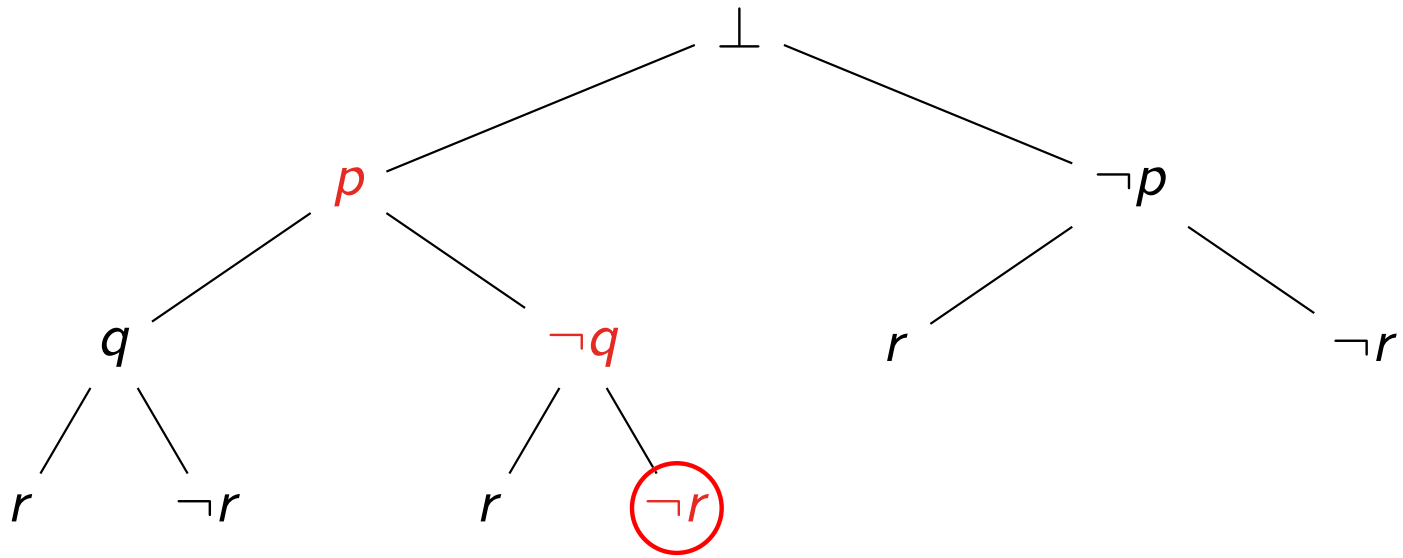
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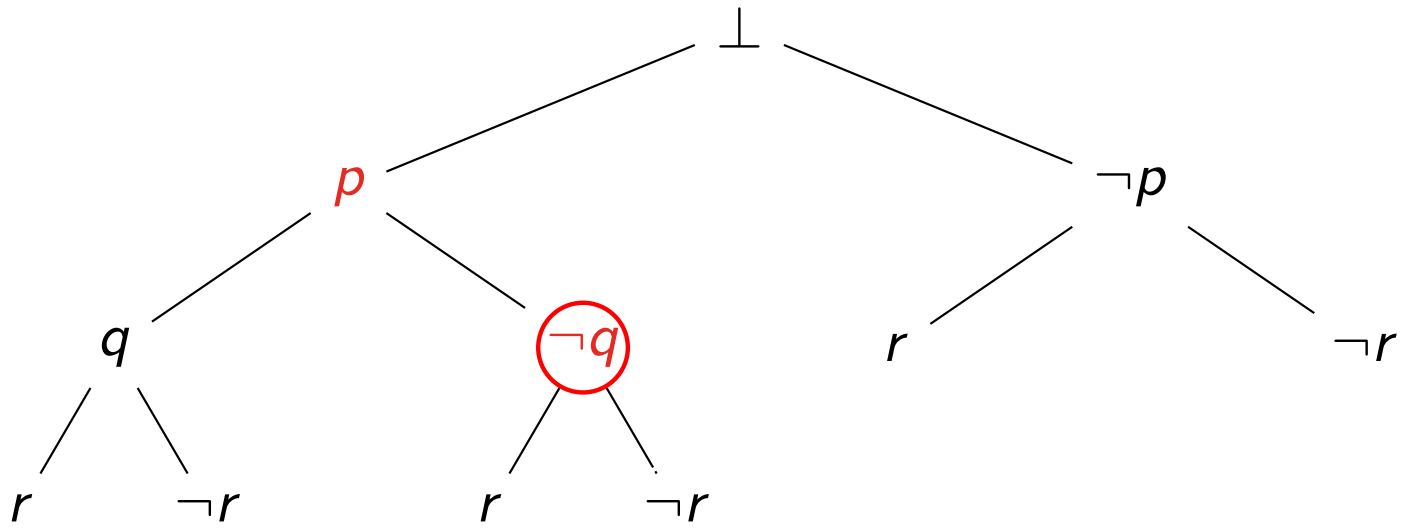
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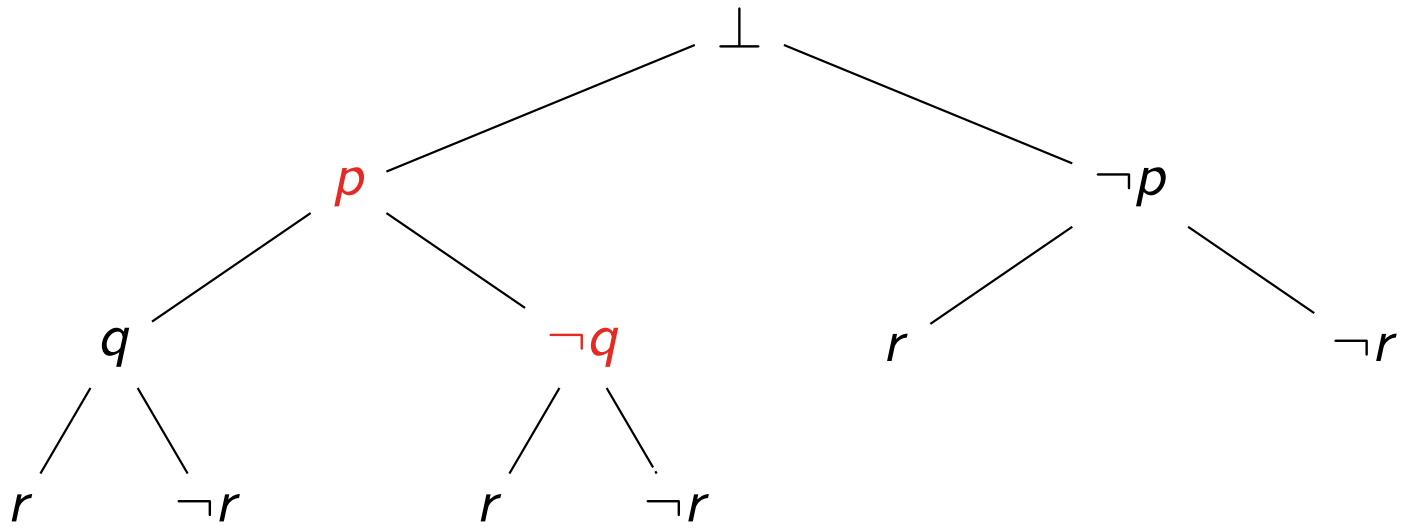
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- It remains false further down.

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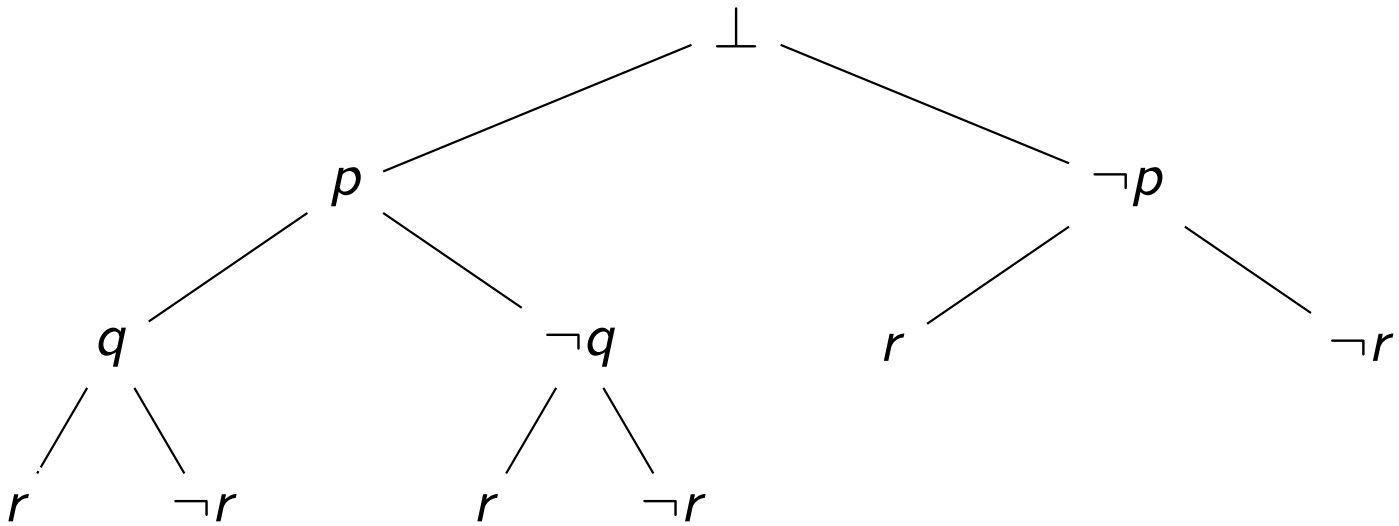
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Note: A has the root as a failure node iff $\square \in S$.

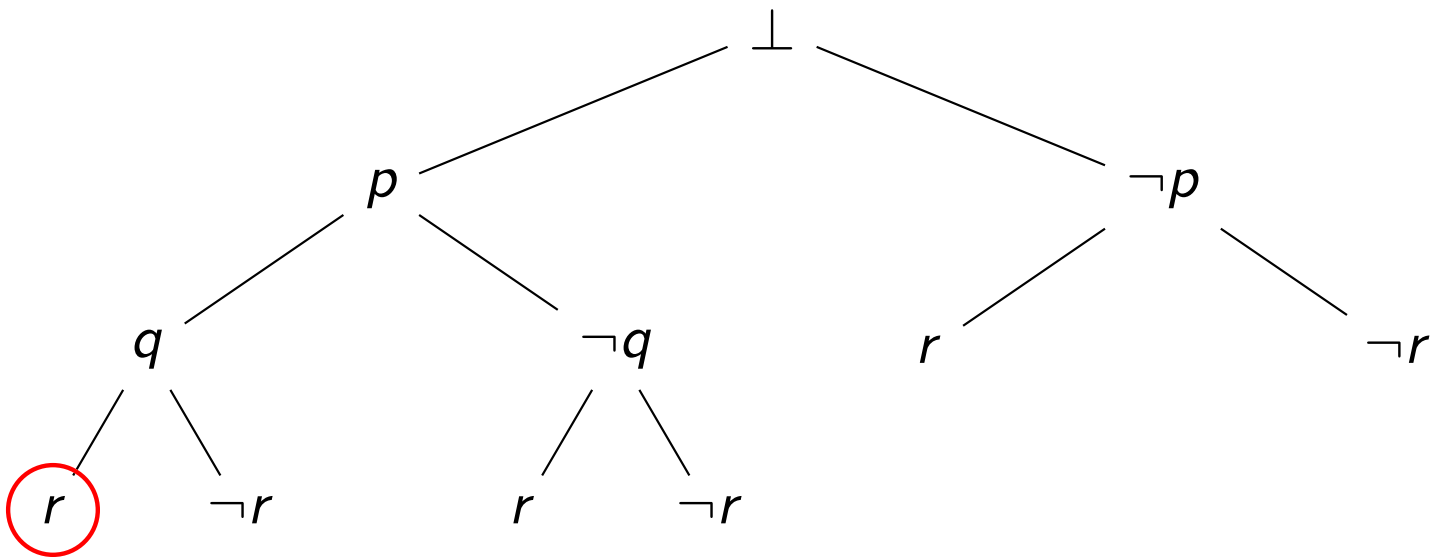
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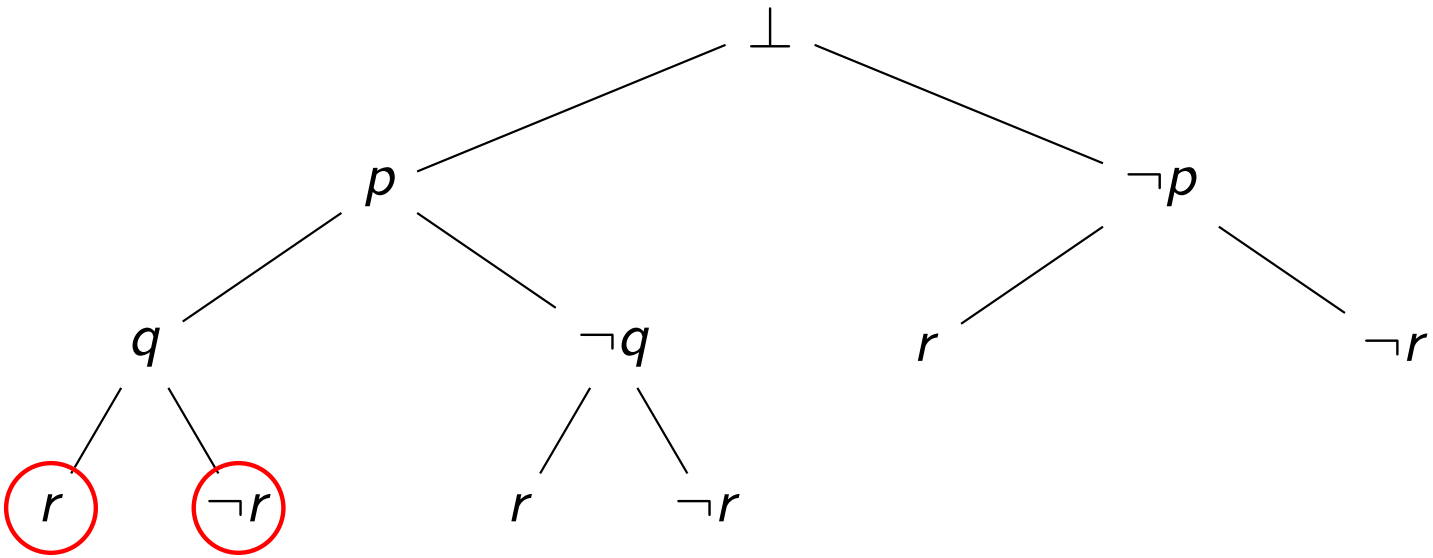
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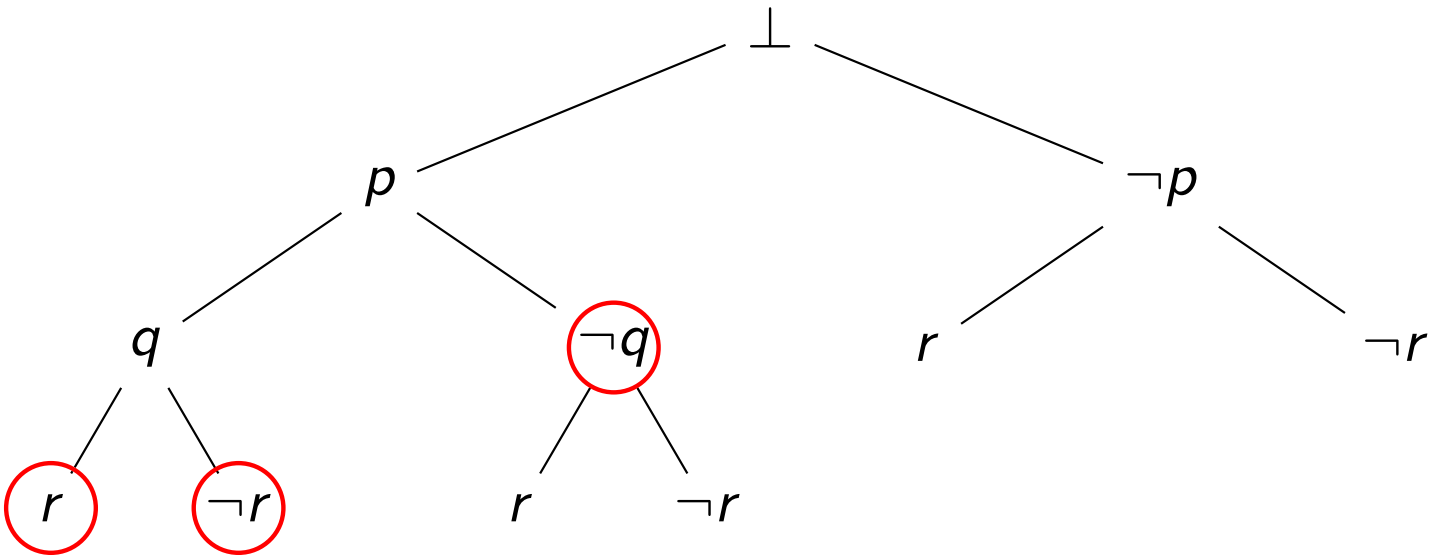
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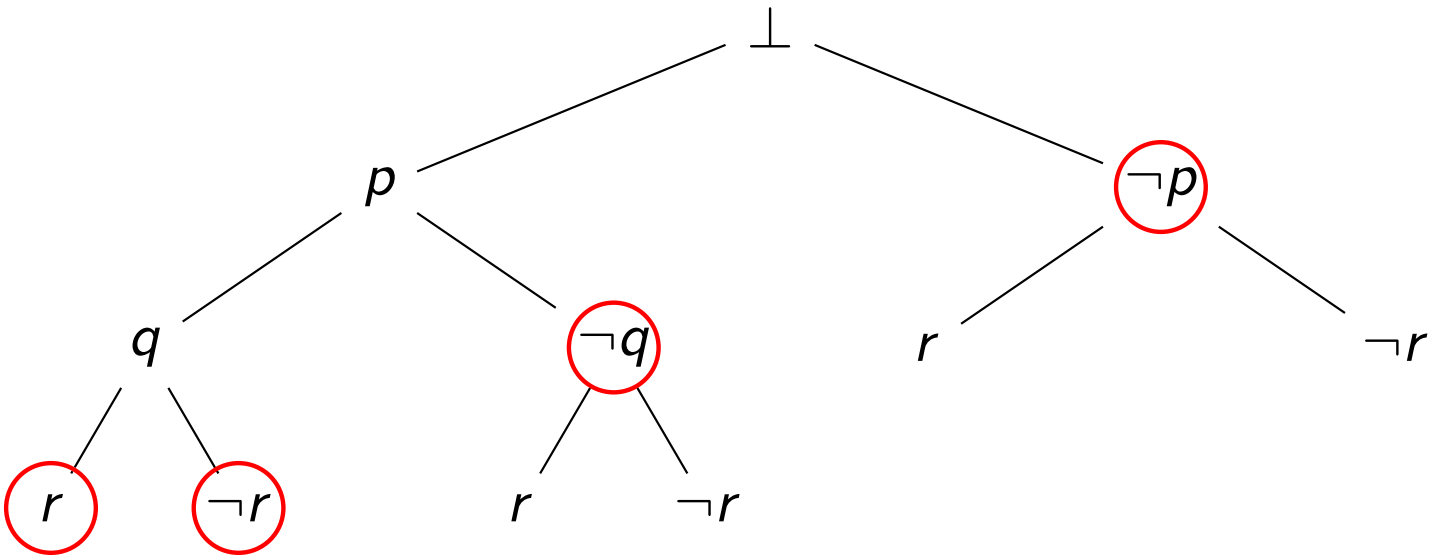
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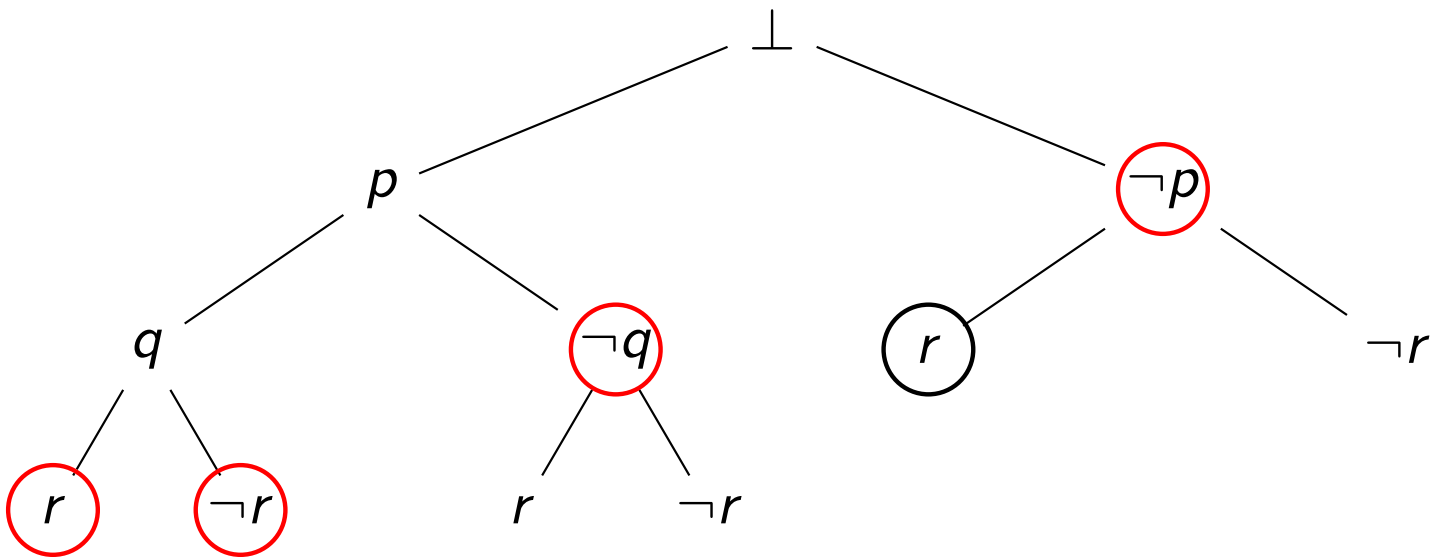
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2. $\neg q \vee r$
3. $\neg p \vee q$
4. p
5. $p \vee \neg r$



Failure Nodes – Example

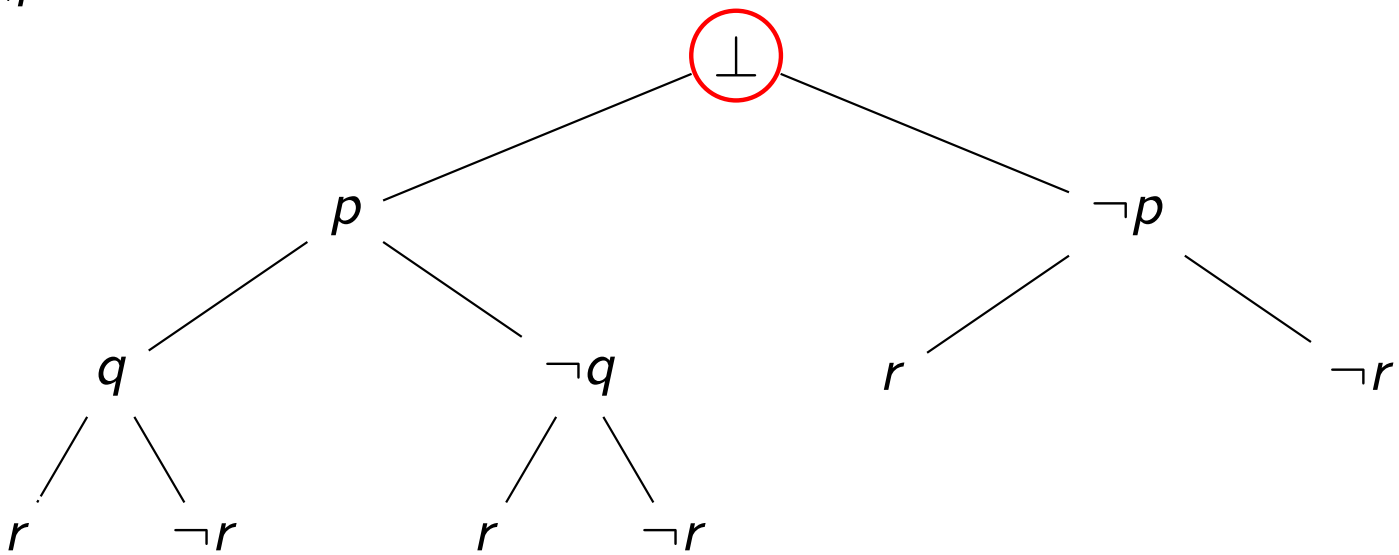
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Not a failure node: parent node falsifies clause 4.

Failure Nodes – Example

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2. $\neg q \vee r$
3. $\neg p \vee q$
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5. $p \vee \neg r$
6. $\square$



The empty clause is falsified by the root node

Closed Semantic Trees

Definition 7.4.

Given a semantic tree and a clause set S, a branch of the tree is closed if it contains a failure node.

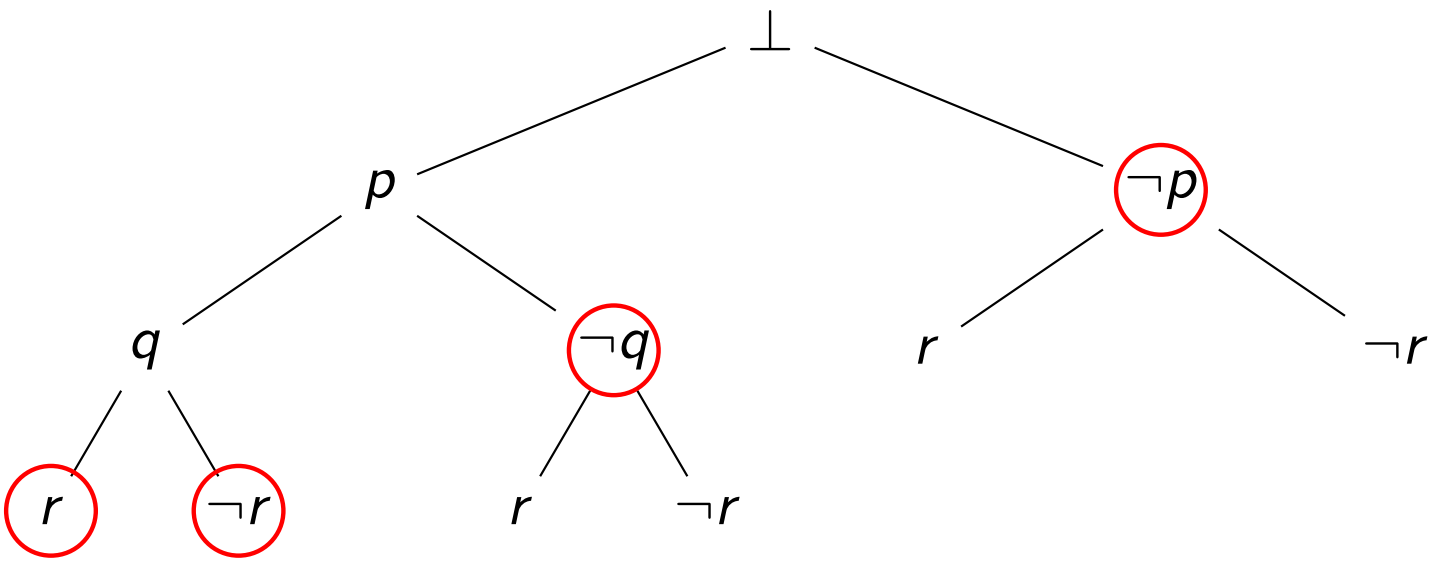
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Given a semantic tree and a clause set S, a branch of the tree is closed if it contains a failure node.
The semantic tree is closed if all branches contain failure nodes.

Closed Semantic Tree – Example

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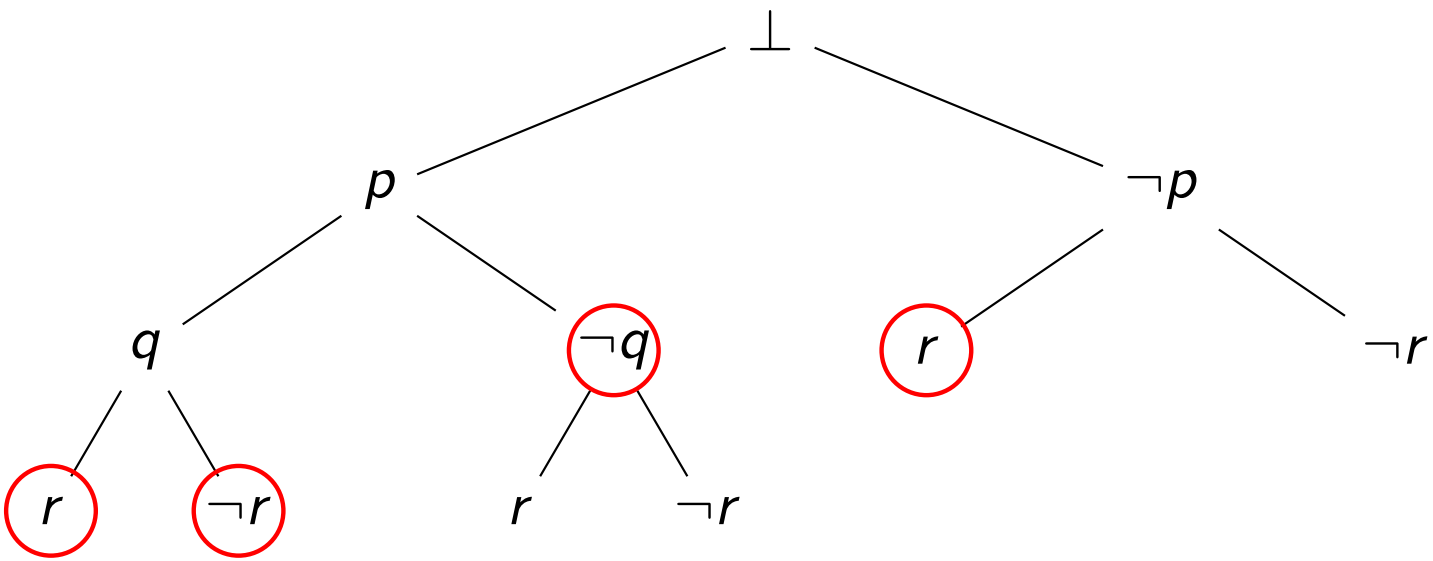


The semantic tree is closed for these 5 clauses.

Closed Semantic Tree – Example

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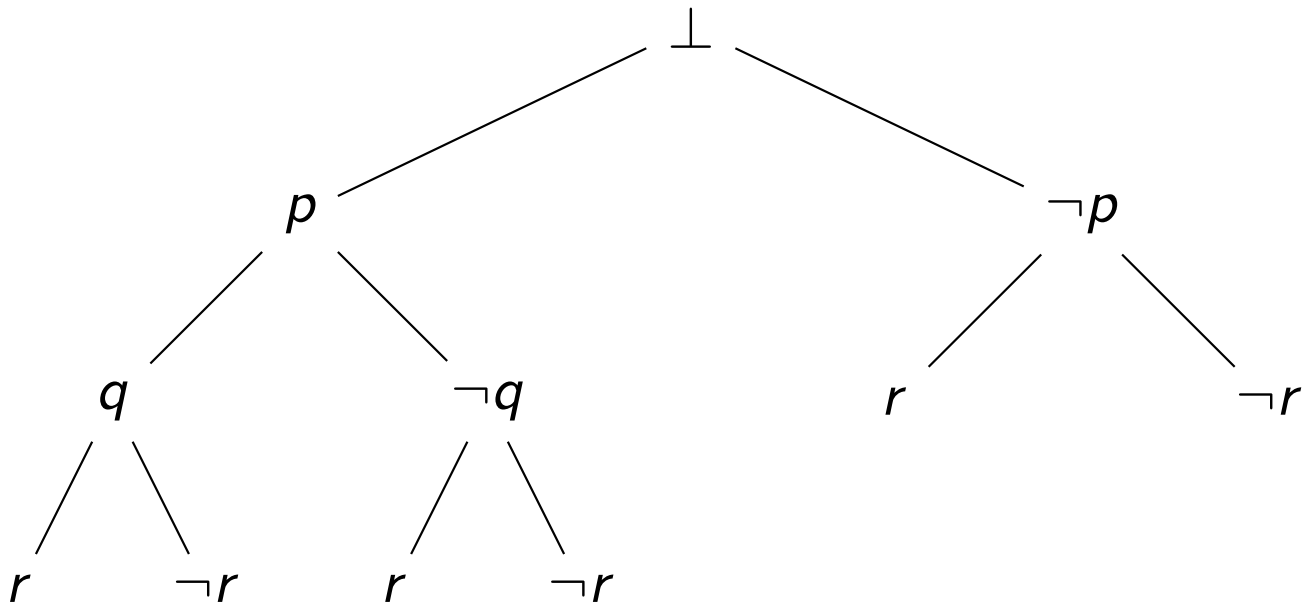
Without p, it is not closed.

Complete Semantic Trees

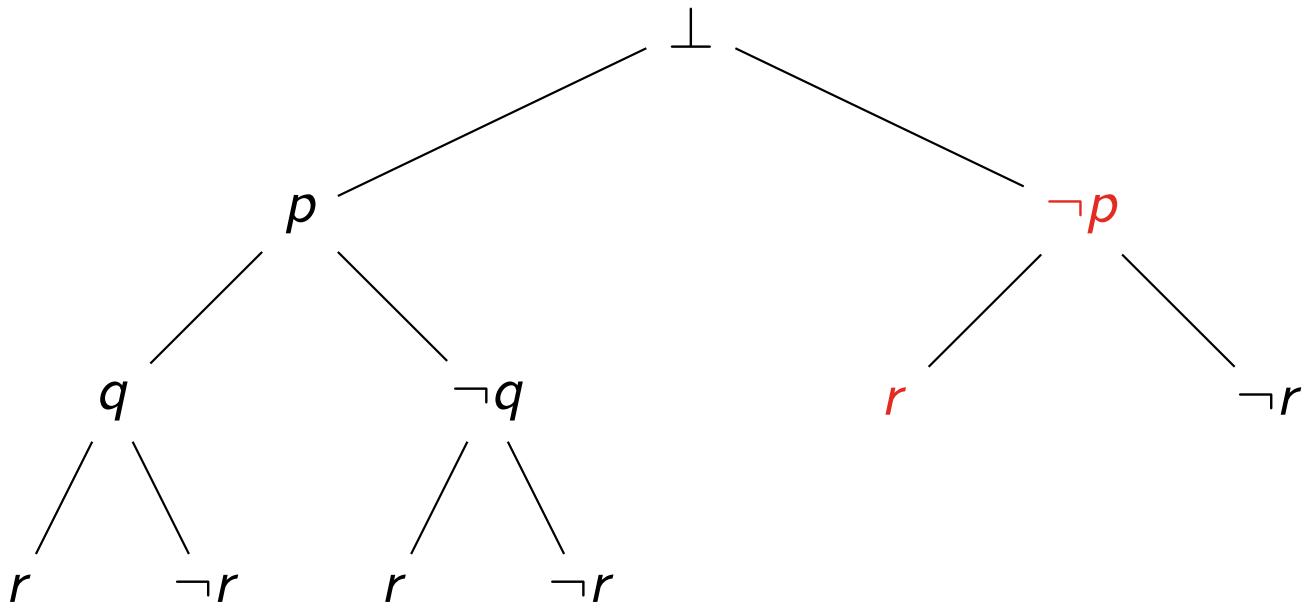
Definition 7.5.

A semantic tree is complete if for every atomic formula A and every branch (from root to leaf) either A or $\neg A$ occurs

Example: Complete Semantic Tree

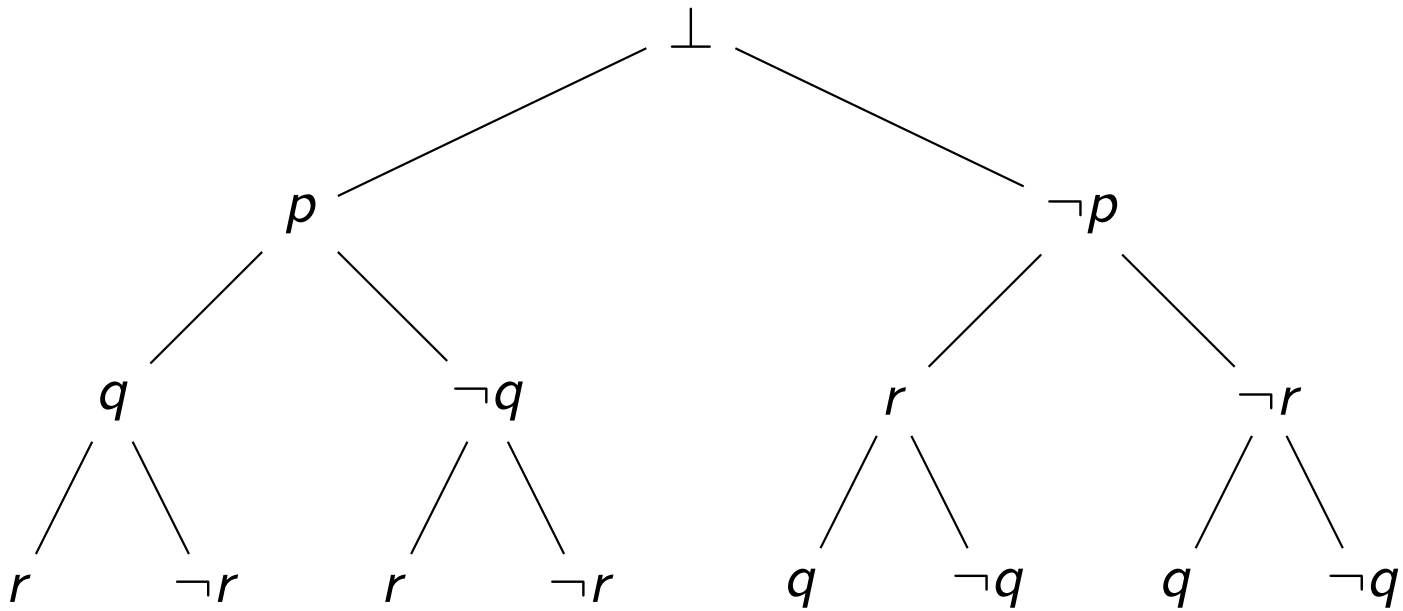


Example: Complete Semantic Tree



Not complete, since neither q nor $\neg q$ on branch

Example: Complete Semantic Tree



Complete for vocabulary $\{p, q, r\}$

Complete Semantic Trees

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For every interpretation $\mathcal{I}$ there is a branch $\mathcal{B}$ in a complete semantic tree with $\mathcal{I} = \mathcal{I}_{\mathcal{B}}$.

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A complete semantic tree 'enumerates' all possible interpretations.

Unsatisfiable Clause Sets close Semantic Trees

Theorem 7.1.

A clause set is unsatisfiable iff there is a closed semantic tree for it.

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$\Leftarrow$ Let S be a clause set and let a closed semantic tree be given. For any interpretation $\mathcal{I}$, there is a branch in the tree such that $\mathcal{I} \models L$ for all literals L on that branch. Since there is a failure node for some clause $C \in S$ on that branch,

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The falsifying nodes highest up on each branch are failure nodes. So the semantic tree is closed.

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This holds for arbitrary interpretations $\mathcal{I}$, so S is unsatisfiable. □

Resolution Steps from Closed Semantic Trees

Lemma 7.2.

Let S be an unsatisfiable clause set, with a closed semantic tree

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Resolution Steps from Closed Semantic Trees

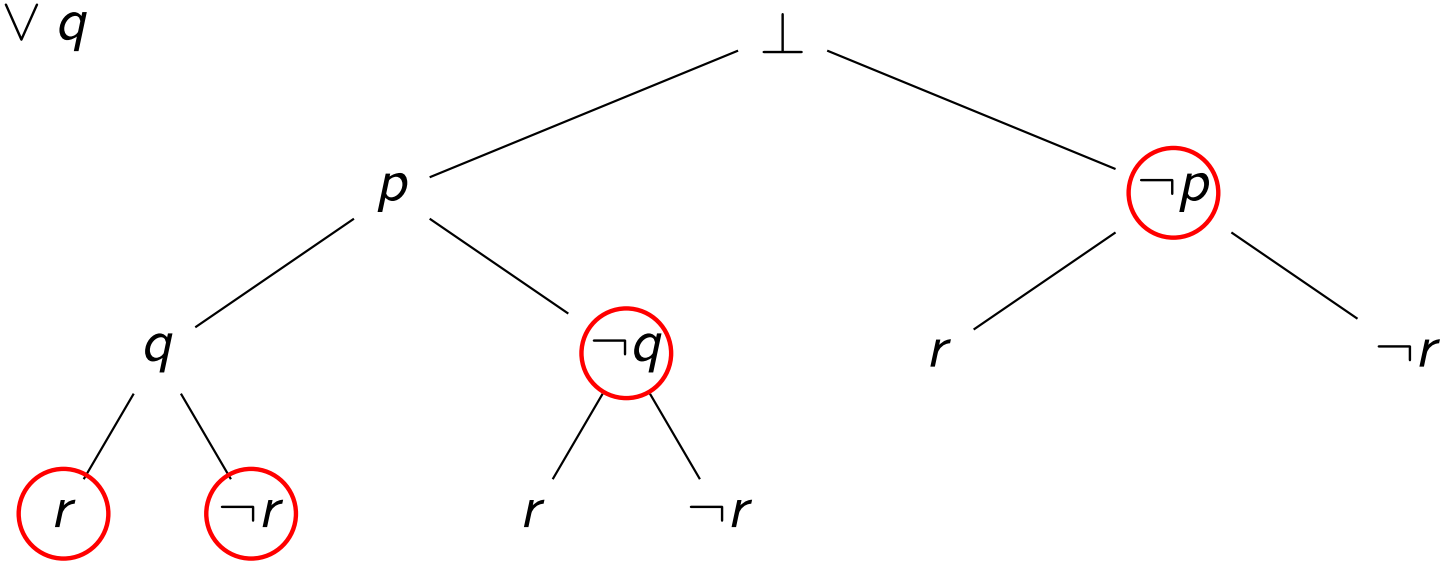
Lemma 7.2.

Let S be an unsatisfiable clause set, with a closed semantic tree, and $\square \notin S$. Then

- *a resolution step is possible from S,*
- *and the resulting clause set S' has a smaller closed semantic tree*

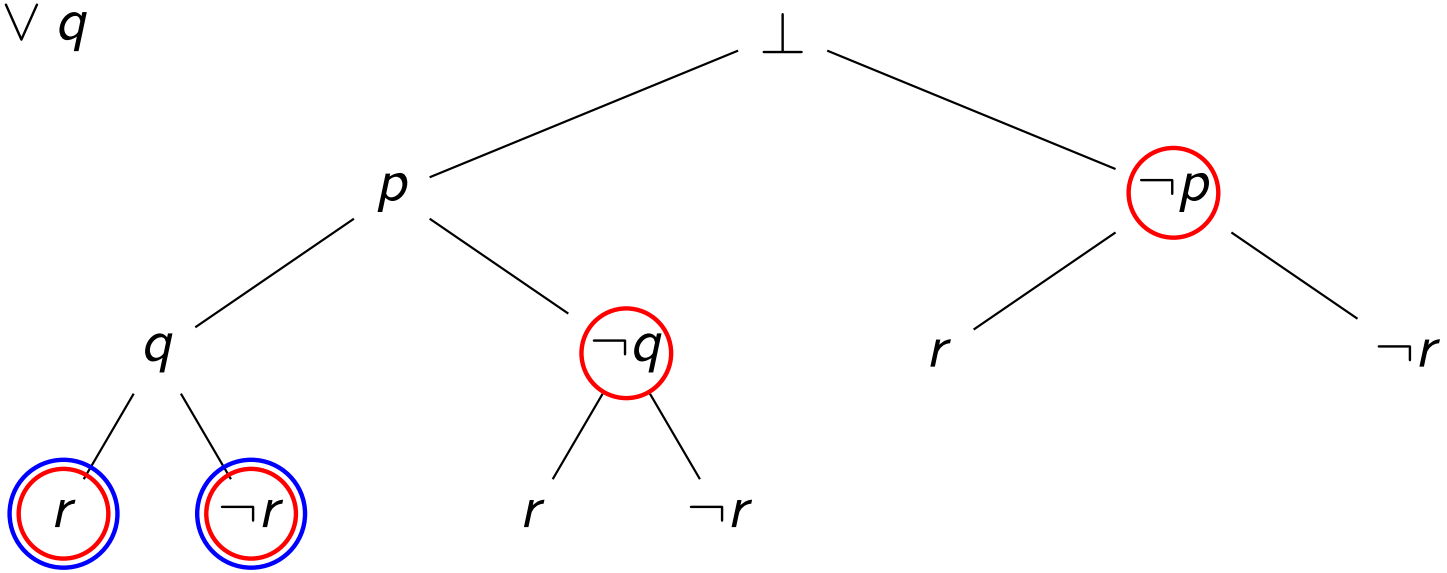
Idea of proof

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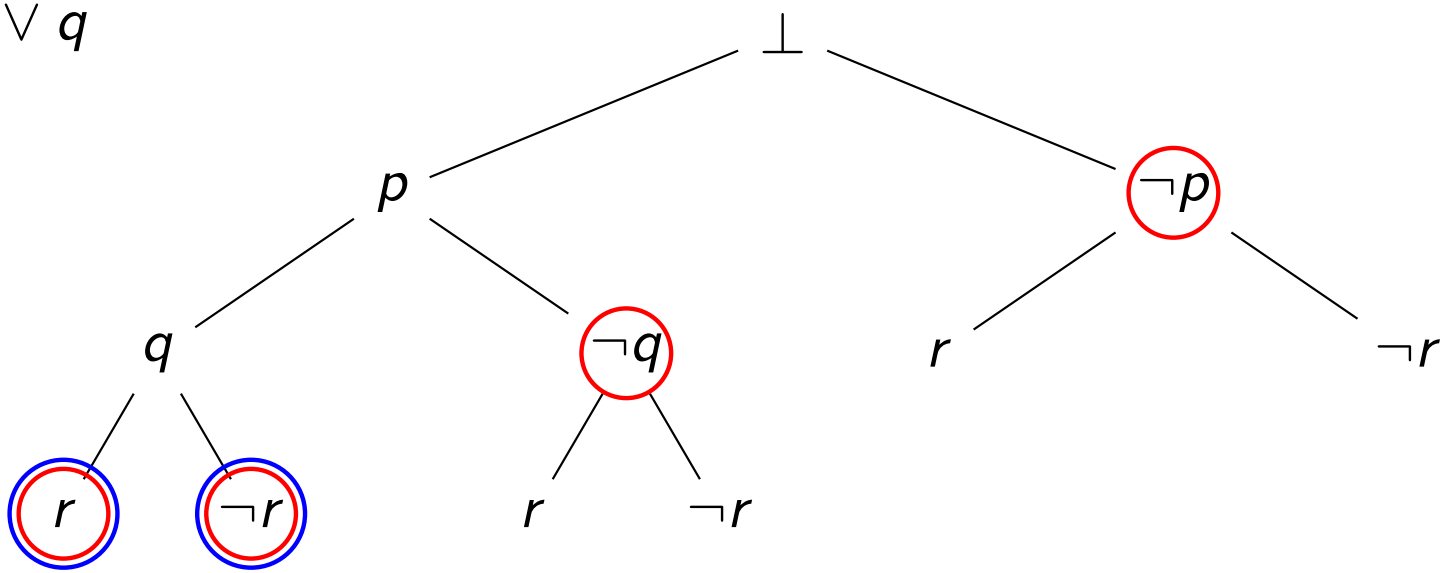
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- There are two sibling failure nodes

Idea of proof

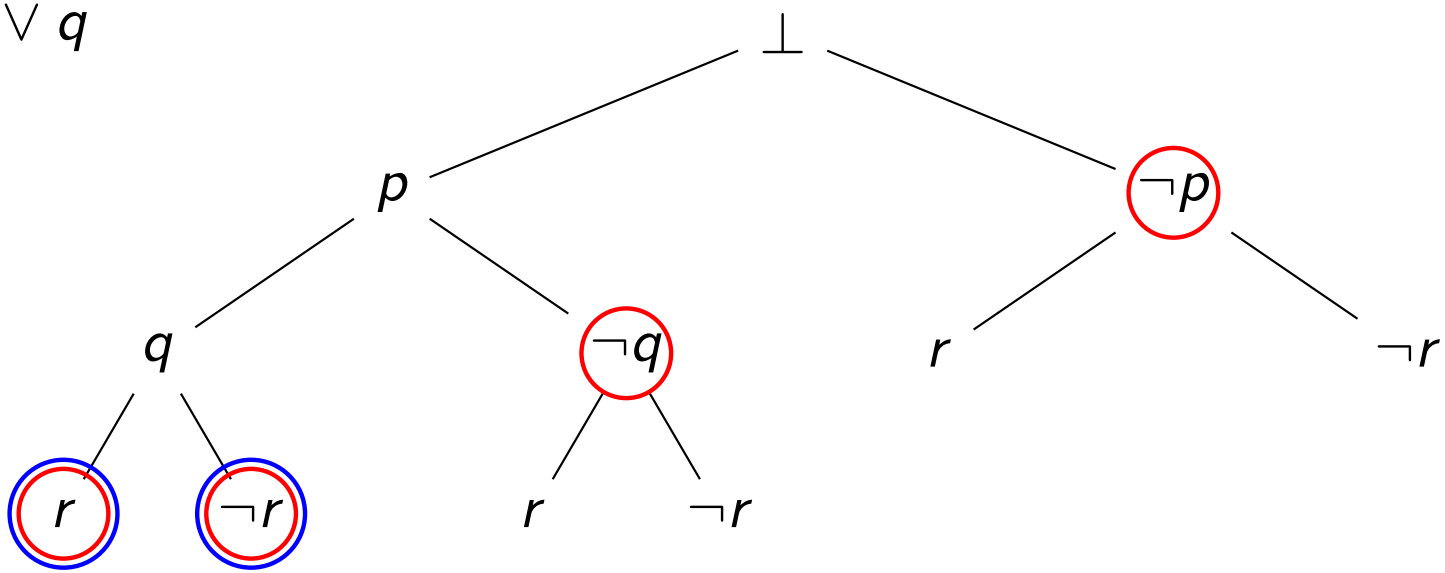
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- There are two sibling failure nodes
- They falsify two clauses

Idea of proof

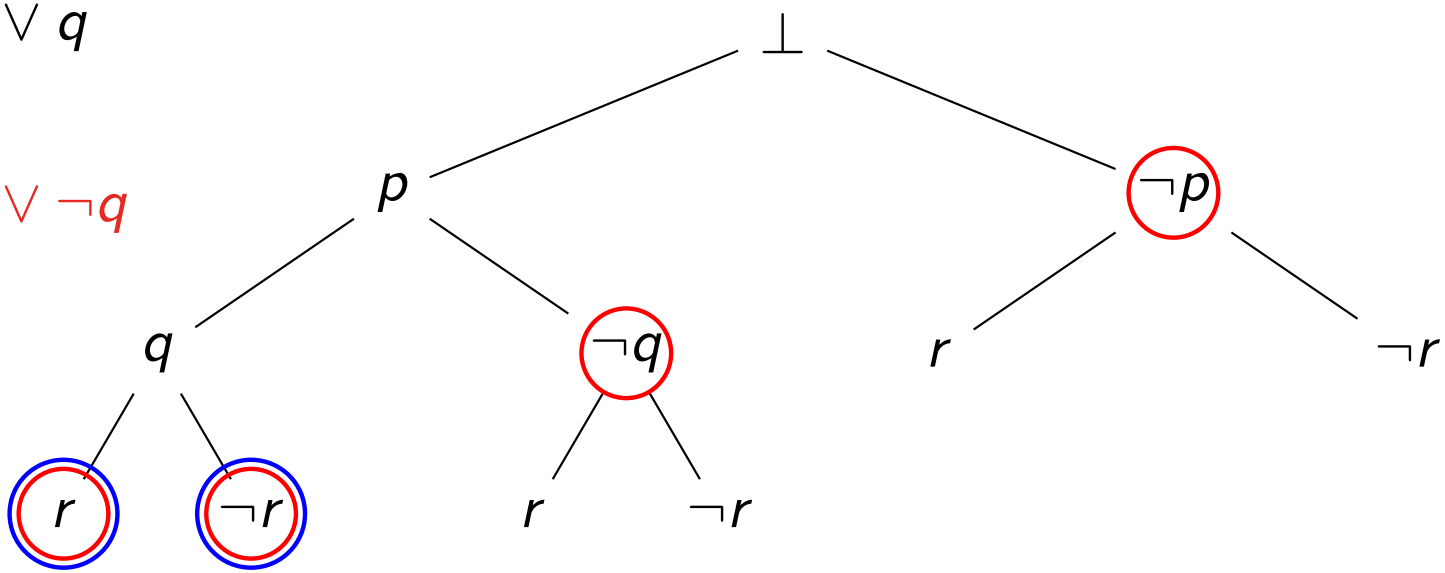
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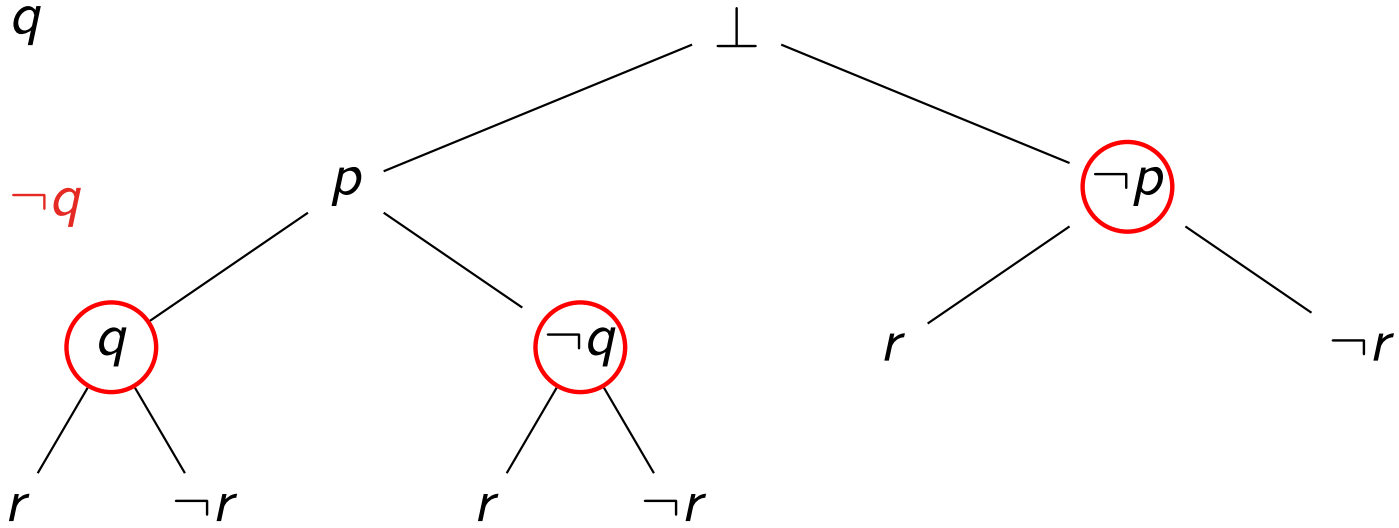
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- There are two sibling failure nodes
- They falsify two clauses with complementary literals
- They can be resolved to a new clause $\neg p \vee \neg q$
- Which is falsified by the parent node

There are two sibling failure nodes

- Let n_0 be the root.

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- ...
- This either finds sibling failure nodes...
- or it constructs a path in the tree without a failure node, but that is not possible.

Sibling Failure Nodes give Resolution Opportunities

- Let n_1 and n_2 be sibling failure nodes

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- Either n or a node between n and the root becomes a failure node.

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  - labeled A and $\neg A$.
- The parent node n of n_1 and n_2 does not falsify C_1 and C_2.
- Let N be the set of literals on the nodes up to and including n.
- Every literal in C_1 has its negation in $N \cup \{A\}$
- But *not every* literal in C_1 has its negation in N
- Therefore $\neg A \in C_1$
- Similarly $A \in C_2$
- C_1 and C_2 can be resolved to $C := (C_1 \setminus \{\neg A\}) \cup (C_2 \setminus \{A\})$
- Every literal in C has its negation in N, so C is falisfied by n.
- Either n or a node between n and the root becomes a failure node.
- This gives a closed semantic tree with ≥ 2 nodes less than before.

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- ...and S contain the empty clause $\square$.

$\square$