

IN3070/4070 – Logic – Autumn 2025

Lecture 2: Propositional Logic & Sequent Calculus

Martin Giese

25th August 2025



DEPARTMENT OF
INFORMATICS



UNIVERSITY OF
OSLO

Today's Plan

- ▶ Motivation
- ▶ Syntax
- ▶ Semantics
- ▶ Logical Equivalence
- ▶ Satisfiability & Validity
- ▶ Summary
- ▶ Motivation
- ▶ Sequent Calculus
- ▶ Decision Procedure
- ▶ Summary

Outline

- ▶ **Motivation**
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- ▶ logical systems **formalize reasoning** similar to programming languages that formalize computation
- ▶ consequent separation of **syntactical** notions (formulae, proofs) and **semantical** notions (truth values, models)
- ▶ **syntax** defines what strings of symbols are “legal” formulae
- ▶ **semantics** assign meanings to legal formulae (through an interpretation of its symbols)

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Definition 2.1 (Atomic Formulae).

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Definition 2.2 (Propositional Formulae).

The **propositional formulae**, denoted A, B, C, F, G, H , are inductively defined as follows:

1. Every atom $A \in \mathcal{P}$ is a formula.
2. If A and B are formulae, then $(\neg A)$, $(A \wedge B)$, $(A \vee B)$ and $(A \rightarrow B)$ are formulae.

Let $\mathcal{F}$ be the set of all (legal) formulae.

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Examples:

$((p \rightarrow q) \leftrightarrow ((\neg p) \rightarrow (\neg q)))$ is a (legal) formula, identical to $(p \rightarrow q) \leftrightarrow (\neg p \rightarrow \neg q)$ and $p \rightarrow q \leftrightarrow \neg p \rightarrow \neg q$

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Alternative connectives: $\Rightarrow$ and $\supset$ (for $\rightarrow$), $\Leftrightarrow$ (for $\leftrightarrow$), $\&$ (for $\wedge$)

Formula Trees

Definition 2.4 (Formula Tree).

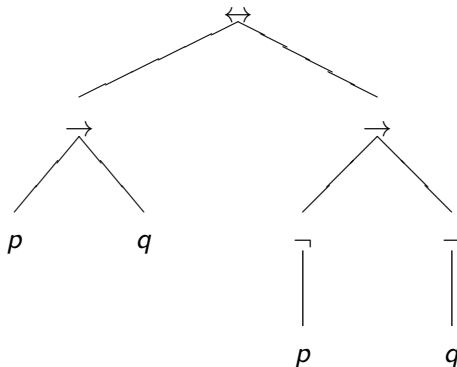
*A formula can be presented as **formula tree**.*

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Example: $(p \rightarrow q) \leftrightarrow (\neg p \rightarrow \neg q)$



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Semantics—Interpretation

Truth values are assigned to the atoms of a formula in order to evaluate the truth value of the formula.

Definition 3.1 (Interpretation).

Let $\mathcal{P}$ be a set of atoms.

*An **interpretation** is a total function $\mathcal{I} : \mathcal{P} \rightarrow \{T, F\}$ that assigns one of the truth values T or F to every atom in $\mathcal{P}$.*

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Simplifies some places, complicates things in others.

Semantics—Truth Value

Definition 3.2 (Truth Value).

Let $\mathcal{I}$ be an interpretation. The *truth value* $v_{\mathcal{I}}(A)$ of A under $\mathcal{I}$ is defined inductively as follows.

- ▶ If $A \in \mathcal{P}$ is an atom, then $v_{\mathcal{I}}(A) = \mathcal{I}(A)$
- ▶ $v_{\mathcal{I}}(\neg A) = T$ if $v_{\mathcal{I}}(A) = F$ and F otherwise.
- ▶ $v_{\mathcal{I}}(A \wedge B) = T$ if $v_{\mathcal{I}}(A) = T$ and $v_{\mathcal{I}}(B) = T$, and F otherwise.
- ▶ $v_{\mathcal{I}}(A \vee B) = F$ if $v_{\mathcal{I}}(A) = F$ and $v_{\mathcal{I}}(B) = F$, and T otherwise.
- ▶ $v_{\mathcal{I}}(A \rightarrow B) = F$ if $v_{\mathcal{I}}(A) = T$ and $v_{\mathcal{I}}(B) = F$, and T otherwise.

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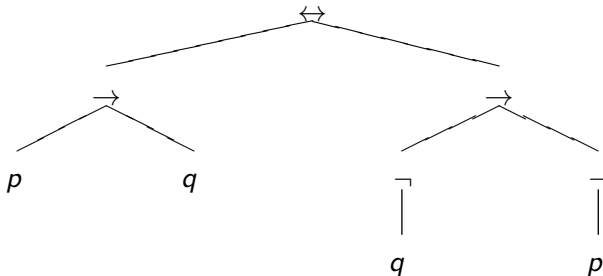
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- ▶ $v_{\mathcal{I}}(A \rightarrow B) = F$ if $v_{\mathcal{I}}(A) = T$ and $v_{\mathcal{I}}(B) = F$, and T otherwise.

Note: For the equivalence connective, it follows that

- ▶ $v_{\mathcal{I}}(A \leftrightarrow B) = T$ if $v_{\mathcal{I}}(A) = v_{\mathcal{I}}(B)$, and F otherwise.

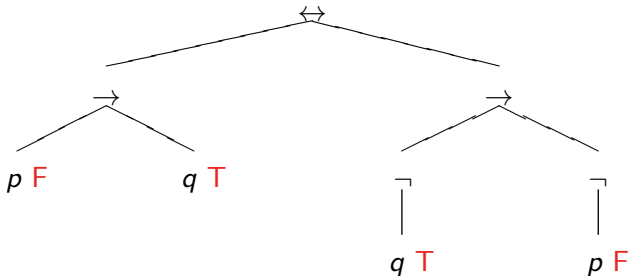
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Example: Let $A = (p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p)$
with $\mathcal{I}(p) = F$ and $\mathcal{I}(q) = T$.



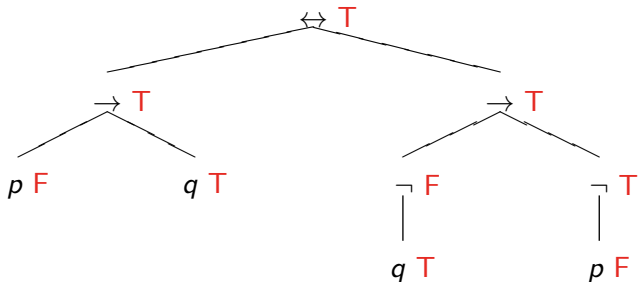
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Truth Tables

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Definition 3.3 (Truth Table).

Let $A \in \mathcal{F}$ with n atoms. A **truth table** has $n + 1$ columns and 2^n rows. There is a column for each atom in A , plus a column for the formula A . The first n columns specify all possible interpretations $\mathcal{I}$ that map atoms in A to $\{T, F\}$. The last column shows $v_{\mathcal{I}}(A)$, the truth value of A for each interpretation $\mathcal{I}$.

p_1	p_2	$\dots$	p_n	A
T	T	$\dots$	T	$v_{\mathcal{I}}(A)$
T	T	$\dots$	F	$v_{\mathcal{I}}(A)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
F	F	$\dots$	F	$v_{\mathcal{I}}(A)$

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p	q	$p \rightarrow q$	$\neg p$	$\neg q$	$\neg q \rightarrow \neg p$	$(p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p)$
T	T					
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p	q	$p \rightarrow q$	$\neg p$	$\neg q$	$\neg q \rightarrow \neg p$	$(p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p)$
T	T	T	F	F	T	T
T	F	F	F	T	F	T
F	T	T	T	F	T	T
F	F	T	T	T	T	T

Material Implication

The operator of $p \rightarrow q$ is called **material implication**.

- ▶ p is the antecedent and q is the consequent
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Example:

“Earth is farther from the sun than Venus” $\rightarrow$ “ $1 + 1 = 3$ ”

is **false** since the antecedent is true and the consequent is false, but:

“Earth is farther from the sun than Mars” $\rightarrow$ “ $1 + 1 = 3$ ”

is **true(!)** as the falsity of the antecedent by itself is sufficient to ensure the truth of the implication

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Let $A_1, A_2 \in \mathcal{F}$. If $v_{\mathcal{I}}(A_1) = v_{\mathcal{I}}(A_2)$ for all interpretations $\mathcal{I}$, then A_1 is *logically equivalent* to A_2 , denoted $A_1 \equiv A_2$.

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Since $\mathcal{I}$ was chosen arbitrarily, $v_{\mathcal{I}}(A \vee B) = v_{\mathcal{I}}(B \vee A)$ for all interpretations. $\square$

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The following formulae are **logical equivalent** (more in [Ben-Ari, 2.3.3]):

$$A \vee \text{true} \equiv \text{true}$$

$$A \vee \text{false} \equiv A$$

$$A \rightarrow \text{true} \equiv \text{true}$$

$$A \rightarrow \text{false} \equiv \neg A$$

$$A \equiv A \wedge A$$

$$A \vee B \equiv B \vee A$$

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$$A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$$

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$$\text{true} \rightarrow A \equiv A$$

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Contrapositive: $A \rightarrow B \equiv \neg B \rightarrow \neg A$

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- ▶ **Satisfiability & Validity**
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Satisfiability and Validity

Definition 5.1 (Satisfiable, Model, Valid, Unsatisfiable, Invalid).

Let $A \in \mathcal{F}$.

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- ▶ A **set** of formulae $U = \{A_1, \dots\}$ is **(simultaneously) satisfiable** iff there exists an interpretation $\mathcal{I}$ such that $v_{\mathcal{I}}(A_i) = T$ for all i ; otherwise U is **unsatisfiable**. The satisfying interpretation is a **model** of U .

Satisfiability and Validity

There is a close relation between these four semantical concepts.

Theorem 5.1 (Satisfiable, Valid, Unsatisfiable, Invalid).

Let $A \in F$. A is *valid* iff $\neg A$ is *unsatisfiable*. A is *satisfiable* iff $\neg A$ is *invalid*.

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Proof.

Let $\mathcal{I}$ be an arbitrary interpretation. $v_{\mathcal{I}}(A) = T$ if and only if $v_{\mathcal{I}}(\neg A) = F$ by definition of the truth value of negation. Since $\mathcal{I}$ was arbitrary, $v_{\mathcal{I}}(A) = T$ for all interpretations if and only if $v_{\mathcal{I}}(\neg A) = F$ for all interpretations, that is, iff $\neg A$ is unsatisfiable.

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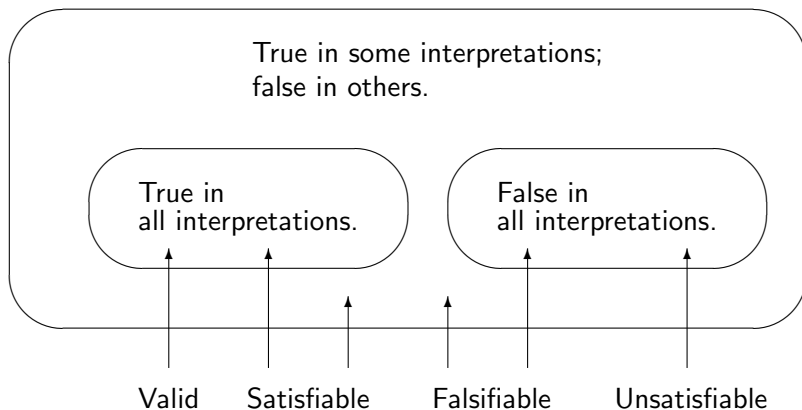
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If A is satisfiable then for some interpretation $\mathcal{I}$, $v_{\mathcal{I}}(A) = T$. By definition of the truth value of negation, $v_{\mathcal{I}}(\neg A) = F$ so that $\neg A$ is invalid. Conversely, if $v_{\mathcal{I}}(\neg A) = F$ then $v_{\mathcal{I}}(A) = T$. □

Satisfiability and Validity



Decidability

Definition 5.2 (Decision Procedure).

Let $\mathcal{U} \subseteq \mathcal{F}$ be a set of (propositional) formulae. An algorithm is a *decision procedure* for $\mathcal{U}$ if given a formula $A \in \mathcal{F}$, it terminates and returns the answer “yes” if $A \in \mathcal{U}$ and the answer “no” if $A \notin \mathcal{U}$.

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For a given formula A with n atoms, use truth tables to evaluate truth values for A . If $v_{\mathcal{I}}(A) = T$ for all 2^n possible interpretations $\mathcal{I}$, then answer “yes”; otherwise answer “no”. □

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This method is **not very efficient**; more efficient procedures will be introduced later.

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Example: Let $A = (p \vee r) \wedge (\neg q \vee \neg r)$. A is a logical consequence of $\{p, \neg q\}$, denoted $\{p, \neg q\} \models A$, as $v_{\mathcal{I}}(A) = T$ for all interpretations $\mathcal{I}$ such that $\mathcal{I}(p) = T$ and $\mathcal{I}(q) = F$. But A is not valid, as $v_{\mathcal{I}}(A) = F$ for the interpretation $\mathcal{I}$ where $\mathcal{I}(p) = F$, $\mathcal{I}(q) = T$, $\mathcal{I}(r) = T$.

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Proof. Left as an exercise.

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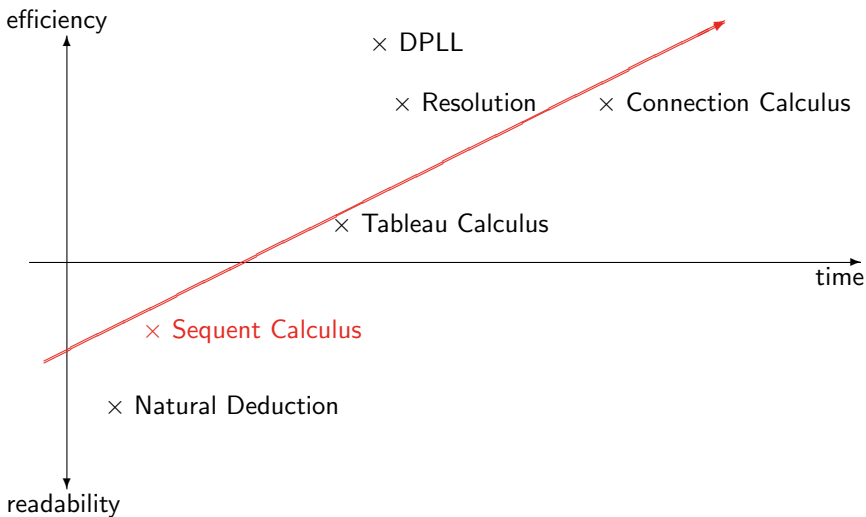
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Proof Search Calculi



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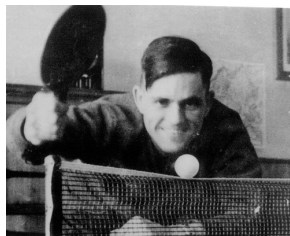
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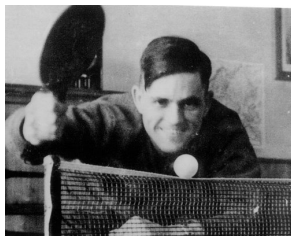


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- ▶ So let's look at the rules in detail!

LK — Rules for Conjunction and Disjunction

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Calculus and Proof — General Definitions

Definition 8.2 (Calculus/Deductive System).

A *calculus* consists of axioms and inference rules.

Axioms have the form $\frac{}{w}$; *rules* have the form $\frac{w_1 \cdots w_n}{w}$
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Definition 8.3 (Proof, Derivation).

Let $\mathcal{A} = \{A_1, \dots\}$ be axioms and $\mathcal{R} = \{R_1, \dots\}$ be rules of a calculus.

1. Let $\frac{}{w}$ be an instance of an axiom $A_i \in \mathcal{A}$. Then $\frac{}{w}$ is a *proof* of w .
2. Let $\frac{w_1 \cdots w_n}{w}$ be an instance of a rule $R_i \in \mathcal{R}$ and $\mathcal{D}_1, \dots, \mathcal{D}_n$ proofs of $w_1, \dots, w_n$. Then $\frac{\mathcal{D}_1 \cdots \mathcal{D}_n}{w}$ is a *proof* of w .

A *derivation* is defined similarly, but leaves do not need to be axioms.

The Sequent Calculus LK

Definition 8.4 (Proofs in LK).

A *proof* of a formula A in the LK calculus is a proof of the sequent $\Rightarrow A$ using the rules and axiom of LK. A formula A is *provable*, written $\vdash A$, iff there is a proof for A .

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Theorem 8.1 (Soundness and Completeness of LK).

The calculus of natural deduction LK is sound and complete, i.e.

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Proof.

Next week!



Outline

- ▶ Motivation
- ▶ Syntax
- ▶ Semantics
- ▶ Logical Equivalence
- ▶ Satisfiability & Validity
- ▶ Summary
- ▶ Motivation
- ▶ Sequent Calculus
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- ▶ **Next week**: Soundness and Completeness proofs