IN3070
IN4070

Autumn 2025

Exercises for the Course

Logic

Week 3



Exercise 3.1 (Variables & Assignments)

a) Which variables occur free in the following formulae? Which variables occur bound?

b) Apply the following substitution to both formulae: $\{x \backslash a, y \backslash f(a,b), z \backslash g(x,c)\}$.

c) Is the substitution capture-free for these formulae?

1. $\forall x\,(p(y) \wedge \forall y\,(p(x,z) \rightarrow q(y)))$
2. $\forall x\,\exists x\,\forall y\,(p(x) \vee \exists y\,q(y)) \rightarrow r(x)$

Exercise 3.2 (Formalization & Interpretations)

a) "If there is a man in town who shaves all the men in town who do not shave themselves, then some man in town shaves himself." Formalize these statements by a single first-order formula F_S.

b) What is the value of the term $(5+3) * (8-5)$ under the interpretation $\mathcal{I} = (\mathbb{N}, \iota)$ with $+^\iota = *$, $*^\iota = -$, $-^\iota = \div$ (division), $3^\iota = 8$, $5^\iota = 6$, $8^\iota = 36$.

c) Show that the following formulae are satisfiable or invalid (or both) by providing a model and/or a counter-model (i.e. an interpretation that falsifies the formula).

1. $\forall x\,p(f(x),a) \rightarrow \exists x\,p(g(x),x)$
2. $\forall y\,q(y,b) \rightarrow \exists x\,q(a,x)$

Exercise 3.3 (Sequent Calculus LK & Eigenvariables)

Try to prove the validity of the following formulae in the sequent calculus LK. If you cannot find a proof in LK then povide a counter-model.

1. The first-order formula F_S from Exercise 3.2 a).
2. $\forall x\,\exists y\,p(x,y) \rightarrow \forall u\,\exists v\,p(u,v)$
3. $\forall x\,\exists y\,p(x,y) \rightarrow \exists v\,\forall u\,p(u,v)$

Exercise 3.4 (Symmetry of LK)

Let A be a formula containing only $\forall$, $\exists$ and the connectives $\neg$, $\vee$, and $\wedge$. The dual formula A' of A is obtained by exchanging $\forall$ and $\exists$, and exchanging $\vee$ and $\wedge$. Prove that $\vdash A$ iff $\vdash \neg A'$.

Hint: try to explain how to transform an LK proof for A into a proof for $\neg A'$, and vice versa

Exercise 3.5 **(Variable Assignments and Closed Formulas)**

The term value, resp. truth value of a *closed* term t, resp. formula A in an interpretation $\mathcal{I} = (D, \iota)$ is independent of the variable assignment. I.e. if α and β are two variable assignments for $\mathcal{I}$, and t is closed, then

$$v_{\mathcal{I}}(\alpha, t) = v_{\mathcal{I}}(\beta, t)$$

and if A is closed then

$$v_{\mathcal{I}}(\alpha, A) = v_{\mathcal{I}}(\beta, A)$$

Prove these facts by structural induction on t and A.

Hint: when you try to prove this for $\forall x\, A$ and $\exists x\, A$, you will run into the obstacle that the subformula A is not necessarily closed, so it's not possible to apply the induction hypothesis to it. You have to find and prove a more general statement about variable assignments and the free variables occurring in formulae and terms.