

IN3070/4070 – Logic – Autumn 2024

Lecture 3: LK: Soundness & Completeness

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DEPARTMENT OF
INFORMATICS



UNIVERSITY OF
OSLO

Today's Plan

- ▶ Semantics for Sequents
- ▶ Soundness
- ▶ Completeness

Outline

- ▶ Semantics for Sequents
- ▶ Soundness
- ▶ Completeness

Semantics for Sequents

Definition 1.1 (Valid sequent).

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Definition 1.2 (Countermodel/falsifiable sequent).

- An interpretation $\mathcal{I}$ is a *countermodel* for the sequent $\Gamma \implies \Delta$ if $v_{\mathcal{I}}(A) = T$ for all formulae $A \in \Gamma$ and $v_{\mathcal{I}}(B) = F$ for all formulae $B \in \Delta$

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- ▶ $p \Longrightarrow$ Countermodel: $\mathcal{I}(p) = T$
- ▶ $\Longrightarrow$ Countermodel: *all interpretations!*

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then $\mathcal{I} \models q$.
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Not provable

$$\frac{\frac{q \implies p, p}{\neg p \implies p, \neg q} \quad \frac{q, q \implies p}{q, \neg p \implies \neg q}}{\neg p, p \rightarrow q \implies \neg q}$$

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Theorem 2.1.

The sequent calculus LK is sound.

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 - ▶ Show that P is preserved by the induction steps

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- ▶ We let Γ , Δ , A and B in the rule stand for arbitrary (sets of) propositional formulae

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- ▶ Thus, $\mathcal{I}$ falsifies the premiss.



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- ▶ Then $\mathcal{I}$ satisfies $\Gamma \cup \{A \rightarrow B\}$ and falsifies all formulae in Δ .
- ▶ Since $\mathcal{I}$ satisfies $A \rightarrow B$, by definition of model semantics,
 - (1) $\mathcal{I} \not\models A$, or
 - (2) $\mathcal{I} \models B$.



Proof for $\rightarrow$ -left

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$$\frac{\Gamma \implies A, \Delta \quad \Gamma, B \implies \Delta}{\Gamma, A \rightarrow B \implies \Delta} \rightarrow\text{-left}$$

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- ▶ In case (2), $\mathcal{I}$ falsifies the right premiss.



How to show the Soundness Theorem?

We show the following lemmas:

1. All LK-rules preserve falsifiability upwards.
2. An LK-derivation with a falsifiable root sequent has at least one falsifiable leaf sequent
3. All axioms are valid

Finally, we use these lemmas to show the soundness theorem.

Reminder: LK derivation

Definition 2.3 (LK Derivation).

1. Let $\Gamma \Rightarrow \Delta$ be a sequent. Then

$$\Gamma \Rightarrow \Delta$$

is an **LK-derivation** of $\Gamma \Rightarrow \Delta$.

2. Let $\frac{w_1 \quad \dots \quad w_n}{\Gamma \Rightarrow \Delta}$ be an instance of an LK rule, and $\mathcal{D}_1, \dots, \mathcal{D}_n$ derivations of $w_1, \dots, w_n$. Then

$$\frac{\mathcal{D}_1 \quad \dots \quad \mathcal{D}_n}{\Gamma \Rightarrow \Delta}$$

is an **LK-derivation** of $\Gamma \Rightarrow \Delta$.

Existence of a falsifiable leaf sequent

Lemma 2.2.

If an interpretation $\mathcal{I}$ falsifies the root sequent of an LK-derivation δ , then $\mathcal{I}$ falsifies at least one of the leaf sequents of δ .

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- ▶ Here, $\Gamma \Longrightarrow \Delta$ is both root sequent and (only) leaf sequent.
- ▶ Assume $\mathcal{I}$ falsifies $\Gamma \Longrightarrow \Delta$.
- ▶ Then $\mathcal{I}$ falsifies a leaf sequent in δ , namely $\Gamma \Longrightarrow \Delta$.



Continued.

Induction step: δ is a derivation of the form

$$\frac{\begin{array}{ccc} \mathcal{D}_1 & & \mathcal{D}_n \\ \vdots & & \vdots \\ \Gamma_1 \Longrightarrow \Delta_1 & \cdots & \Gamma_n \Longrightarrow \Delta_n \end{array}}{\Gamma \Longrightarrow \Delta} r$$

for some smaller derivations $\mathcal{D}_i$ with roots $\Gamma_i \Longrightarrow \Delta_i$.

Continued.

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- ▶ Therefore $\mathcal{I}$ falsifies $\Gamma_i \Rightarrow \Delta_i$ for some $i \in \{1, \dots, n\}$.



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- ▶ This is also a leaf sequent of δ



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- ▶ Then $\mathcal{I}$ satisfies the formula A in the succedent.



Proof of the Soundness Theorem for LK

Proof of soundness.

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- ▶ Then there is a countermodel $\mathcal{I}$ that falsifies $\Gamma \Longrightarrow \Delta$.
- ▶ We know from the previous Lemma that $\mathcal{I}$ falsifies at least one leaf sequent of $\mathcal{P}$.
- ▶ Then $\mathcal{P}$ has a leaf sequent that is not an axiom, since axioms are not falsifiable.
- ▶ So $\mathcal{P}$ cannot be an LK-proof.



Analysis

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- ▶ Shown individually for each rule
- ▶ Can add new rules, and just show “soundness” for those

Outline

- ▶ Semantics for Sequents
- ▶ Soundness
- ▶ Completeness

Completeness — Introduction

Definition 3.1 (Soundness).

*The calculus LK is **sound** if any LK-provable sequent is valid.*

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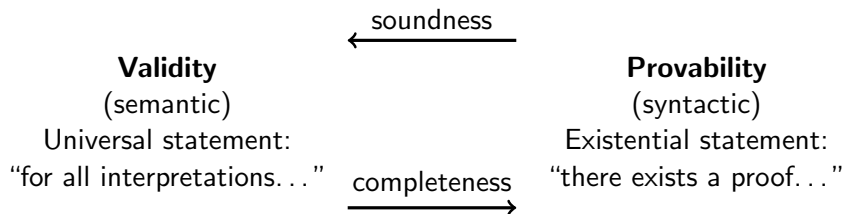
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An LK-machine?



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Soundness

All that is printed is valid.

An LK-machine?



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All that is valid will get printed.

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- Something can be sound without being complete.

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- ▶ Something can be sound without being complete.
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Theorem 3.1 (Completeness).

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If $\Gamma \implies \Delta$ is not provable in LK, then it is falsifiable.

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To show completeness of our calculus, we show the equivalent statement:

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This means that there is an interpretation that makes all formulae in Γ true and all formulae in Δ false.

Proof of Completeness

Assume $\Gamma \Rightarrow \Delta$ is not provable.

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- ▶ We construct an interpretation that falsifies $\Gamma \implies \Delta$. Let

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 - ▶ the leaf sequent of $\mathcal{B}$ is not an axiom.
- ▶ We construct an interpretation that falsifies $\Gamma \implies \Delta$. Let $\mathcal{B}^\top$ be the set of formulae that occur in an antecedent on $\mathcal{B}$, and

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Assume $\Gamma \implies \Delta$ is not provable.

- ▶ Construct a derivation $\mathcal{D}$ from $\Gamma \implies \Delta$ such that no further rule applications are possible. “A maximal derivation.”
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$\mathcal{B}^\top$ be the set of formulae that occur in an antecedent on $\mathcal{B}$, and
 $\mathcal{B}^\perp$ be the set of formulae that occur in a succedent on $\mathcal{B}$, and
 $\mathcal{I}_{\mathcal{B}}$ be the interpretation that makes all atomic formulae in $\mathcal{B}^\top$ true and all other atomic formulae (in particular those in $\mathcal{B}^\perp$) false.

Example

$$\begin{array}{c}
 \frac{p \Rightarrow q, p}{p \rightarrow q, p \Rightarrow q} \quad \frac{q, p \Rightarrow q}{p \rightarrow q, p \Rightarrow q} \quad \frac{r \Rightarrow q, p}{p \rightarrow q, r \Rightarrow q} \quad \frac{q, r \Rightarrow q}{p \rightarrow q, r \Rightarrow q} \\
 \hline
 p \rightarrow q, p \vee r \Rightarrow q \\
 \hline
 p \rightarrow q \Rightarrow (p \vee r) \rightarrow q
 \end{array}$$

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 \frac{p \Rightarrow q, p}{p \rightarrow q, p \Rightarrow q} \quad \frac{q, p \Rightarrow q}{p \rightarrow q, p \Rightarrow q} \quad \frac{r \Rightarrow q, p}{p \rightarrow q, r \Rightarrow q} \quad \frac{q, r \Rightarrow q}{p \rightarrow q, r \Rightarrow q} \\
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 \frac{p \implies q, p}{p \rightarrow q, p \implies q} \quad \frac{q, p \implies q}{p \rightarrow q, p \implies q} \quad \frac{r \implies q, p}{p \rightarrow q, r \implies q} \quad \frac{q, r \implies q}{p \rightarrow q, r \implies q} \\
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 \frac{\overline{p \Rightarrow q, p}}{p \rightarrow q, p \Rightarrow q} \quad \frac{\overline{q, p \Rightarrow q}}{p \rightarrow q, p \Rightarrow q} \quad \frac{\overline{r \Rightarrow q, p} \quad \overline{q, r \Rightarrow q}}{p \rightarrow q, r \Rightarrow q} \\
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Example

$$\begin{array}{c}
 \frac{p \Rightarrow q, p}{p \rightarrow q, p \Rightarrow q} \quad \frac{\textcolor{red}{q}, p \Rightarrow \textcolor{red}{q}}{p \rightarrow q, p \Rightarrow q} \quad \frac{r \Rightarrow q, p}{p \rightarrow q, r \Rightarrow q} \quad \frac{q, r \Rightarrow q}{p \rightarrow q, r \Rightarrow q} \\
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We see that the branch $\mathcal{B}$ with leaf sequent $r \Rightarrow q, p$ is not closed.

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$$\mathcal{B}^\top = \{r, p \rightarrow q, p \vee r\}$$

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$$\mathcal{B}^\perp = \{q, p, (p \vee r) \rightarrow q\}$$

Example

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To show: this interpretation falsifies the root sequent.

Proof of Completeness, cont.

- We show by structural induction *on propositional formulae* that the interpretation $\mathcal{I}_{\mathcal{B}}$ makes all formulae in $\mathcal{B}^{\top}$ true, and all formulae in $\mathcal{B}^{\perp}$ false.

Proof of Completeness, cont.

- ▶ We show by structural induction *on propositional formulae* that the interpretation $\mathcal{I}_{\mathcal{B}}$ makes all formulae in $\mathcal{B}^{\top}$ true, and all formulae in $\mathcal{B}^{\perp}$ false.
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If $A \in \mathcal{B}^{\top}$, then $\mathcal{I}_{\mathcal{B}} \models A$.

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- ▶ Our statment holds for $A \in \mathcal{B}^{\top}$ because that is how we defined $\mathcal{I}_{\mathcal{B}}$.

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- ▶ For $A \in \mathcal{B}^{\perp}$, $A \notin \mathcal{B}^{\top}$ because atoms do not disappear from a branch and $\mathcal{B}$ contains no axiom.

Proof of Completeness, cont.

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Induction step: From the assumption (IH) that the statement holds for A and B , we must show that it holds for $\neg A$, $(A \wedge B)$, $(A \vee B)$ og $(A \rightarrow B)$. These are four cases, of which we show three here.

Case: Negation in antecedent/succedent

Assume that $\neg A \in \mathcal{B}^\top$.

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- ▶ A appears in a succedent, so we have $A \in \mathcal{B}^\perp$.

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- ▶ A appears in a succedent, so we have $A \in \mathcal{B}^\perp$.
- ▶ By the IH, we have $\mathcal{I}_\mathcal{B} \not\models A$.

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- ▶ A appears in a succedent, so we have $A \in \mathcal{B}^\perp$.
- ▶ By the IH, we have $\mathcal{I}_\mathcal{B} \not\models A$.
- ▶ By definition of model semantics, $\mathcal{I}_\mathcal{B} \models \neg A$.

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- ▶ Since the derivation is maximal, $\neg$ -right *is* eventually applied
- ▶ A appears in an antecedent, so we have $A \in \mathcal{B}^\top$.

Case: Negation in antecedent/succedent

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- ▶ $\neg A$ appears in a succedent, it can't 'go away' unless $\neg$ -right is applied
- ▶ Since the derivation is maximal, $\neg$ -right *is* eventually applied
- ▶ A appears in an antecedent, so we have $A \in \mathcal{B}^\top$.
- ▶ By the IH, we have $\mathcal{I}_\mathcal{B} \models A$.

Case: Negation in antecedent/succedent

Assume that $\neg A \in \mathcal{B}^\top$.

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Assume that $\neg A \in \mathcal{B}^\perp$.

- ▶ $\neg A$ appears in a succedent, it can't 'go away' unless $\neg$ -right is applied
- ▶ Since the derivation is maximal, $\neg$ -right *is* eventually applied
- ▶ A appears in an antecedent, so we have $A \in \mathcal{B}^\top$.
- ▶ By the IH, we have $\mathcal{I}_\mathcal{B} \models A$.
- ▶ By definition of model semantics, $\mathcal{I}_\mathcal{B} \not\models \neg A$.

Case: Conjunction in antecedent/succedent

Assume that $(A \wedge B) \in \mathcal{B}^\top$.

Case: Conjunction in antecedent/succedent

Assume that $(A \wedge B) \in \mathcal{B}^\top$.

- ▶ Since the derivation is maximal, we have $A \in \mathcal{B}^\top$ and $B \in \mathcal{B}^\top$.

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Assume that $(A \wedge B) \in \mathcal{B}^\top$.

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- ▶ By the IH, we have $\mathcal{I}_\mathcal{B} \models A$ and $\mathcal{I}_\mathcal{B} \models B$.
- ▶ By definition of model semantics, $\mathcal{I}_\mathcal{B} \models (A \wedge B)$.

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Assume that $(A \wedge B) \in \mathcal{B}^\perp$.

Case: Conjunction in antecedent/succedent

Assume that $(A \wedge B) \in \mathcal{B}^\top$.

- ▶ Since the derivation is maximal, we have $A \in \mathcal{B}^\top$ and $B \in \mathcal{B}^\top$.
- ▶ By the IH, we have $\mathcal{I}_\mathcal{B} \models A$ and $\mathcal{I}_\mathcal{B} \models B$.
- ▶ By definition of model semantics, $\mathcal{I}_\mathcal{B} \models (A \wedge B)$.

Assume that $(A \wedge B) \in \mathcal{B}^\perp$.

- ▶ Since the derivation is maximal, $\wedge$ -right is eventually applied...

Case: Conjunction in antecedent/succedent

Assume that $(A \wedge B) \in \mathcal{B}^\top$.

- ▶ Since the derivation is maximal, we have $A \in \mathcal{B}^\top$ and $B \in \mathcal{B}^\top$.
- ▶ By the IH, we have $\mathcal{I}_\mathcal{B} \models A$ and $\mathcal{I}_\mathcal{B} \models B$.
- ▶ By definition of model semantics, $\mathcal{I}_\mathcal{B} \models (A \wedge B)$.

Assume that $(A \wedge B) \in \mathcal{B}^\perp$.

- ▶ Since the derivation is maximal, $\wedge$ -right is eventually applied...
- ▶ ...introducing A in the succedent of one branch and B on the other.

Case: Conjunction in antecedent/succedent

Assume that $(A \wedge B) \in \mathcal{B}^\top$.

- ▶ Since the derivation is maximal, we have $A \in \mathcal{B}^\top$ **and** $B \in \mathcal{B}^\top$.
- ▶ By the IH, we have $\mathcal{I}_\mathcal{B} \models A$ **and** $\mathcal{I}_\mathcal{B} \models B$.
- ▶ By definition of model semantics, $\mathcal{I}_\mathcal{B} \models (A \wedge B)$.

Assume that $(A \wedge B) \in \mathcal{B}^\perp$.

- ▶ Since the derivation is maximal, $\wedge$ -right is eventually applied...
- ▶ ...introducing A in the succedent of one branch and B on the other.
- ▶ One of them is our branch $\mathcal{B}$, and therefore $A \in \mathcal{B}^\perp$ **or** $B \in \mathcal{B}^\perp$.

Case: Conjunction in antecedent/succedent

Assume that $(A \wedge B) \in \mathcal{B}^\top$.

- ▶ Since the derivation is maximal, we have $A \in \mathcal{B}^\top$ **and** $B \in \mathcal{B}^\top$.
- ▶ By the IH, we have $\mathcal{I}_\mathcal{B} \models A$ **and** $\mathcal{I}_\mathcal{B} \models B$.
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Assume that $(A \wedge B) \in \mathcal{B}^\perp$.

- ▶ Since the derivation is maximal, $\wedge$ -right is eventually applied...
- ▶ ...introducing A in the succedent of one branch and B on the other.
- ▶ One of them is our branch $\mathcal{B}$, and therefore $A \in \mathcal{B}^\perp$ **or** $B \in \mathcal{B}^\perp$.
- ▶ By the IH, we have $\mathcal{I}_\mathcal{B} \not\models A$ **or** $\mathcal{I}_\mathcal{B} \not\models B$

Case: Conjunction in antecedent/succedent

Assume that $(A \wedge B) \in \mathcal{B}^\top$.

- ▶ Since the derivation is maximal, we have $A \in \mathcal{B}^\top$ **and** $B \in \mathcal{B}^\top$.
- ▶ By the IH, we have $\mathcal{I}_\mathcal{B} \models A$ **and** $\mathcal{I}_\mathcal{B} \models B$.
- ▶ By definition of model semantics, $\mathcal{I}_\mathcal{B} \models (A \wedge B)$.

Assume that $(A \wedge B) \in \mathcal{B}^\perp$.

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- ▶ Not possible to prove completeness ‘one rule at a time’

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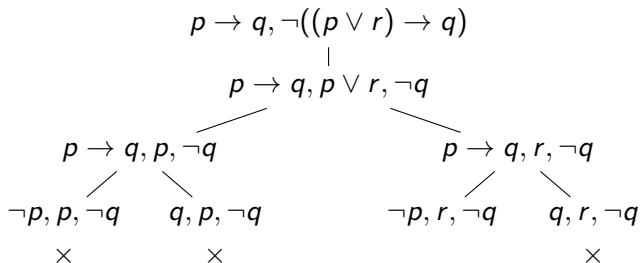
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- Soundness and completeness very similar to two-sided LK.

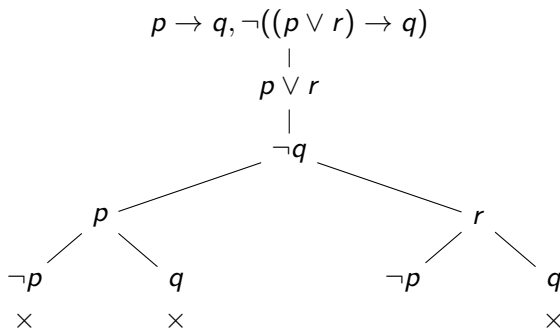
Semantic Tableaux (Ben-Ari 2.6)

- ▶ Others call these 'block tableaux'
- ▶ Sequent arrow $\Rightarrow$ not needed for one-sided calculus
- ▶ More handy to write top-down, like everybody else
- ▶ Mark 'closed' branches (with axioms) with $\times$



Short Hand Notation for Tableaux

- ▶ Only write the new formula in every node.
- ▶ Even more handy to write
- ▶ Close branch using literals A and $\neg A$ anywhere on a branch.
- ▶ Have to make sure that all rules were used on every branch!



Summary and Outlook

Until now:

- ▶ Propositional logic and model semantics
- ▶ LK Calculus
- ▶ Soundness
- ▶ Completeness

Next three weeks:

- ▶ First-order logic and model semantics
- ▶ LK Calculus for first-order logic
- ▶ Soundness
- ▶ Completeness

After that: resolution, DPLL, Prolog,...