IN3070/4070 – Logic – Autumn 2024

Lecture 2: Propositional Logic & Sequent Calculus

Martin Giese

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Department of Informatics

University of Oslo

Today's Plan

- Motivation
- Syntax
- Semantics
- Logical Equivalence
- Satisfiability & Validity
- Summary
- Motivation
- Sequent Calculus
- Decision Procedure
- Summary

Outline

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- logical systems formalize reasoning similar to programming languages that formalize computation
- consequent separation of syntactical notions (formulae, proofs) and semantical notions (truth values, models)
- syntax defines what strings of symbols are "legal" formulae
- semantics assign meanings to legal formulae (through an interpretation of its symbols)

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Definition 2.2 (Propositional Formulae).

The propositional formulae, denoted A, B, C, F, G, H, are inductively defined as follows:

1. *Every atom $A \in \mathcal{P}$ is a formula.*
2. *If A and B are formulae, then $(\neg A)$, $(A \wedge B)$, $(A \vee B)$ and $(A \rightarrow B)$ are formulae.*

Let $\mathcal{F}$ be the set of all (legal) formulae.

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$((p \rightarrow q) \leftrightarrow ((\neg p) \rightarrow (\neg q)))$ is a (legal) formula, identical to $(p \rightarrow q) \leftrightarrow (\neg p \rightarrow \neg q)$ and $p \rightarrow q \leftrightarrow \neg p \rightarrow \neg q$

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Alternative connectives: $\Rightarrow$ and $\supset$ (for $\rightarrow$), $\Leftrightarrow$ (for $\leftrightarrow$), & (for $\wedge$)

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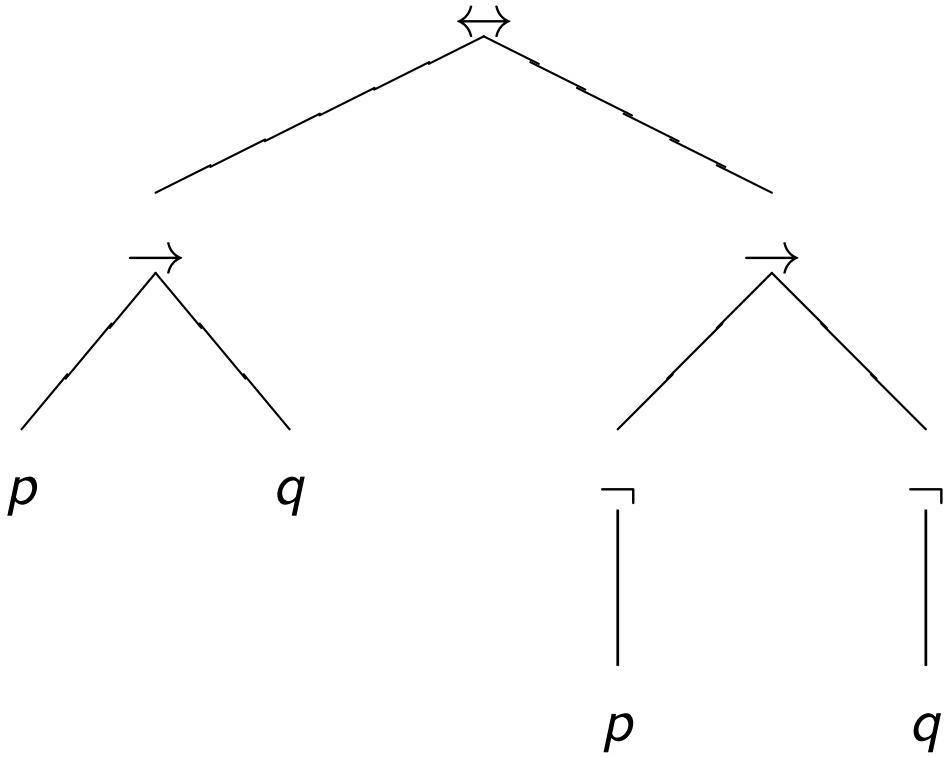

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Truth values are assigned to the atoms of a formula in order to evaluate the truth value of the formula.

Definition 3.1 (Interpretation).

Let $\mathcal{P}$ be a set of atoms.
An interpretation is a total function $\mathcal{I} : \mathcal{P} \to \{T, F\}$ that assigns one of the truth values T or F to every atom in $\mathcal{P}$.

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Simplfies some places, complicates things in others.

Semantics—Truth Value

Definition 3.2 (Truth Value).

Let $\mathcal{I}$ be an interpretation. The truth value $v_{\mathcal{I}}(A)$ of A under $\mathcal{I}$ is defined inductively as follows.

- *If $A \in \mathcal{P}$ is an atom, then $v_{\mathcal{I}}(A) = \mathcal{I}(A)$*
- $v_{\mathcal{I}}(\neg A) = T$ *if* $v_{\mathcal{I}}(A) = F$ *and* F *otherwise.*
- $v_{\mathcal{I}}(A \wedge B) = T$ *if* $v_{\mathcal{I}}(A) = T$ *and* $v_{\mathcal{I}}(B) = T$*, and* F *otherwise.*
- $v_{\mathcal{I}}(A \vee B) = F$ *if* $v_{\mathcal{I}}(A) = F$ *and* $v_{\mathcal{I}}(B) = F$*, and* T *otherwise.*
- $v_{\mathcal{I}}(A \rightarrow B) = F$ *if* $v_{\mathcal{I}}(A) = T$ *and* $v_{\mathcal{I}}(B) = F$*, and* T *otherwise.*

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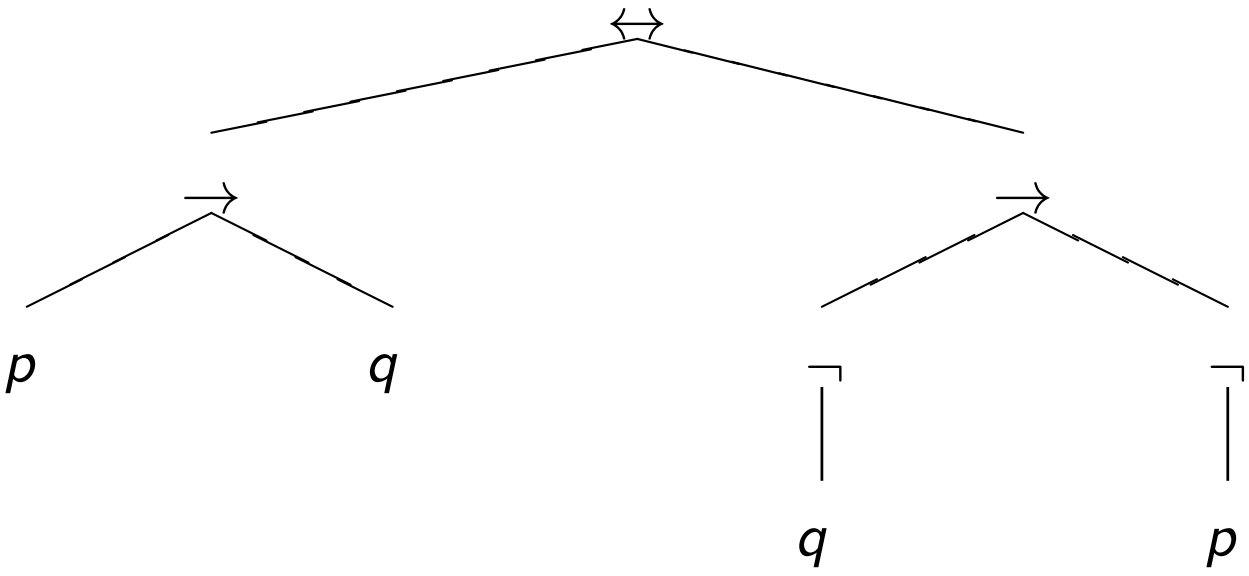
- $v_{\mathcal{I}}(A \rightarrow B) = F$ *if* $v_{\mathcal{I}}(A) = T$ *and* $v_{\mathcal{I}}(B) = F$*, and* T *otherwise.*

Note: For the equivalence connective, it follows that

- $v_{\mathcal{I}}(A \leftrightarrow B) = T$ if $v_{\mathcal{I}}(A) = v_{\mathcal{I}}(B)$, and F otherwise.

Semantics — Truth Value

Example: Let $A = (p \to q) \leftrightarrow (\neg q \to \neg p)$
with $\mathcal{I}(p) = F$ and $\mathcal{I}(q) = T$.



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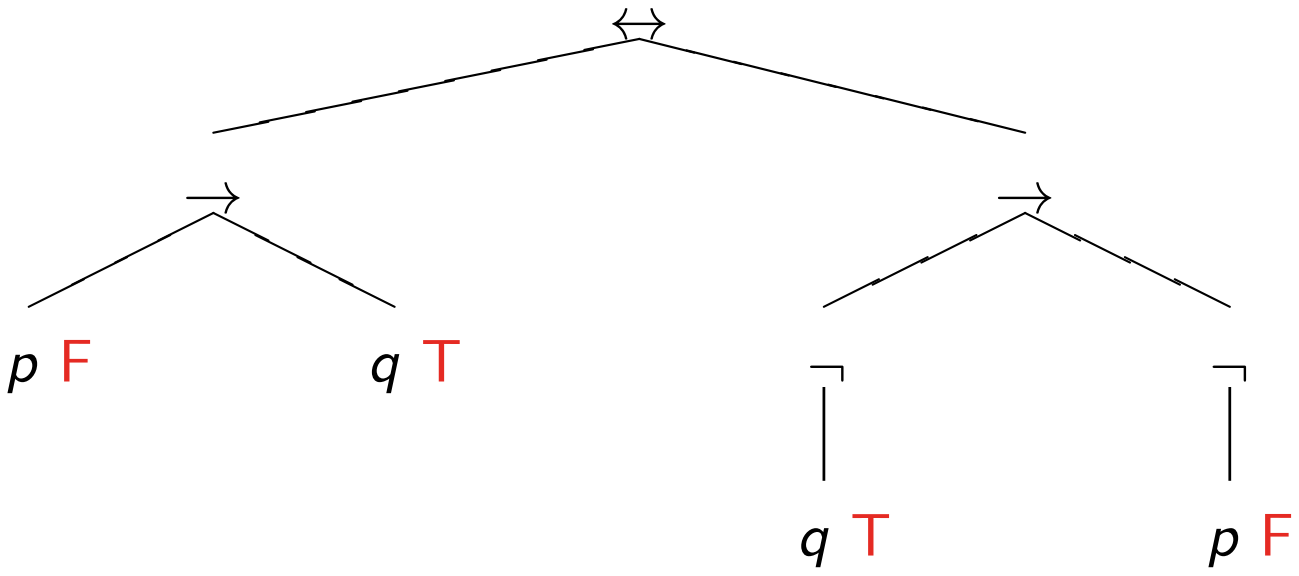

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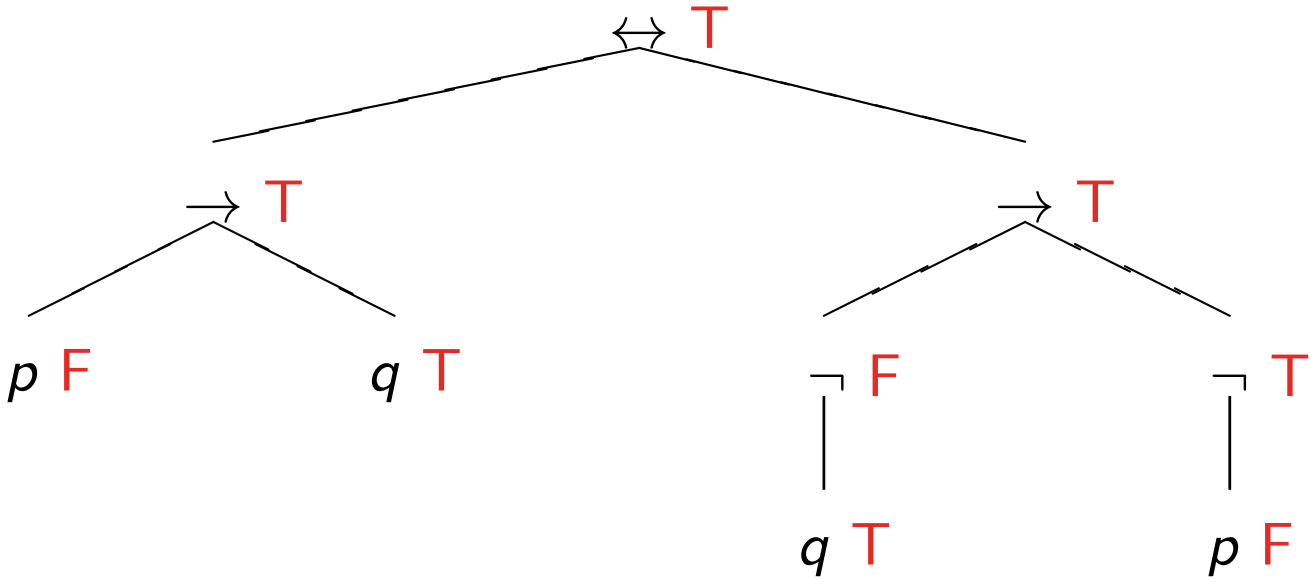

Truth Tables

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Definition 3.3 (Truth Table).

Let $A \in \mathcal{F}$ with n atoms. A truth table has $n+1$ columns and 2^n rows. There is a column for each atom in A, plus a column for the formula A. The first n columns specify all possible interpretations $\mathcal{I}$ that map atoms in A to $\{T, F\}$. The last column shows $v_{\mathcal{I}}(A)$, the truth value of A for each interpretation $\mathcal{I}$.

p_1	p_2	$\dots$	p_n	A
T	T	$\dots$	T	$v_{\mathcal{I}}(A)$
T	T	$\dots$	F	$v_{\mathcal{I}}(A)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
F	F	$\dots$	F	$v_{\mathcal{I}}(A)$

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T	F	
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p	q	$p \to q$	$\neg p$	$\neg q$	$\neg q \to \neg p$	$(p \to q) \leftrightarrow (\neg q \to \neg p)$
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p	q	$p \to q$	$\neg p$	$\neg q$	$\neg q \to \neg p$	$(p \to q) \leftrightarrow (\neg q \to \neg p)$
T	T	T	F	F	T	T
T	F	F	F	T	F	T
F	T	T	T	F	T	T
F	F	T	T	T	T	T

Material Implication

The operator of $p \rightarrow q$ is called material implication.

- p is the antecedent and q is the consequent
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"Earth is farther from the sun than Venus" $\rightarrow$ "$1 + 1 = 3$"
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"Earth is farther from the sun than Mars" $\rightarrow$ "$1 + 1 = 3$"
is true(!) as the falsity of the antecedent by itself is sufficient to ensure the truth of the implication

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Let $A_1, A_2 \in \mathcal{F}$. If $v_{\mathcal{I}}(A_1) = v_{\mathcal{I}}(A_2)$ for all interpretations $\mathcal{I}$, then A_1 is logically equivalent to A_2, denoted $A_1 \equiv A_2$.

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- If $v_{\mathcal{I}}(A \vee B) = F$, then $v_{\mathcal{I}}(A) = F$ and $v_{\mathcal{I}}(B) = F$. Thus, $v_{\mathcal{I}}(B \vee A) = F$.

Since $\mathcal{I}$ was chosen arbitrarily, $v_{\mathcal{I}}(A \vee B) = v_{\mathcal{I}}(B \vee A)$ for all interpretations. $\square$

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Relationship between $\leftrightarrow$ and $\equiv$

- equivalence, $\leftrightarrow$, is a binary connective that appears in formulae
- logical equivalence, $\equiv$, is a property of pairs of formulae
- similar vocabulary, but $\leftrightarrow$ is part of the object language, whereas $\equiv$ is part of the metalanguage that we use to reason about the object language

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The following formulae are logical equivalent (more in [Ben-Ari, 2.3.3]):

$A \vee true \equiv true$

$A \vee false \equiv A$

$A \rightarrow true \equiv true$

$A \rightarrow false \equiv \neg A$

$A \equiv A \wedge A$

$A \vee B \equiv B \vee A$

$A \vee (B \vee C) \equiv (A \vee B) \vee C$

$A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$

$A \wedge true \equiv A$

$A \wedge false \equiv false$

$true \rightarrow A \equiv A$

$false \rightarrow A \equiv true$

$A \equiv A \vee A$

$A \wedge B \equiv B \wedge A$

$A \wedge (B \wedge C) \equiv (A \wedge B) \wedge C$

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Contrapositive: $A \rightarrow B \equiv \neg B \rightarrow \neg A$

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Satisfiability and Validity

Definition 5.1 (Satisfiable, Model, Valid, Unsatisfiable, Invalid).

Let $A \in \mathcal{F}$.

- *A is satisfiable iff $v_{\mathcal{I}}(A) = T$ for some interpretation $\mathcal{I}$.*
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- *A set of formulae $U = \{A_1, ...\}$ is (simultaneously) satisfiable iff there exists an interpretation $\mathcal{I}$ such that $v_{\mathcal{I}}(A_i) = T$ for all i; otherwise U is unsatisfiable. The satisfying interpretation is a model of U.*

Satisfiability and Validity

There is a close relation between these four semantical concepts.

Theorem 5.1 (Satisfiable, Valid, Unsatisfiable, Invalid).

Let $A \in F$. A is valid iff $\neg A$ is unsatisfiable. A is satisfiable iff $\neg A$ is invalid.

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If A is satisfiable then for some interpretation $\mathcal{I}$, $v_{\mathcal{I}}(A) = T$. By definition of the truth value of negation, $v_{\mathcal{I}}(\neg A) = F$ so that $\neg A$ is invalid. Conversely, if $v_{\mathcal{I}}(\neg A) = F$ then $v_{\mathcal{I}}(A) = T$. □

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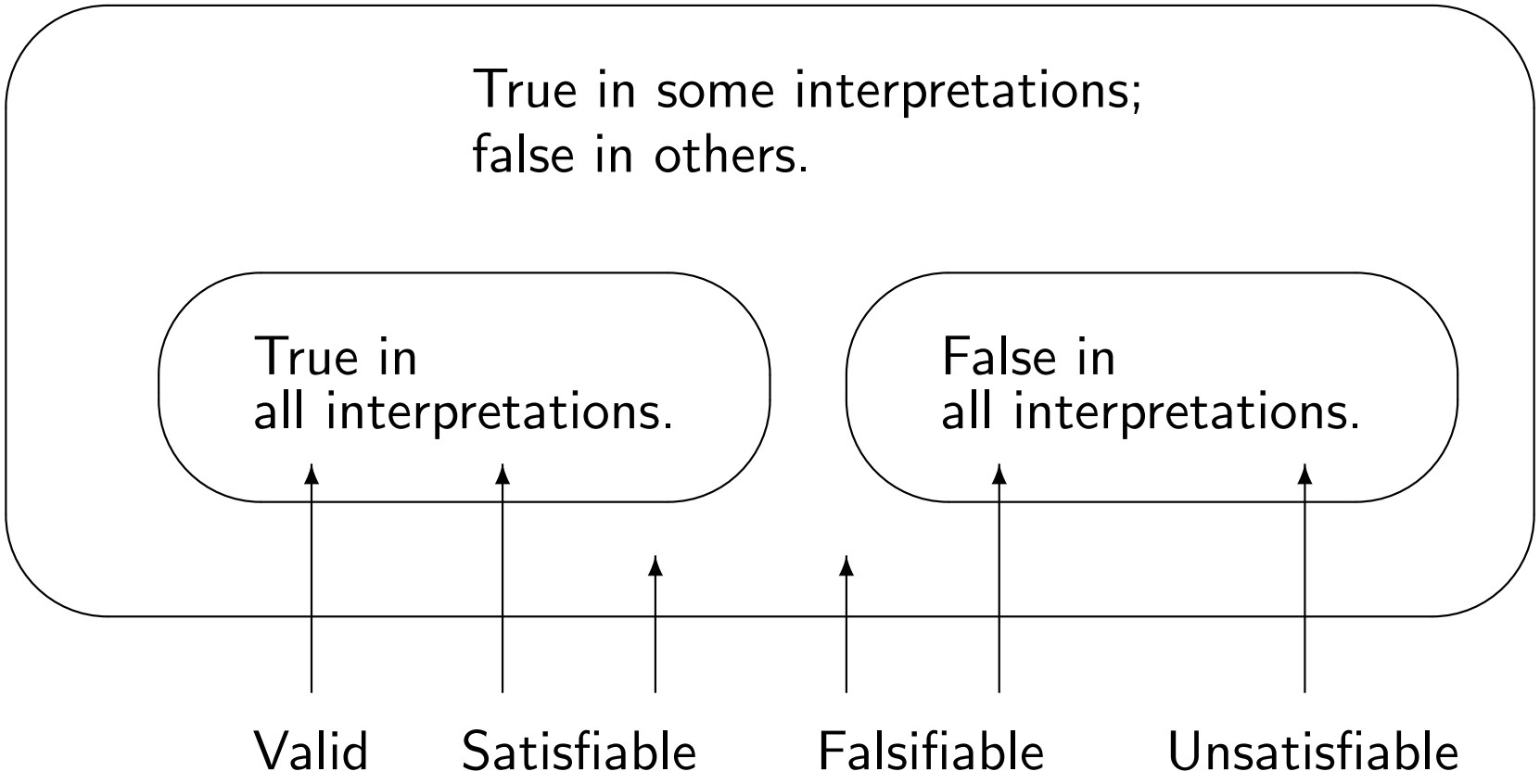

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Let $\mathcal{U} \subseteq \mathcal{F}$ be a set of (propositional) formulae. An algorithm is a decision procedure for $\mathcal{U}$ if given a formula $A \in \mathcal{F}$, it terminates and returns the answer "yes" if $A \in \mathcal{U}$ and the answer "no" if $A \notin \mathcal{U}$.

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For a given formula A with n atoms, use truth tables to evaluate truth values for A. If $v_{\mathcal{I}}(A) = T$ for all 2^n possible interpretations $\mathcal{I}$, then answer "yes"; otherwise answer "no". □

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This method is not very efficient; more efficient procedures will be introduced later.

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Proof. Left as an exercise.

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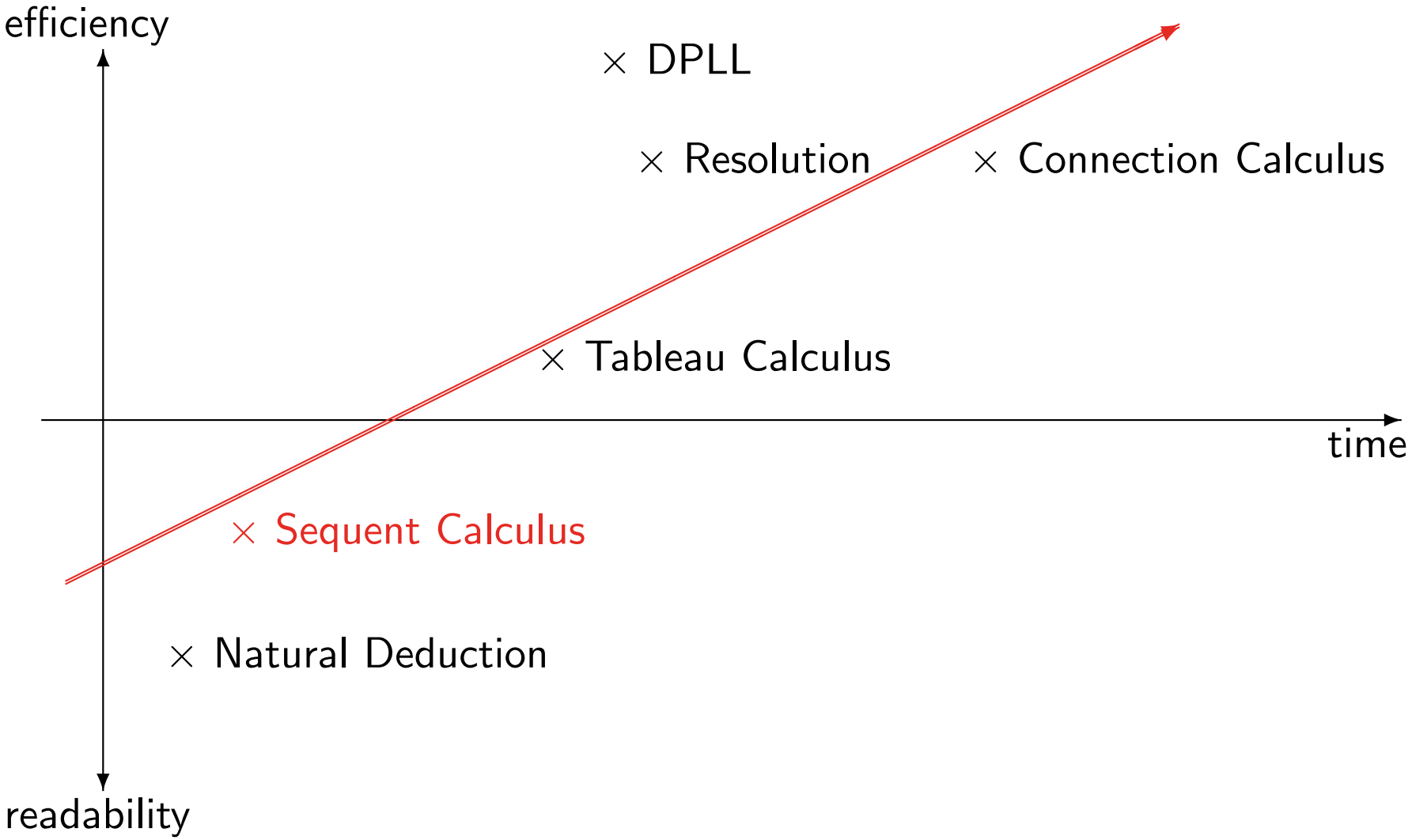
Proof Search Calculi



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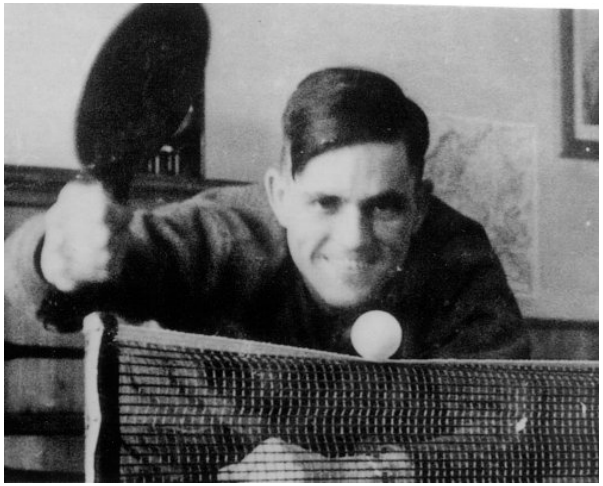

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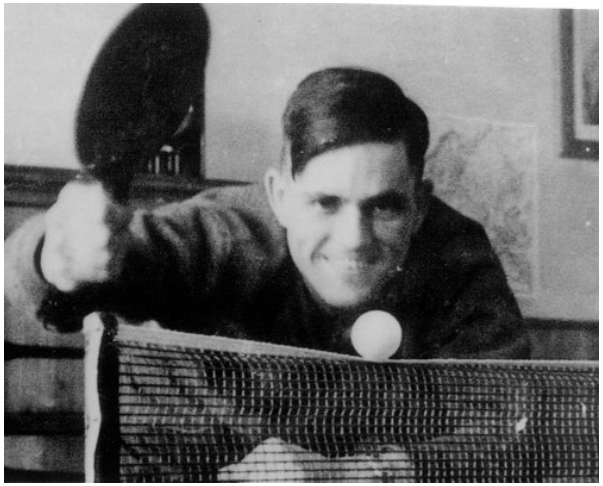

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- Tools for investigating mathematical reasoning.

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The Sequent Calculus LK

- Sequent proofs are trees labeled with sequents.
- Example:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{p \implies p, q}\,\text{axiom} \qquad \dfrac{}{p, q \implies q}\,\text{axiom}}{p, p \to q \implies q}\,\to\text{-left}}{p \wedge (p \to q) \implies q}\,\wedge\text{-left}}{\implies p \wedge (p \to q) \to q}\,\to\text{-right}$$

- The formula we try to show is at the root (bottom)
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- So let's look at the rules in detail!

LK — Rules for Conjunction and Disjunction

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Calculus and Proof — General Definitions

Definition 8.2 (Calculus/Deductive System).

A calculus consists of axioms and inference rules.

Axioms have the form $\frac{}{w}$ *; rules have the form* $\frac{w_1 \cdots w_n}{w}$

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Definition 8.3 (Proof, Derivation).

Let $\mathcal{A}=\{A_1, \ldots\}$ be axioms and $\mathcal{R}=\{R_1, \ldots\}$ be rules of a calculus.

1. *Let* $\frac{}{w}$ *be an instance of an axiom $A_i \in \mathcal{A}$. Then* $\frac{}{w}$ *is a proof of w.*
2. *Let* $\frac{w_1 \cdots w_n}{w}$ *be an instance of a rule $R_i \in \mathcal{R}$ and $\mathcal{D}_1, \ldots, \mathcal{D}_n$ proofs of $w_1, \ldots, w_n$. Then* $\frac{\mathcal{D}_1 \cdots \mathcal{D}_n}{w}$ *is a proof of w.*

A derivation is defined similarly, but leaves do not need to be axioms.

The Sequent Calculus LK

Definition 8.4 (Proofs in LK).

A proof of a formula A in the LK calculus is a proof of the sequent $\Longrightarrow A$ using the rules and axiom of LK. A formula A is provable, written $\vdash A$, iff there is a proof for A.

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Theorem 8.1 (Soundness and Completeness of LK).

The calculus of natural deduction LK is sound and complete, i.e.

- *if A is provable in LK, then A is valid (if $\vdash A$ then $\models A$)*
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Proof.

Next week! □

Outline

- Motivation
- Syntax
- Semantics
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- Satisfiability & Validity
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- Sequent Calculus
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