

IN3070/4070 – Logic – Autumn 2025

Lecture 4: First-order Logic

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8th September 2025



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UNIVERSITY OF
OSLO

Today's Plan

- ▶ Motivation
- ▶ Syntax
- ▶ Variables
- ▶ Semantics
- ▶ The Substitution Lemma
- ▶ Satisfiability & Validity
- ▶ LK for First-order Logic
- ▶ Summary

Outline

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First-order logic: extension of propositional logic

First-Order Logic — Overview

Extending propositional logic by...

Syntax:

- ▶ constants (a, b, c) , functions (f, g, h) , variables (x, y, z)
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Semantics:

- ▶ interpretation of constants, functions, variables
- ▶ interpretation of predicates
- ▶ value of terms
- ▶ truth value of (quantified) formulae
- ▶ satisfiability, validity, logical equivalence,...

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Example: a , x , $f(a, x)$, $f(g(x), b)$, and $g(f(a, g(y)))$ are terms.

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Let $\mathcal{P} = \{p, q, r, \dots\}$ be a countable set of **predicate symbols**. If $p \in \mathcal{P}$ is an n -ary predicate (symbol) $n \geq 0$ and $t_1, \dots, t_n$ are terms, then $p(t_1, \dots, t_n)$, $\top$, and $\perp$ are **atomic formulae** (or **atoms**).

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(First-order) formulae, denoted A, B, C, F, G, H , are inductively defined as follows:

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2. If A and B are formulae and $x \in \mathcal{V}$, then $(\neg A)$, $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$, $\forall x A$, and $\exists x A$ are formulae.

Formula Trees

A formula can be presented as **formula tree**.

Example:

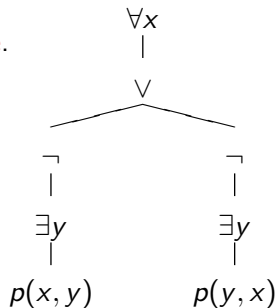
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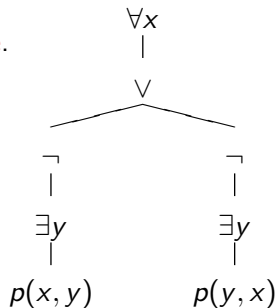


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Definition 2.4 (Subformula, Main Operator).

Formula A is a **(proper) subformula** of formula B iff A is a (proper) subtree of B . If the root of a formula tree of A is a logical connective/quantifier, then it is called the **main operator** of A .

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A **bound variable** in a formula C is a variable that appears in $\forall x$ or $\exists x$ in some subformula of C . A formula/term is **closed** iff it has no free variables.

Scope, Universal and Existential Closure

Definition 3.2 (Scope of Variables).

Let $\forall x A$ or $\exists x A$ be a universally or existentially *quantified formula*. Then x is the *quantified variable* and its *scope* is the formula A .

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If $\{x_1, \dots, x_n\}$ are all the free variables of A , the *universal closure* of A is $\forall x_1 \dots \forall x_n A$ and the *existential closure* of A is $\exists x_1 \dots \exists x_n A$.

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- $p(x, y)$ has the two free variables x and y . Its universal closure is $\forall x \forall y p(x, y)$ and its existential closure is $\exists x \exists y p(x, y)$; $\exists y p(x, y)$ has the only free variable x ; $\forall x \exists y p(x, y)$ is closed

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- ▶ In $\forall x p(x) \wedge q(x)$, the x occurs bound **and** free. The existential closure is $\exists x (\forall x p(x) \wedge q(x))$; **renaming**: $\exists y (\forall x p(x) \wedge q(y))$

Substitutions

Free variables in a first-order formula can be substituted by terms.

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*Let $\mathcal{V}$ be a set of variables, $\mathcal{T}$ be the set of terms. A **substitution** $\sigma : \mathcal{V} \rightarrow \mathcal{T}$ assigns each variable a term.*

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Others: $[a/x, f(z, b)/y]$

Application of substitutions

Definition 3.5 (Application of Substitutions, informally).

Let σ be a substitution. The *application* of σ to a term t or formula A , written $\sigma(t)$ or $\sigma(A)$, replaces every free variable in t or A according to its image under σ .

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Definition 3.6 (Capture-free substitution).

A substitution σ is *capture-free* for a formula A if for every free variable x in A , none of the variables in $\sigma(x)$ is bound in A .

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Definition 3.7 (Application of Substitutions, formally).

The application of a substitution σ to a term or formula is defined by structural induction:

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- ▶ ... similarly for $\neg A, A \vee B, A \rightarrow B$...
- ▶ $\sigma(\exists x A) = \exists x \sigma_x(A), \quad \sigma(\forall x A) = \forall x \sigma_x(A)$

where we define σ_x by: $\sigma_x(x) = x$, and $\sigma_x(y) = \sigma(y)$ for all $y \neq x$

Outline

- ▶ Motivation
- ▶ Syntax
- ▶ Variables
- ▶ **Semantics**
- ▶ The Substitution Lemma
- ▶ Satisfiability & Validity
- ▶ LK for First-order Logic
- ▶ Summary

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- ▶ $+(1, *(4, 2))$ with $\mathcal{I} = (\mathbb{Z}, \iota)$ with $+^{\iota} = *$ (multiplication), $*^{\iota} = -$ (subtraction), $1^{\iota} = -20$, $2^{\iota} = 0$, $4^{\iota} = 10$;

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Examples:

- ▶ $f(a, f(a, b))$ with $\mathcal{I} = (\mathbb{N}, \iota)$ with $f^{\iota} = +$, $a^{\iota} = 20$, $b^{\iota} = 2$; then $v_{\mathcal{I}}(f(a, f(a, b))) = 42$
- ▶ $+(1, *(4, 2))$ with $\mathcal{I} = (\mathbb{Z}, \iota)$ with $+^{\iota} = *$ (multiplication), $*^{\iota} = -$ (subtraction), $1^{\iota} = -20$, $2^{\iota} = 0$, $4^{\iota} = 10$; then $v_{\mathcal{I}}(+(1, *(4, 2))) = -200$

Semantics — Variable Assignments, Value of Terms

The interpretation doesn't tell what to do about variables.
We need something additional.

Definition 4.3 (Variable Assignment).

Given the set of variables $\mathcal{V}$, and an interpretation $\mathcal{I} = (D, \iota)$, a variable assignment α for $\mathcal{I}$ is a function $\alpha : \mathcal{V} \rightarrow D$.

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Definition 4.4 (Term Value).

*Let $\mathcal{I} = (D, \iota)$ be an interpretation, and α an variable assignment for $\mathcal{I}$. The **term value** $v_{\mathcal{I}}(\alpha, t)$ of a term $t \in \mathcal{T}$ under $\mathcal{I}$ and α is inductively defined:*

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1. $v_{\mathcal{I}}(\alpha, x) = \alpha(x)$ for a variable $x \in \mathcal{V}$
2. $v_{\mathcal{I}}(\alpha, a) = a^{\iota}$ for a constant symbol $a \in \mathcal{A}$

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1. $v_{\mathcal{I}}(\alpha, x) = \alpha(x)$ for a variable $x \in \mathcal{V}$
2. $v_{\mathcal{I}}(\alpha, a) = a^{\iota}$ for a constant symbol $a \in \mathcal{A}$
3. $v_{\mathcal{I}}(\alpha, f(t_1, \dots, t_n)) = f^{\iota}(v_{\mathcal{I}}(\alpha, t_1), \dots, v_{\mathcal{I}}(\alpha, t_n))$ for an n -ary $f \in \mathcal{F}$

Semantics — Term value Examples

► $\mathcal{I} = (\mathbb{N}, \iota)$ with $f^\iota = +$, $a^\iota = 10$

Semantics — Term value Examples

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 - ▶ $v_{\mathcal{I}}(\alpha, f(a, f(a, x))) = 23$

Semantics — Term value Examples

- ▶ $\mathcal{I} = (\mathbb{N}, \iota)$ with $f^\iota = +$, $a^\iota = 10$
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 - ▶ $\alpha(x) = 3 \in \mathbb{N}$ and $\alpha(y) = 5 \in \mathbb{N}$ is an assignment for $\mathcal{I}$
 - ▶ $v_{\mathcal{I}}(\alpha, f(a, f(a, x))) = 23$
- ▶ $\mathcal{I} = (\text{Strings}, \iota)$ with $g^\iota = \text{concatenation}$, $a^\iota = \text{"Hello"}$

Semantics — Term value Examples

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 - ▶ $v_{\mathcal{I}}(\alpha, f(a, f(a, x))) = 23$
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 - ▶ $v_{\mathcal{I}}(\alpha, f(a, f(a, x))) = 23$
- ▶ $\mathcal{I} = (\text{Strings}, \iota)$ with $g^\iota = \text{concatenation}$, $a^\iota = \text{"Hello"}$
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 - ▶ $\alpha(y) = \text{"students"}$

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- ▶ $\mathcal{I} = (\mathbb{N}, \iota)$ with $f^\iota = +$, $a^\iota = 10$
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 - ▶ $\mathcal{V} = \{y\}$
 - ▶ $\alpha(y) = \text{"students"}$
 - ▶ $v_{\mathcal{I}}(\alpha, f(a, f(y, a))) = \text{"HellostudentsHello"}$

Semantics — Modification of an assignment

Definition 4.5 (Modification of a variable assignment).

Given an interpretation $\mathcal{I} = (D, \iota)$ and a variable assignment α for $\mathcal{I}$.

Given also a variable $y \in \mathcal{V}$ and a domain element $d \in D$.

The modified variable assignment $\alpha\{y \leftarrow d\}$ is defined as

$$\alpha\{y \leftarrow d\}(x) = \begin{cases} d & \text{if } x = y \\ \alpha(x) & \text{otherwise} \end{cases}$$

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- ▶ $\mathcal{I} = (\mathbb{N}, \iota)$
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- ▶ $\alpha(x) = 3 \in \mathbb{N}$ and $\alpha(y) = 5 \in \mathbb{N}$ is an assignment for $\mathcal{I}$

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$$\alpha\{y \leftarrow d\}(x) = \begin{cases} d & \text{if } x = y \\ \alpha(x) & \text{otherwise} \end{cases}$$

- ▶ $\mathcal{I} = (\mathbb{N}, \iota)$
- ▶ $\mathcal{V} = \{x, y\}$
- ▶ $\alpha(x) = 3 \in \mathbb{N}$ and $\alpha(y) = 5 \in \mathbb{N}$ is an assignment for $\mathcal{I}$
- ▶ $\alpha\{y \leftarrow 7\}(x) =$

Semantics — Modification of an assignment

Definition 4.5 (Modification of a variable assignment).

Given an interpretation $\mathcal{I} = (D, \iota)$ and a variable assignment α for $\mathcal{I}$.
Given also a variable $y \in \mathcal{V}$ and a domain element $d \in D$.
The modified variable assignment $\alpha\{y \leftarrow d\}$ is defined as

$$\alpha\{y \leftarrow d\}(x) = \begin{cases} d & \text{if } x = y \\ \alpha(x) & \text{otherwise} \end{cases}$$

- ▶ $\mathcal{I} = (\mathbb{N}, \iota)$
- ▶ $\mathcal{V} = \{x, y\}$
- ▶ $\alpha(x) = 3 \in \mathbb{N}$ and $\alpha(y) = 5 \in \mathbb{N}$ is an assignment for $\mathcal{I}$
- ▶ $\alpha\{y \leftarrow 7\}(x) =$

Semantics — Modification of an assignment

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Given an interpretation $\mathcal{I} = (D, \iota)$ and a variable assignment α for $\mathcal{I}$.

Given also a variable $y \in \mathcal{V}$ and a domain element $d \in D$.

The modified variable assignment $\alpha\{y \leftarrow d\}$ is defined as

$$\alpha\{y \leftarrow d\}(x) = \begin{cases} d & \text{if } x = y \\ \alpha(x) & \text{otherwise} \end{cases}$$

- ▶ $\mathcal{I} = (\mathbb{N}, \iota)$
- ▶ $\mathcal{V} = \{x, y\}$
- ▶ $\alpha(x) = 3 \in \mathbb{N}$ and $\alpha(y) = 5 \in \mathbb{N}$ is an assignment for $\mathcal{I}$
- ▶ $\alpha\{y \leftarrow 7\}(x) = 3$

Semantics — Modification of an assignment

Definition 4.5 (Modification of a variable assignment).

Given an interpretation $\mathcal{I} = (D, \iota)$ and a variable assignment α for $\mathcal{I}$.

Given also a variable $y \in \mathcal{V}$ and a domain element $d \in D$.

The modified variable assignment $\alpha\{y \leftarrow d\}$ is defined as

$$\alpha\{y \leftarrow d\}(x) = \begin{cases} d & \text{if } x = y \\ \alpha(x) & \text{otherwise} \end{cases}$$

- ▶ $\mathcal{I} = (\mathbb{N}, \iota)$
- ▶ $\mathcal{V} = \{x, y\}$
- ▶ $\alpha(x) = 3 \in \mathbb{N}$ and $\alpha(y) = 5 \in \mathbb{N}$ is an assignment for $\mathcal{I}$
- ▶ $\alpha\{y \leftarrow 7\}(x) = 3$ and $\alpha\{y \leftarrow 7\}(y) =$

Semantics — Modification of an assignment

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Given an interpretation $\mathcal{I} = (D, \iota)$ and a variable assignment α for $\mathcal{I}$.

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$$\alpha\{y \leftarrow d\}(x) = \begin{cases} d & \text{if } x = y \\ \alpha(x) & \text{otherwise} \end{cases}$$

- ▶ $\mathcal{I} = (\mathbb{N}, \iota)$
- ▶ $\mathcal{V} = \{x, y\}$
- ▶ $\alpha(x) = 3 \in \mathbb{N}$ and $\alpha(y) = 5 \in \mathbb{N}$ is an assignment for $\mathcal{I}$
- ▶ $\alpha\{y \leftarrow 7\}(x) = 3$ and $\alpha\{y \leftarrow 7\}(y) = 7$

Semantics — Truth Value

Definition 4.6 (Truth Value).

Let $\mathcal{I} = (D, \iota)$ be an interpretation and α an assignment for $\mathcal{I}$. The **truth value** $v_{\mathcal{I}}(\alpha, A) \in \{T, F\}$ of a formula A under $\mathcal{I}$ and α is defined inductively as follows:

Semantics — Truth Value

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1. $v_{\mathcal{I}}(\alpha, p) = T$ for 0-ary $p \in \mathcal{P}$ iff $p^{\iota} = T$, otherwise $v_{\mathcal{I}}(\alpha, p) = F$

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1. $v_{\mathcal{I}}(\alpha, p) = T$ for 0-ary $p \in \mathcal{P}$ iff $p^{\iota} = T$, otherwise $v_{\mathcal{I}}(\alpha, p) = F$
2. $v_{\mathcal{I}}(\alpha, p(t_1, \dots, t_n)) = T$ for $p \in \mathcal{P}$, $n > 0$, iff $(v_{\mathcal{I}}(\alpha, t_1), \dots, v_{\mathcal{I}}(\alpha, t_n)) \in p^{\iota}$, otherwise $v_{\mathcal{I}}(\alpha, p(t_1, \dots, t_n)) = F$

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2. $v_{\mathcal{I}}(\alpha, p(t_1, \dots, t_n)) = T$ for $p \in \mathcal{P}$, $n > 0$, iff $(v_{\mathcal{I}}(\alpha, t_1), \dots, v_{\mathcal{I}}(\alpha, t_n)) \in p^{\iota}$, otherwise $v_{\mathcal{I}}(\alpha, p(t_1, \dots, t_n)) = F$
3. $v_{\mathcal{I}}(\alpha, \neg A) = T$ iff $v_{\mathcal{I}}(\alpha, A) = F$, otherwise $v_{\mathcal{I}}(\alpha, \neg A) = F$

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2. $v_{\mathcal{I}}(\alpha, p(t_1, \dots, t_n)) = T$ for $p \in \mathcal{P}$, $n > 0$, iff $(v_{\mathcal{I}}(\alpha, t_1), \dots, v_{\mathcal{I}}(\alpha, t_n)) \in p^{\iota}$, otherwise $v_{\mathcal{I}}(\alpha, p(t_1, \dots, t_n)) = F$
3. $v_{\mathcal{I}}(\alpha, \neg A) = T$ iff $v_{\mathcal{I}}(\alpha, A) = F$, otherwise $v_{\mathcal{I}}(\alpha, \neg A) = F$
4. $v_{\mathcal{I}}(\alpha, A \wedge B) = T$ iff $v_{\mathcal{I}}(\alpha, A) = T$ and $v_{\mathcal{I}}(\alpha, B) = T$, otherwise $v_{\mathcal{I}}(\alpha, A \wedge B) = F$

Semantics — Truth Value

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1. $v_{\mathcal{I}}(\alpha, p) = T$ for 0-ary $p \in \mathcal{P}$ iff $p^{\iota} = T$, otherwise $v_{\mathcal{I}}(\alpha, p) = F$
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3. $v_{\mathcal{I}}(\alpha, \neg A) = T$ iff $v_{\mathcal{I}}(\alpha, A) = F$, otherwise $v_{\mathcal{I}}(\alpha, \neg A) = F$
4. $v_{\mathcal{I}}(\alpha, A \wedge B) = T$ iff $v_{\mathcal{I}}(\alpha, A) = T$ and $v_{\mathcal{I}}(\alpha, B) = T$, otherwise $v_{\mathcal{I}}(\alpha, A \wedge B) = F$
5. $v_{\mathcal{I}}(\alpha, A \vee B) = T$ iff $v_{\mathcal{I}}(\alpha, A) = T$ or $v_{\mathcal{I}}(\alpha, B) = T$, otherwise $v_{\mathcal{I}}(\alpha, A \vee B) = F$

Semantics — Truth Value

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Let $\mathcal{I} = (D, \iota)$ be an interpretation and α an assignment for $\mathcal{I}$. The **truth value** $v_{\mathcal{I}}(\alpha, A) \in \{T, F\}$ of a formula A under $\mathcal{I}$ and α is defined inductively as follows:

1. $v_{\mathcal{I}}(\alpha, p) = T$ for 0-ary $p \in \mathcal{P}$ iff $p^{\iota} = T$, otherwise $v_{\mathcal{I}}(\alpha, p) = F$
2. $v_{\mathcal{I}}(\alpha, p(t_1, \dots, t_n)) = T$ for $p \in \mathcal{P}$, $n > 0$, iff $(v_{\mathcal{I}}(\alpha, t_1), \dots, v_{\mathcal{I}}(\alpha, t_n)) \in p^{\iota}$, otherwise $v_{\mathcal{I}}(\alpha, p(t_1, \dots, t_n)) = F$
3. $v_{\mathcal{I}}(\alpha, \neg A) = T$ iff $v_{\mathcal{I}}(\alpha, A) = F$, otherwise $v_{\mathcal{I}}(\alpha, \neg A) = F$
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5. $v_{\mathcal{I}}(\alpha, A \vee B) = T$ iff $v_{\mathcal{I}}(\alpha, A) = T$ or $v_{\mathcal{I}}(\alpha, B) = T$, otherwise $v_{\mathcal{I}}(\alpha, A \vee B) = F$
6. $v_{\mathcal{I}}(\alpha, A \rightarrow B) = T$ iff $v_{\mathcal{I}}(\alpha, A) = F$ or $v_{\mathcal{I}}(\alpha, B) = T$, otherwise $v_{\mathcal{I}}(\alpha, A \rightarrow B) = F$

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Let $\mathcal{I} = (D, \iota)$ be an interpretation and α an assignment for $\mathcal{I}$. The **truth value** $v_{\mathcal{I}}(\alpha, A) \in \{T, F\}$ of a formula A under $\mathcal{I}$ and α is defined inductively as follows:

1. $v_{\mathcal{I}}(\alpha, p) = T$ for 0-ary $p \in \mathcal{P}$ iff $p^{\iota} = T$, otherwise $v_{\mathcal{I}}(\alpha, p) = F$
2. $v_{\mathcal{I}}(\alpha, p(t_1, \dots, t_n)) = T$ for $p \in \mathcal{P}$, $n > 0$, iff $(v_{\mathcal{I}}(\alpha, t_1), \dots, v_{\mathcal{I}}(\alpha, t_n)) \in p^{\iota}$, otherwise $v_{\mathcal{I}}(\alpha, p(t_1, \dots, t_n)) = F$
3. $v_{\mathcal{I}}(\alpha, \neg A) = T$ iff $v_{\mathcal{I}}(\alpha, A) = F$, otherwise $v_{\mathcal{I}}(\alpha, \neg A) = F$
4. $v_{\mathcal{I}}(\alpha, A \wedge B) = T$ iff $v_{\mathcal{I}}(\alpha, A) = T$ and $v_{\mathcal{I}}(\alpha, B) = T$, otherwise $v_{\mathcal{I}}(\alpha, A \wedge B) = F$
5. $v_{\mathcal{I}}(\alpha, A \vee B) = T$ iff $v_{\mathcal{I}}(\alpha, A) = T$ or $v_{\mathcal{I}}(\alpha, B) = T$, otherwise $v_{\mathcal{I}}(\alpha, A \vee B) = F$
6. $v_{\mathcal{I}}(\alpha, A \rightarrow B) = T$ iff $v_{\mathcal{I}}(\alpha, A) = F$ or $v_{\mathcal{I}}(\alpha, B) = T$, otherwise $v_{\mathcal{I}}(\alpha, A \rightarrow B) = F$
7. $v_{\mathcal{I}}(\alpha, \forall x A) = T$ iff $v_{\mathcal{I}}(\alpha\{x \leftarrow d\}, A) = T$ **for all** $d \in D$, otherwise $v_{\mathcal{I}}(\alpha, \forall x A) = F$

Semantics — Truth Value

Definition 4.6 (Truth Value).

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1. $v_{\mathcal{I}}(\alpha, p) = T$ for 0-ary $p \in \mathcal{P}$ iff $p^{\iota} = T$, otherwise $v_{\mathcal{I}}(\alpha, p) = F$
2. $v_{\mathcal{I}}(\alpha, p(t_1, \dots, t_n)) = T$ for $p \in \mathcal{P}$, $n > 0$, iff $(v_{\mathcal{I}}(\alpha, t_1), \dots, v_{\mathcal{I}}(\alpha, t_n)) \in p^{\iota}$, otherwise $v_{\mathcal{I}}(\alpha, p(t_1, \dots, t_n)) = F$
3. $v_{\mathcal{I}}(\alpha, \neg A) = T$ iff $v_{\mathcal{I}}(\alpha, A) = F$, otherwise $v_{\mathcal{I}}(\alpha, \neg A) = F$
4. $v_{\mathcal{I}}(\alpha, A \wedge B) = T$ iff $v_{\mathcal{I}}(\alpha, A) = T$ and $v_{\mathcal{I}}(\alpha, B) = T$, otherwise $v_{\mathcal{I}}(\alpha, A \wedge B) = F$
5. $v_{\mathcal{I}}(\alpha, A \vee B) = T$ iff $v_{\mathcal{I}}(\alpha, A) = T$ or $v_{\mathcal{I}}(\alpha, B) = T$, otherwise $v_{\mathcal{I}}(\alpha, A \vee B) = F$
6. $v_{\mathcal{I}}(\alpha, A \rightarrow B) = T$ iff $v_{\mathcal{I}}(\alpha, A) = F$ or $v_{\mathcal{I}}(\alpha, B) = T$, otherwise $v_{\mathcal{I}}(\alpha, A \rightarrow B) = F$
7. $v_{\mathcal{I}}(\alpha, \forall x A) = T$ iff $v_{\mathcal{I}}(\alpha \{x \leftarrow d\}, A) = T$ **for all** $d \in D$, otherwise $v_{\mathcal{I}}(\alpha, \forall x A) = F$
8. $v_{\mathcal{I}}(\alpha, \exists x A) = T$ iff $v_{\mathcal{I}}(\alpha \{x \leftarrow d\}, A) = T$ **for some** $d \in D$, otherwise $v_{\mathcal{I}}(\alpha, \exists x A) = F$

Semantics — Truth Value

Definition 4.6 (Truth Value).

Let $\mathcal{I} = (D, \iota)$ be an interpretation and α an assignment for $\mathcal{I}$. The **truth value** $v_{\mathcal{I}}(\alpha, A) \in \{T, F\}$ of a formula A under $\mathcal{I}$ and α is defined inductively as follows:

1. $v_{\mathcal{I}}(\alpha, p) = T$ for 0-ary $p \in \mathcal{P}$ iff $p^{\iota} = T$, otherwise $v_{\mathcal{I}}(\alpha, p) = F$
2. $v_{\mathcal{I}}(\alpha, p(t_1, \dots, t_n)) = T$ for $p \in \mathcal{P}$, $n > 0$, iff $(v_{\mathcal{I}}(\alpha, t_1), \dots, v_{\mathcal{I}}(\alpha, t_n)) \in p^{\iota}$, otherwise $v_{\mathcal{I}}(\alpha, p(t_1, \dots, t_n)) = F$
3. $v_{\mathcal{I}}(\alpha, \neg A) = T$ iff $v_{\mathcal{I}}(\alpha, A) = F$, otherwise $v_{\mathcal{I}}(\alpha, \neg A) = F$
4. $v_{\mathcal{I}}(\alpha, A \wedge B) = T$ iff $v_{\mathcal{I}}(\alpha, A) = T$ and $v_{\mathcal{I}}(\alpha, B) = T$, otherwise $v_{\mathcal{I}}(\alpha, A \wedge B) = F$
5. $v_{\mathcal{I}}(\alpha, A \vee B) = T$ iff $v_{\mathcal{I}}(\alpha, A) = T$ or $v_{\mathcal{I}}(\alpha, B) = T$, otherwise $v_{\mathcal{I}}(\alpha, A \vee B) = F$
6. $v_{\mathcal{I}}(\alpha, A \rightarrow B) = T$ iff $v_{\mathcal{I}}(\alpha, A) = F$ or $v_{\mathcal{I}}(\alpha, B) = T$, otherwise $v_{\mathcal{I}}(\alpha, A \rightarrow B) = F$
7. $v_{\mathcal{I}}(\alpha, \forall x A) = T$ iff $v_{\mathcal{I}}(\alpha\{x \leftarrow d\}, A) = T$ **for all** $d \in D$, otherwise $v_{\mathcal{I}}(\alpha, \forall x A) = F$
8. $v_{\mathcal{I}}(\alpha, \exists x A) = T$ iff $v_{\mathcal{I}}(\alpha\{x \leftarrow d\}, A) = T$ **for some** $d \in D$, otherwise $v_{\mathcal{I}}(\alpha, \exists x A) = F$
9. $v_{\mathcal{I}}(\alpha, \top) = T$ and $v_{\mathcal{I}}(\alpha, \perp) = F$

Semantics — Truth Value

Theorem 4.1 (Value of closed formulae).

*For a **closed** term or formula, the assignment has no influence on the term value or truth value. We can write $v_I(A)$ instead of $v_I(\alpha, A)$.*

Example: $A = \forall x p(a, x)$ with the interpretations

Semantics — Truth Value

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Example: $A = \forall x p(a, x)$ with the interpretations

1. $\mathcal{I} = (\mathbb{N}, \iota)$ with $p^{\iota} = \leq$ and $a^{\iota} = 0$

Semantics — Truth Value

Theorem 4.1 (Value of closed formulae).

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Example: $A = \forall x p(a, x)$ with the interpretations

1. $\mathcal{I} = (\mathbb{N}, \iota)$ with $p^{\iota} = \leq$ and $a^{\iota} = 0 \rightsquigarrow v_{\mathcal{I}}(A) = T$

Semantics — Truth Value

Theorem 4.1 (Value of closed formulae).

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Example: $A = \forall x p(a, x)$ with the interpretations

1. $\mathcal{I} = (\mathbb{N}, \iota)$ with $p^{\iota} = \leq$ and $a^{\iota} = 0 \rightsquigarrow v_{\mathcal{I}}(A) = T$
2. $\mathcal{I} = (\mathbb{N}, \iota)$ with $p^{\iota} = \leq$ and $a^{\iota} = 3$

Semantics — Truth Value

Theorem 4.1 (Value of closed formulae).

*For a **closed** term or formula, the assignment has no influence on the term value or truth value. We can write $v_{\mathcal{I}}(A)$ instead of $v_{\mathcal{I}}(\alpha, A)$.*

Example: $A = \forall x p(a, x)$ with the interpretations

1. $\mathcal{I} = (\mathbb{N}, \iota)$ with $p^{\iota} = \leq$ and $a^{\iota} = 0 \rightsquigarrow v_{\mathcal{I}}(A) = T$
2. $\mathcal{I} = (\mathbb{N}, \iota)$ with $p^{\iota} = \leq$ and $a^{\iota} = 3 \rightsquigarrow v_{\mathcal{I}}(A) = F$

Semantics — Truth Value

Theorem 4.1 (Value of closed formulae).

*For a **closed** term or formula, the assignment has no influence on the term value or truth value. We can write $v_{\mathcal{I}}(A)$ instead of $v_{\mathcal{I}}(\alpha, A)$.*

Example: $A = \forall x p(a, x)$ with the interpretations

1. $\mathcal{I} = (\mathbb{N}, \iota)$ with $p^{\iota} = \leq$ and $a^{\iota} = 0 \rightsquigarrow v_{\mathcal{I}}(A) = T$
2. $\mathcal{I} = (\mathbb{N}, \iota)$ with $p^{\iota} = \leq$ and $a^{\iota} = 3 \rightsquigarrow v_{\mathcal{I}}(A) = F$
3. $\mathcal{I} = (\mathbb{Z}, \iota)$ with $p^{\iota} = \leq$ and $a^{\iota} = 0$

Semantics — Truth Value

Theorem 4.1 (Value of closed formulae).

*For a **closed** term or formula, the assignment has no influence on the term value or truth value. We can write $v_{\mathcal{I}}(A)$ instead of $v_{\mathcal{I}}(\alpha, A)$.*

Example: $A = \forall x p(a, x)$ with the interpretations

1. $\mathcal{I} = (\mathbb{N}, \iota)$ with $p^{\iota} = \leq$ and $a^{\iota} = 0 \quad \rightsquigarrow v_{\mathcal{I}}(A) = T$
2. $\mathcal{I} = (\mathbb{N}, \iota)$ with $p^{\iota} = \leq$ and $a^{\iota} = 3 \quad \rightsquigarrow v_{\mathcal{I}}(A) = F$
3. $\mathcal{I} = (\mathbb{Z}, \iota)$ with $p^{\iota} = \leq$ and $a^{\iota} = 0 \quad \rightsquigarrow v_{\mathcal{I}}(A) = F$

Semantics — Truth Value

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*For a **closed** term or formula, the assignment has no influence on the term value or truth value. We can write $v_{\mathcal{I}}(A)$ instead of $v_{\mathcal{I}}(\alpha, A)$.*

Example: $A = \forall x p(a, x)$ with the interpretations

1. $\mathcal{I} = (\mathbb{N}, \iota)$ with $p^{\iota} = \leq$ and $a^{\iota} = 0 \rightsquigarrow v_{\mathcal{I}}(A) = T$
2. $\mathcal{I} = (\mathbb{N}, \iota)$ with $p^{\iota} = \leq$ and $a^{\iota} = 3 \rightsquigarrow v_{\mathcal{I}}(A) = F$
3. $\mathcal{I} = (\mathbb{Z}, \iota)$ with $p^{\iota} = \leq$ and $a^{\iota} = 0 \rightsquigarrow v_{\mathcal{I}}(A) = F$
4. $\mathcal{I} = (\{c, d, e, f\}, \iota)$ with $p^{\iota} = \leq_{lexi}$ and $a^{\iota} = c$

Semantics — Truth Value

Theorem 4.1 (Value of closed formulae).

For a *closed* term or formula, the assignment has no influence on the term value or truth value. We can write $v_{\mathcal{I}}(A)$ instead of $v_{\mathcal{I}}(\alpha, A)$.

Example: $A = \forall x p(a, x)$ with the interpretations

1. $\mathcal{I} = (\mathbb{N}, \iota)$ with $p^{\iota} = \leq$ and $a^{\iota} = 0 \rightsquigarrow v_{\mathcal{I}}(A) = T$
2. $\mathcal{I} = (\mathbb{N}, \iota)$ with $p^{\iota} = \leq$ and $a^{\iota} = 3 \rightsquigarrow v_{\mathcal{I}}(A) = F$
3. $\mathcal{I} = (\mathbb{Z}, \iota)$ with $p^{\iota} = \leq$ and $a^{\iota} = 0 \rightsquigarrow v_{\mathcal{I}}(A) = F$
4. $\mathcal{I} = (\{c, d, e, f\}, \iota)$ with $p^{\iota} = \leq_{lexi}$ and $a^{\iota} = c \rightsquigarrow v_{\mathcal{I}}(A) = T$

Semantics — Truth Value

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For a *closed* term or formula, the assignment has no influence on the term value or truth value. We can write $v_{\mathcal{I}}(A)$ instead of $v_{\mathcal{I}}(\alpha, A)$.

Example: $A = \forall x p(a, x)$ with the interpretations

1. $\mathcal{I} = (\mathbb{N}, \iota)$ with $p^{\iota} = \leq$ and $a^{\iota} = 0 \rightsquigarrow v_{\mathcal{I}}(A) = T$
2. $\mathcal{I} = (\mathbb{N}, \iota)$ with $p^{\iota} = \leq$ and $a^{\iota} = 3 \rightsquigarrow v_{\mathcal{I}}(A) = F$
3. $\mathcal{I} = (\mathbb{Z}, \iota)$ with $p^{\iota} = \leq$ and $a^{\iota} = 0 \rightsquigarrow v_{\mathcal{I}}(A) = F$
4. $\mathcal{I} = (\{c, d, e, f\}, \iota)$ with $p^{\iota} = \leq_{lexi}$ and $a^{\iota} = c \rightsquigarrow v_{\mathcal{I}}(A) = T$

Example: $B = \forall x \forall y (p(x, y) \rightarrow p(f(x, a), f(y, a)))$ with interpretations

Semantics — Truth Value

Theorem 4.1 (Value of closed formulae).

For a **closed** term or formula, the assignment has no influence on the term value or truth value. We can write $v_{\mathcal{I}}(A)$ instead of $v_{\mathcal{I}}(\alpha, A)$.

Example: $A = \forall x p(a, x)$ with the interpretations

1. $\mathcal{I} = (\mathbb{N}, \iota)$ with $p^{\iota} = \leq$ and $a^{\iota} = 0 \rightsquigarrow v_{\mathcal{I}}(A) = T$
2. $\mathcal{I} = (\mathbb{N}, \iota)$ with $p^{\iota} = \leq$ and $a^{\iota} = 3 \rightsquigarrow v_{\mathcal{I}}(A) = F$
3. $\mathcal{I} = (\mathbb{Z}, \iota)$ with $p^{\iota} = \leq$ and $a^{\iota} = 0 \rightsquigarrow v_{\mathcal{I}}(A) = F$
4. $\mathcal{I} = (\{c, d, e, f\}, \iota)$ with $p^{\iota} = \leq_{lexi}$ and $a^{\iota} = c \rightsquigarrow v_{\mathcal{I}}(A) = T$

Example: $B = \forall x \forall y (p(x, y) \rightarrow p(f(x, a), f(y, a)))$ with interpretations

1. $\mathcal{I} = (\mathbb{Z}, \iota)$ with $p^{\iota} = \leq$, $f^{\iota} = +$, and $a^{\iota} = 1$

Semantics — Truth Value

Theorem 4.1 (Value of closed formulae).

For a *closed* term or formula, the assignment has no influence on the term value or truth value. We can write $v_{\mathcal{I}}(A)$ instead of $v_{\mathcal{I}}(\alpha, A)$.

Example: $A = \forall x p(a, x)$ with the interpretations

1. $\mathcal{I} = (\mathbb{N}, \iota)$ with $p^{\iota} = \leq$ and $a^{\iota} = 0 \rightsquigarrow v_{\mathcal{I}}(A) = T$
2. $\mathcal{I} = (\mathbb{N}, \iota)$ with $p^{\iota} = \leq$ and $a^{\iota} = 3 \rightsquigarrow v_{\mathcal{I}}(A) = F$
3. $\mathcal{I} = (\mathbb{Z}, \iota)$ with $p^{\iota} = \leq$ and $a^{\iota} = 0 \rightsquigarrow v_{\mathcal{I}}(A) = F$
4. $\mathcal{I} = (\{c, d, e, f\}, \iota)$ with $p^{\iota} = \leq_{lexi}$ and $a^{\iota} = c \rightsquigarrow v_{\mathcal{I}}(A) = T$

Example: $B = \forall x \forall y (p(x, y) \rightarrow p(f(x, a), f(y, a)))$ with interpretations

1. $\mathcal{I} = (\mathbb{Z}, \iota)$ with $p^{\iota} = \leq$, $f^{\iota} = +$, and $a^{\iota} = 1$
 $\rightsquigarrow v_{\mathcal{I}}(B) = T$

Semantics — Truth Value

Theorem 4.1 (Value of closed formulae).

For a **closed** term or formula, the assignment has no influence on the term value or truth value. We can write $v_{\mathcal{I}}(A)$ instead of $v_{\mathcal{I}}(\alpha, A)$.

Example: $A = \forall x p(a, x)$ with the interpretations

1. $\mathcal{I} = (\mathbb{N}, \iota)$ with $p^{\iota} = \leq$ and $a^{\iota} = 0 \rightsquigarrow v_{\mathcal{I}}(A) = T$
2. $\mathcal{I} = (\mathbb{N}, \iota)$ with $p^{\iota} = \leq$ and $a^{\iota} = 3 \rightsquigarrow v_{\mathcal{I}}(A) = F$
3. $\mathcal{I} = (\mathbb{Z}, \iota)$ with $p^{\iota} = \leq$ and $a^{\iota} = 0 \rightsquigarrow v_{\mathcal{I}}(A) = F$
4. $\mathcal{I} = (\{c, d, e, f\}, \iota)$ with $p^{\iota} = \leq_{lexi}$ and $a^{\iota} = c \rightsquigarrow v_{\mathcal{I}}(A) = T$

Example: $B = \forall x \forall y (p(x, y) \rightarrow p(f(x, a), f(y, a)))$ with interpretations

1. $\mathcal{I} = (\mathbb{Z}, \iota)$ with $p^{\iota} = \leq$, $f^{\iota} = +$, and $a^{\iota} = 1$
 $\rightsquigarrow v_{\mathcal{I}}(B) = T$
2. $\mathcal{I} = (\mathbb{Z}, \iota)$ with $p^{\iota} = >$, $f^{\iota} = *$, and $a^{\iota} = -1$

Semantics — Truth Value

Theorem 4.1 (Value of closed formulae).

For a **closed** term or formula, the assignment has no influence on the term value or truth value. We can write $v_{\mathcal{I}}(A)$ instead of $v_{\mathcal{I}}(\alpha, A)$.

Example: $A = \forall x p(a, x)$ with the interpretations

1. $\mathcal{I} = (\mathbb{N}, \iota)$ with $p^\iota = \leq$ and $a^\iota = 0 \rightsquigarrow v_{\mathcal{I}}(A) = T$
2. $\mathcal{I} = (\mathbb{N}, \iota)$ with $p^\iota = \leq$ and $a^\iota = 3 \rightsquigarrow v_{\mathcal{I}}(A) = F$
3. $\mathcal{I} = (\mathbb{Z}, \iota)$ with $p^\iota = \leq$ and $a^\iota = 0 \rightsquigarrow v_{\mathcal{I}}(A) = F$
4. $\mathcal{I} = (\{c, d, e, f\}, \iota)$ with $p^\iota = \leq_{lexi}$ and $a^\iota = c \rightsquigarrow v_{\mathcal{I}}(A) = T$

Example: $B = \forall x \forall y (p(x, y) \rightarrow p(f(x, a), f(y, a)))$ with interpretations

1. $\mathcal{I} = (\mathbb{Z}, \iota)$ with $p^\iota = \leq$, $f^\iota = +$, and $a^\iota = 1$
 $\rightsquigarrow v_{\mathcal{I}}(B) = T$
2. $\mathcal{I} = (\mathbb{Z}, \iota)$ with $p^\iota = >$, $f^\iota = *$, and $a^\iota = -1$
 $\rightsquigarrow v_{\mathcal{I}}(B) = F$

Outline

- ▶ Motivation
- ▶ Syntax
- ▶ Variables
- ▶ Semantics
- ▶ **The Substitution Lemma**
- ▶ Satisfiability & Validity
- ▶ LK for First-order Logic
- ▶ Summary

The Substitution Lemma for Terms

Theorem 5.1 (Substitution Lemma for Terms).

*Given an interpretation $\mathcal{I} = (D, \iota)$ and a variable assignment α for $\mathcal{I}$.
Given also a variable $y \in \mathcal{V}$, and terms $t, s \in \mathcal{T}$*

$$v_{\mathcal{I}}(\alpha, t[y \setminus s]) = v_{\mathcal{I}}(\alpha\{y \leftarrow v_{\mathcal{I}}(\alpha, s)\}, t)$$

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Proof.

By structural induction on t .

The Substitution Lemma for Terms

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$$v_{\mathcal{I}}(\alpha, t[y \setminus s]) = v_{\mathcal{I}}(\alpha\{y \leftarrow v_{\mathcal{I}}(\alpha, s)\}, t)$$

Proof.

By structural induction on t . We abbreviate: $\alpha' := \alpha\{y \leftarrow v_{\mathcal{I}}(\alpha, s)\}$

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*Given an interpretation $\mathcal{I} = (D, \iota)$ and a variable assignment α for $\mathcal{I}$.
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$$v_{\mathcal{I}}(\alpha, t[y \setminus s]) = v_{\mathcal{I}}(\alpha\{y \leftarrow v_{\mathcal{I}}(\alpha, s)\}, t)$$

Proof.

By structural induction on t . We abbreviate: $\alpha' := \alpha\{y \leftarrow v_{\mathcal{I}}(\alpha, s)\}$

For a constant a , $a[y \setminus s] = a$, so $v_{\mathcal{I}}(\alpha, a[y \setminus s]) = v_{\mathcal{I}}(\alpha, a) = a^{\iota} = v_{\mathcal{I}}(\alpha', a)$

The Substitution Lemma for Terms

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*Given an interpretation $\mathcal{I} = (D, \iota)$ and a variable assignment α for $\mathcal{I}$.
Given also a variable $y \in \mathcal{V}$, and terms $t, s \in \mathcal{T}$*

$$v_{\mathcal{I}}(\alpha, t[y \setminus s]) = v_{\mathcal{I}}(\alpha\{y \leftarrow v_{\mathcal{I}}(\alpha, s)\}, t)$$

Proof.

By structural induction on t . We abbreviate: $\alpha' := \alpha\{y \leftarrow v_{\mathcal{I}}(\alpha, s)\}$

For a constant a , $a[y \setminus s] = a$, so $v_{\mathcal{I}}(\alpha, a[y \setminus s]) = v_{\mathcal{I}}(\alpha, a) = a^{\iota} = v_{\mathcal{I}}(\alpha', a)$

For a variable $x \neq y$, $x[y \setminus s] = x$, so

$$v_{\mathcal{I}}(\alpha, x[y \setminus s]) = v_{\mathcal{I}}(\alpha, x) = \alpha(x) = \alpha'(x) = v_{\mathcal{I}}(\alpha', x)$$

Proof of substitution lemma, continued

Proof.

For the variable y , $y[y \backslash s] = s$, so

$$v_I(\alpha, y[y \backslash s]) = v_I(\alpha, s) = v_I(\alpha\{y \leftarrow v_I(\alpha, s)\}, y)$$

Proof of substitution lemma, continued

Proof.

For the variable y , $y[y \setminus s] = s$, so

$$v_I(\alpha, y[y \setminus s]) = v_I(\alpha, s) = v_I(\alpha \{y \leftarrow v_I(\alpha, s)\}, y)$$

For a complex term, $f(\dots t_i \dots)[y \setminus s] = f(\dots t_i[y \setminus s] \dots)$, so

$$v_I(\alpha, f(\dots t_i \dots)[y \setminus s])$$

Proof of substitution lemma, continued

Proof.

For the variable y , $y[y \backslash s] = s$, so

$$v_{\mathcal{I}}(\alpha, y[y \backslash s]) = v_{\mathcal{I}}(\alpha, s) = v_{\mathcal{I}}(\alpha \{y \leftarrow v_{\mathcal{I}}(\alpha, s)\}, y)$$

For a complex term, $f(\dots t_i \dots)[y \backslash s] = f(\dots t_i[y \backslash s] \dots)$, so

$$\begin{aligned} & v_{\mathcal{I}}(\alpha, f(\dots t_i \dots)[y \backslash s]) \\ &= v_{\mathcal{I}}(\alpha, f(\dots t_i[y \backslash s] \dots)) \quad \text{by def. of substitution} \end{aligned}$$

Proof of substitution lemma, continued

Proof.

For the variable y , $y[y \backslash s] = s$, so

$$v_{\mathcal{I}}(\alpha, y[y \backslash s]) = v_{\mathcal{I}}(\alpha, s) = v_{\mathcal{I}}(\alpha \{y \leftarrow v_{\mathcal{I}}(\alpha, s)\}, y)$$

For a complex term, $f(\dots t_i \dots)[y \backslash s] = f(\dots t_i[y \backslash s] \dots)$, so

$$\begin{aligned} & v_{\mathcal{I}}(\alpha, f(\dots t_i \dots)[y \backslash s]) \\ &= v_{\mathcal{I}}(\alpha, f(\dots t_i[y \backslash s] \dots)) \quad \text{by def. of substitution} \\ &= f^v(\dots v_{\mathcal{I}}(\alpha, t_i[y \backslash s]) \dots) \quad \text{by model semantics} \end{aligned}$$

Proof of substitution lemma, continued

Proof.

For the variable y , $y[y \backslash s] = s$, so

$$v_{\mathcal{I}}(\alpha, y[y \backslash s]) = v_{\mathcal{I}}(\alpha, s) = v_{\mathcal{I}}(\alpha \{y \leftarrow v_{\mathcal{I}}(\alpha, s)\}, y)$$

For a complex term, $f(\dots t_i \dots)[y \backslash s] = f(\dots t_i[y \backslash s] \dots)$, so

$$\begin{aligned} & v_{\mathcal{I}}(\alpha, f(\dots t_i \dots)[y \backslash s]) \\ &= v_{\mathcal{I}}(\alpha, f(\dots t_i[y \backslash s] \dots)) \quad \text{by def. of substitution} \\ &= f^{\iota}(\dots v_{\mathcal{I}}(\alpha, t_i[y \backslash s]) \dots) \quad \text{by model semantics} \\ &= f^{\iota}(\dots v_{\mathcal{I}}(\alpha', t_i) \dots) \quad \text{by the induction hypothesis} \end{aligned}$$

Proof of substitution lemma, continued

Proof.

For the variable y , $y[y \backslash s] = s$, so

$$v_{\mathcal{I}}(\alpha, y[y \backslash s]) = v_{\mathcal{I}}(\alpha, s) = v_{\mathcal{I}}(\alpha\{y \leftarrow v_{\mathcal{I}}(\alpha, s)\}, y)$$

For a complex term, $f(\dots t_i \dots)[y \backslash s] = f(\dots t_i[y \backslash s] \dots)$, so

$$\begin{aligned} & v_{\mathcal{I}}(\alpha, f(\dots t_i \dots)[y \backslash s]) \\ &= v_{\mathcal{I}}(\alpha, f(\dots t_i[y \backslash s] \dots)) \quad \text{by def. of substitution} \\ &= f^{\iota}(\dots v_{\mathcal{I}}(\alpha, t_i[y \backslash s]) \dots) \quad \text{by model semantics} \\ &= f^{\iota}(\dots v_{\mathcal{I}}(\alpha', t_i) \dots) \quad \text{by the induction hypothesis} \\ &= v_{\mathcal{I}}(\alpha', f(\dots t_i \dots)) \quad \text{by model semantics} \end{aligned}$$



The Substitution Lemma for Formulae

Theorem 5.2 (Substitution Lemma for Formulae).

*Given an interpretation $\mathcal{I} = (D, \iota)$ and a variable assignment α for $\mathcal{I}$.
Given also a variable $y \in \mathcal{V}$, a formula A and a term $s \in \mathcal{T}$, such that
 $\{y \setminus s\}$ is capture-free for A .*

$$v_{\mathcal{I}}(\alpha, A[y \setminus s]) = v_{\mathcal{I}}(\alpha\{y \leftarrow v_{\mathcal{I}}(\alpha, s)\}, A)$$

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Satisfiability and Validity

Definition 6.1 (Satisfiable, Model, Unsatisfiable, Valid, Invalid).

Let A be a *closed* (first-order) formula and $U = \{A_1, \dots\}$ be a set of *closed* (first-order) formulae A_i .

- ▶ A is *satisfiable* iff $v_{\mathcal{I}}(A) = T$ for some interpretation $\mathcal{I}$.
- ▶ A satisfying interpretation $\mathcal{I}$ for A is called a *model* for A .

Satisfiability and Validity

Definition 6.1 (Satisfiable, Model, Unsatisfiable, Valid, Invalid).

Let A be a *closed* (first-order) formula and $U=\{A_1, \dots\}$ be a set of *closed* (first-order) formulae A_i .

- ▶ A is *satisfiable* iff $v_{\mathcal{I}}(A) = T$ for some interpretation $\mathcal{I}$.
- ▶ A satisfying interpretation $\mathcal{I}$ for A is called a *model* for A .
- ▶ $U=\{A_1, \dots\}$ is *satisfiable* iff there is (common) model for all A_i .

Satisfiability and Validity

Definition 6.1 (Satisfiable, Model, Unsatisfiable, Valid, Invalid).

Let A be a *closed* (first-order) formula and $U = \{A_1, \dots\}$ be a set of *closed* (first-order) formulae A_i .

- ▶ A is *satisfiable* iff $v_{\mathcal{I}}(A) = T$ for some interpretation $\mathcal{I}$.
- ▶ A satisfying interpretation $\mathcal{I}$ for A is called a *model* for A .
- ▶ $U = \{A_1, \dots\}$ is *satisfiable* iff there is (common) model for all A_i .
- ▶ A (resp. U) is *unsatisfiable* iff A (resp. U) is *not* satisfiable.

Satisfiability and Validity

Definition 6.1 (Satisfiable, Model, Unsatisfiable, Valid, Invalid).

Let A be a *closed* (first-order) formula and $U = \{A_1, \dots\}$ be a set of *closed* (first-order) formulae A_i .

- ▶ A is *satisfiable* iff $v_{\mathcal{I}}(A) = T$ for some interpretation $\mathcal{I}$.
- ▶ A satisfying interpretation $\mathcal{I}$ for A is called a *model* for A .
- ▶ $U = \{A_1, \dots\}$ is *satisfiable* iff there is (common) model for all A_i .
- ▶ A (resp. U) is *unsatisfiable* iff A (resp. U) is *not* satisfiable.
- ▶ A is *valid*, denoted $\models A$, iff $v_{\mathcal{I}}(A) = T$ for all interpretations $\mathcal{I}$.
- ▶ A is *invalid/falsifiable* iff A is *not* valid.

Satisfiability and Validity

Definition 6.1 (Satisfiable, Model, Unsatisfiable, Valid, Invalid).

Let A be a **closed** (first-order) formula and $U = \{A_1, \dots\}$ be a set of **closed** (first-order) formulae A_i .

- ▶ A is **satisfiable** iff $v_{\mathcal{I}}(A) = T$ for some interpretation $\mathcal{I}$.
- ▶ A satisfying interpretation $\mathcal{I}$ for A is called a **model** for A .
- ▶ $U = \{A_1, \dots\}$ is **satisfiable** iff there is (common) model for all A_i .
- ▶ A (resp. U) is **unsatisfiable** iff A (resp. U) is **not** satisfiable.
- ▶ A is **valid**, denoted $\models A$, iff $v_{\mathcal{I}}(A) = T$ for all interpretations $\mathcal{I}$.
- ▶ A is **invalid/falsifiable** iff A is **not** valid.

Theorem 6.1 (Satisfiable, Valid, Unsatisfiable, Invalid).

A is **valid** iff $\neg A$ is **unsatisfiable**. A is **satisfiable** iff $\neg A$ is **invalid**.

Examples for Satisfiable and Invalid Formulae

Example: $A = \forall x p(a, x)$

Examples for Satisfiable and Invalid Formulae

Example: $A = \forall x p(a, x)$

1. $\mathcal{I} = (\mathbb{N}, \iota)$ with $p^\iota = \leq$ and $a^\iota = 3$

Examples for Satisfiable and Invalid Formulae

Example: $A = \forall x p(a, x)$

1. $\mathcal{I} = (\mathbb{N}, \iota)$ with $p^\iota = \leq$ and $a^\iota = 3 \rightsquigarrow v_{\mathcal{I}}(A) = F$
 $\rightsquigarrow A$ is invalid

Examples for Satisfiable and Invalid Formulae

Example: $A = \forall x p(a, x)$

1. $\mathcal{I} = (\mathbb{N}, \iota)$ with $p^\iota = \leq$ and $a^\iota = 3 \rightsquigarrow v_{\mathcal{I}}(A) = F$
 $\rightsquigarrow A$ is invalid
2. $\mathcal{I} = (\{c, d, e, f\}, \iota)$ with $p^\iota = \leq_{lexi}$ and $a^\iota = c$

Examples for Satisfiable and Invalid Formulae

Example: $A = \forall x p(a, x)$

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 $\rightsquigarrow A$ is invalid
2. $\mathcal{I} = (\{c, d, e, f\}, \iota)$ with $p^\iota = \leq_{lexi}$ and $a^\iota = c \rightsquigarrow v_{\mathcal{I}}(A) = T$
 $\rightsquigarrow A$ is satisfiable ($\mathcal{I}$ is a model)

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Example: $B = \forall x \forall y (p(x, y) \rightarrow p(f(x, a), f(y, a)))$

1. $\mathcal{I} = (\mathbb{Z}, \iota)$ with $p^\iota = \leq$, $f^\iota = +$, and $a^\iota = 1$

Examples for Satisfiable and Invalid Formulae

Example: $A = \forall x p(a, x)$

1. $\mathcal{I} = (\mathbb{N}, \iota)$ with $p^\iota = \leq$ and $a^\iota = 3 \rightsquigarrow v_{\mathcal{I}}(A) = F$
 $\rightsquigarrow A$ is invalid
2. $\mathcal{I} = (\{c, d, e, f\}, \iota)$ with $p^\iota = \leq_{lexi}$ and $a^\iota = c \rightsquigarrow v_{\mathcal{I}}(A) = T$
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Example: $B = \forall x \forall y (p(x, y) \rightarrow p(f(x, a), f(y, a)))$

1. $\mathcal{I} = (\mathbb{Z}, \iota)$ with $p^\iota = \leq$, $f^\iota = +$, and $a^\iota = 1 \rightsquigarrow v_{\mathcal{I}}(B) = T$
 $\rightsquigarrow$ satisfiable ($\mathcal{I}$ is a model)

Examples for Satisfiable and Invalid Formulae

Example: $A = \forall x p(a, x)$

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Example: $B = \forall x \forall y (p(x, y) \rightarrow p(f(x, a), f(y, a)))$

1. $\mathcal{I} = (\mathbb{Z}, \iota)$ with $p^\iota = \leq$, $f^\iota = +$, and $a^\iota = 1 \rightsquigarrow v_{\mathcal{I}}(B) = T$
 $\rightsquigarrow$ satisfiable ($\mathcal{I}$ is a model)
2. $\mathcal{I} = (\mathbb{Z}, \iota)$ with $p^\iota = >$, $f^\iota = *$, and $a^\iota = -1$

Examples for Satisfiable and Invalid Formulae

Example: $A = \forall x p(a, x)$

1. $\mathcal{I} = (\mathbb{N}, \iota)$ with $p^\iota = \leq$ and $a^\iota = 3 \rightsquigarrow v_{\mathcal{I}}(A) = F$
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2. $\mathcal{I} = (\mathbb{Z}, \iota)$ with $p^\iota = >$, $f^\iota = *$, and $a^\iota = -1 \rightsquigarrow v_{\mathcal{I}}(B) = F$
 $\rightsquigarrow$ invalid ($\mathcal{I}$ is a “counter-model”)

Examples for Satisfiable and Invalid Formulae

Example: $A = \forall x p(a, x)$

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$\rightsquigarrow$ satisfiable (e.g. $p^\iota = “=”$),

Examples for Satisfiable and Invalid Formulae

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Example: $\forall x \forall y (p(x, y) \rightarrow p(y, x))$

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Example: $\exists x \exists y (p(x) \wedge \neg p(y))$

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Example: $A = \forall x p(a, x)$

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Example: $\forall x \forall y (p(x, y) \rightarrow p(y, x))$

$\rightsquigarrow$ satisfiable (e.g. $p^\iota = “=”$), but invalid (e.g. $p^\iota = “<”$)

Example: $\exists x \exists y (p(x) \wedge \neg p(y))$

$\rightsquigarrow$ only satisfiable for $|D| \geq 2$, invalid (e.g. $D = \{a\}$)

Logical Equivalence

The concept of **logical equivalence** can be adapted to first-order logic, i.e. to closed first-order formulae.

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Definition 6.2 (Logical Equivalence).

*Let A_1, A_2 be two closed formulae. A_1 is **logically equivalent** to A_2 , denoted $A_1 \equiv A_2$ iff $v_{\mathcal{I}}(A_1) = v_{\mathcal{I}}(A_2)$ for all interpretations $\mathcal{I}$.*

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Theorem 6.2 (Relation $\equiv$ and $\leftrightarrow$).

Let A, B be two closed formulae. Then $A \equiv B$ iff $\models A \leftrightarrow B$.

Remark: $A \leftrightarrow B := (A \rightarrow B) \wedge (B \rightarrow A)$

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Important: even though $\equiv$ and $\leftrightarrow$ are closely related, they are different relations. Whereas $\leftrightarrow$ is part of the object language (i.e. the definition of formulae), $\equiv$ is used in the meta-language to talk about or relate formulae.

Logically Equivalent Formulae

Duality: $\forall$ can be expressed with $\exists$, and vice versa

► $\models \forall x A(x) \leftrightarrow \neg \exists x \neg A(x)$

► $\models \exists x A(x) \leftrightarrow \neg \forall x \neg A(x)$

Logically Equivalent Formulae

Duality: $\forall$ can be expressed with $\exists$, and vice versa

▶ $\models \forall x A(x) \leftrightarrow \neg \exists x \neg A(x)$

▶ $\models \exists x A(x) \leftrightarrow \neg \forall x \neg A(x)$

Commutativity:

▶ $\models \forall x \forall y A(x, y) \leftrightarrow \forall y \forall x A(x, y)$

▶ $\models \exists x \exists y A(x, y) \leftrightarrow \exists y \exists x A(x, y)$

▶ $\models \exists x \forall y A(x, y) \rightarrow \forall y \exists x A(x, y)$ (other direction is not valid!)

Logically Equivalent Formulae

Duality: $\forall$ can be expressed with $\exists$, and vice versa

$$\models \forall x A(x) \leftrightarrow \neg \exists x \neg A(x)$$

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Commutativity:

$$\models \forall x \forall y A(x, y) \leftrightarrow \forall y \forall x A(x, y)$$

$$\models \exists x \exists y A(x, y) \leftrightarrow \exists y \exists x A(x, y)$$

$$\models \exists x \forall y A(x, y) \rightarrow \forall y \exists x A(x, y) \quad (\text{other direction is not valid!})$$

Distributivity:

$$\models \exists x (A(x) \vee B(x)) \leftrightarrow \exists x A(x) \vee \exists x B(x)$$

$$\models \forall x (A(x) \wedge B(x)) \leftrightarrow \forall x A(x) \wedge \forall x B(x)$$

$$\models \forall x A(x) \vee \forall x B(x) \rightarrow \forall x (A(x) \vee B(x)) \quad (\text{other direction not valid!})$$

$$\models \exists x (A(x) \wedge B(x)) \rightarrow \exists x A(x) \wedge \exists x B(x) \quad (\text{other direction not valid!})$$

See [Ben-Ari 2012] for **more equivalences** involving quantifiers.

Logical Consequence

Definition 6.3 (Logical Consequence).

Let A be a closed formula and U be a set of closed formulae. A is a *logical consequence* of U , denoted $U \models A$, iff every model of U is a model of A , i.e. $v_{\mathcal{I}}(A_i) = T$ for all $A_i \in U$ implies $v_{\mathcal{I}}(A) = T$.

Logical Consequence

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Let A be a closed formula and U be a set of closed formulae. A is a **logical consequence** of U , denoted $U \models A$, iff every model of U is a model of A , i.e. $v_{\mathcal{I}}(A_i) = T$ for all $A_i \in U$ implies $v_{\mathcal{I}}(A) = T$.

Theorem 6.3 (Logical Consequence and Validity).

Let A be a closed formula and $U = \{A_1, \dots, A_n\}$ be a set of closed formulae. Then $U \models A$ iff $\models (A_1 \wedge \dots \wedge A_n) \rightarrow A$.

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Let A be a closed formula and $U = \{A_1, \dots, A_n\}$ be a set of closed formulae. Then $U \models A$ iff $\models (A_1 \wedge \dots \wedge A_n) \rightarrow A$.

- ▶ again, we can **reduce** the problem of “logical consequence” to the problem of determining if a formula is **valid**
- ▶ hence, we need methods or **proof search calculi** that can deal with **first-order formulae**

Outline

- ▶ Motivation
- ▶ Syntax
- ▶ Variables
- ▶ Semantics
- ▶ The Substitution Lemma
- ▶ Satisfiability & Validity
- ▶ **LK for First-order Logic**
- ▶ Summary

LK — Axiom and Propositional Rules

► axiom

$$\frac{}{\Gamma, A \Rightarrow A, \Delta} \text{ axiom}$$

► rules for $\wedge$ (conjunction)

$$\frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta} \wedge\text{-left}$$

$$\frac{\Gamma \Rightarrow A, \Delta \quad \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \wedge B, \Delta} \wedge\text{-right}$$

► rules for $\vee$ (disjunction)

$$\frac{\Gamma, A \Rightarrow \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \vee B \Rightarrow \Delta} \vee\text{-left}$$

$$\frac{\Gamma \Rightarrow A, B, \Delta}{\Gamma \Rightarrow A \vee B, \Delta} \vee\text{-right}$$

► rules for $\rightarrow$ (implication)

$$\frac{\Gamma \Rightarrow A, \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \rightarrow B \Rightarrow \Delta} \rightarrow\text{-left}$$

$$\frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \rightarrow B, \Delta} \rightarrow\text{-right}$$

► rules for $\neg$ (negation)

$$\frac{\Gamma \Rightarrow A, \Delta}{\Gamma, \neg A \Rightarrow \Delta} \neg\text{-left}$$

$$\frac{\Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \neg A, \Delta} \neg\text{-right}$$

LK — Rules for Universal and Existential Quantifier

- rules for $\forall$ (universal quantifier)

LK — Rules for Universal and Existential Quantifier

► rules for $\forall$ (universal quantifier)

$$\frac{\Gamma, A[x \backslash t], \forall x A \implies \Delta}{\Gamma, \forall x A \implies \Delta} \forall\text{-left}$$

LK — Rules for Universal and Existential Quantifier

► rules for $\forall$ (universal quantifier)

$$\frac{\Gamma, A[x \backslash t], \forall x A \implies \Delta}{\Gamma, \forall x A \implies \Delta} \forall\text{-left} \qquad \frac{\Gamma \implies A[x \backslash a], \Delta}{\Gamma \implies \forall x A, \Delta} \forall\text{-right}^*$$

LK — Rules for Universal and Existential Quantifier

► rules for $\forall$ (universal quantifier)

$$\frac{\Gamma, A[x \backslash t], \forall x A \implies \Delta}{\Gamma, \forall x A \implies \Delta} \forall\text{-left} \qquad \frac{\Gamma \implies A[x \backslash a], \Delta}{\Gamma \implies \forall x A, \Delta} \forall\text{-right}^*$$

- t is an arbitrary closed term
- **Eigenvariable condition** for the rule $\forall\text{-right}^*$: a must not occur in the conclusion, i.e. in Γ , Δ , or A
- the formula $\forall x A$ is preserved in the premise of the rule $\forall\text{-left}$

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- ## ► rules for $\exists$ (existential quantifier)

$$\frac{\Gamma, A[x \backslash a] \implies \Delta}{\Gamma, \exists x A \implies \Delta} \exists\text{-left}^*$$

LK — Rules for Universal and Existential Quantifier

► rules for $\forall$ (universal quantifier)

$$\frac{\Gamma, A[x \backslash t], \forall x A \implies \Delta}{\Gamma, \forall x A \implies \Delta} \forall\text{-left} \qquad \frac{\Gamma \implies A[x \backslash a], \Delta}{\Gamma \implies \forall x A, \Delta} \forall\text{-right}^*$$

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- **Eigenvariable condition** for the rule $\forall\text{-right}^*$: a must not occur in the conclusion, i.e. in Γ , Δ , or A
- the formula $\forall x A$ is preserved in the premise of the rule $\forall\text{-left}$

► rules for $\exists$ (existential quantifier)

$$\frac{\Gamma, A[x \backslash a] \implies \Delta}{\Gamma, \exists x A \implies \Delta} \exists\text{-left}^* \qquad \frac{\Gamma \implies \exists x A, A[x \backslash t], \Delta}{\Gamma \implies \exists x A, \Delta} \exists\text{-right}$$

LK — Rules for Universal and Existential Quantifier

► rules for $\forall$ (universal quantifier)

$$\frac{\Gamma, A[x \backslash t], \forall x A \implies \Delta}{\Gamma, \forall x A \implies \Delta} \forall\text{-left} \qquad \frac{\Gamma \implies A[x \backslash a], \Delta}{\Gamma \implies \forall x A, \Delta} \forall\text{-right}^*$$

- t is an arbitrary closed term
- **Eigenvariable condition** for the rule $\forall\text{-right}^*$: a must not occur in the conclusion, i.e. in Γ , Δ , or A
- the formula $\forall x A$ is preserved in the premise of the rule $\forall\text{-left}$

► rules for $\exists$ (existential quantifier)

$$\frac{\Gamma, A[x \backslash a] \implies \Delta}{\Gamma, \exists x A \implies \Delta} \exists\text{-left}^* \qquad \frac{\Gamma \implies \exists x A, A[x \backslash t], \Delta}{\Gamma \implies \exists x A, \Delta} \exists\text{-right}$$

- t is an arbitrary closed term
- **Eigenvariable condition** for the rule $\exists\text{-left}^*$: a must not occur in the conclusion, i.e. in Γ , Δ , or A
- the formula $\exists x A$ is preserved in the premise of the rule $\exists\text{-right}$

Soundness and Completeness

Theorem 7.1 (Soundness and Completeness of LK).

The calculus of natural deduction LK is sound and complete, i.e.

- ▶ *if A is provable in LK, then A is valid (if $\vdash A$ then $\models A$)*
- ▶ *if A is valid, then A is provable in LK (if $\models A$ then $\vdash A$)*

Proof.

Next week.



Examples of LK Proofs

Example: $p(a) \rightarrow \exists x p(x)$

Examples of LK Proofs

Example: $p(a) \rightarrow \exists x p(x)$

$$\implies p(a) \rightarrow \exists x p(x)$$

Examples of LK Proofs

Example: $p(a) \rightarrow \exists x p(x)$

$$\frac{p(a) \implies \exists x p(x)}{\implies p(a) \rightarrow \exists x p(x)} \rightarrow\text{-right}$$

Examples of LK Proofs

Example: $p(a) \rightarrow \exists x p(x)$

$$\frac{\frac{p(a) \implies p(a), \exists x p(x)}{p(a) \implies \exists x p(x)} \exists\text{-right}}{\implies p(a) \rightarrow \exists x p(x)} \rightarrow\text{-right}$$

Examples of LK Proofs

Example: $p(a) \rightarrow \exists x p(x)$

$$\begin{array}{c}
 \frac{}{p(a) \Rightarrow p(a), \exists x p(x)} \text{ axiom} \\
 \frac{}{p(a) \Rightarrow \exists x p(x)} \exists\text{-right} \\
 \frac{}{\Rightarrow p(a) \rightarrow \exists x p(x)} \rightarrow\text{-right}
 \end{array}$$

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Example: $p(a) \rightarrow \exists x p(x)$

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 \frac{p(a) \implies p(a), \exists x p(x)}{} \text{axiom} \\
 \frac{}{p(a) \implies \exists x p(x)} \exists\text{-right} \\
 \frac{p(a) \implies \exists x p(x)}{\implies p(a) \rightarrow \exists x p(x)} \rightarrow\text{-right}
 \end{array}$$

Example: $p(a) \rightarrow p(b)$

Examples of LK Proofs

Example: $p(a) \rightarrow \exists x p(x)$

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 \frac{p(a) \implies p(a), \exists x p(x)}{} \text{axiom} \\
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 \implies p(a) \rightarrow \exists x p(x) \quad \rightarrow\text{-right}
 \end{array}$$

Example: $p(a) \rightarrow p(b)$

$$\frac{p(a) \implies p(b)}{\implies p(a) \rightarrow p(b)} \rightarrow\text{-right}$$

Examples of LK Proofs

Example: $p(a) \rightarrow \exists x p(x)$

$$\begin{array}{c}
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 \Rightarrow p(a) \rightarrow \exists x p(x) \rightarrow\text{-right}
 \end{array}$$

Example: $p(a) \rightarrow p(b)$

$$\begin{array}{c}
 \frac{p(a) \Rightarrow p(b)}{\Rightarrow p(a) \rightarrow p(b)} \text{ (?) } \rightarrow\text{-right}
 \end{array}$$

Examples of LK Proofs

Example: $p(a) \rightarrow \exists x p(x)$

$$\frac{\frac{\frac{p(a)}{p(a) \Rightarrow p(a), \exists x p(x)} \text{ axiom}}{p(a) \Rightarrow \exists x p(x)} \exists\text{-right}}{\Rightarrow p(a) \rightarrow \exists x p(x)} \rightarrow\text{-right}$$

Example: $\forall x p(x) \rightarrow \exists x p(x)$

Example: $p(a) \rightarrow p(b)$

$$\frac{\frac{p(a)}{p(a) \Rightarrow p(b)} (?)}{\Rightarrow p(a) \rightarrow p(b)} \rightarrow\text{-right}$$

Examples of LK Proofs

Example: $p(a) \rightarrow \exists x p(x)$

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Example: $p(a) \rightarrow p(b)$

$$\frac{\frac{p(a) \Rightarrow p(b)}{\Rightarrow p(a) \rightarrow p(b)} (?)}{\Rightarrow p(a) \rightarrow p(b)} \rightarrow\text{-right}$$

Example: $\forall x p(x) \rightarrow \exists x p(x)$

$$\Rightarrow \forall x p(x) \rightarrow \exists x p(x)$$

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Example: $\forall x p(x) \rightarrow \exists x p(x)$

$$\frac{\frac{p(c), \forall x p(x) \Rightarrow \exists x p(x)}{\forall x p(x) \Rightarrow \exists x p(x)} \forall\text{-left}}{\Rightarrow \forall x p(x) \rightarrow \exists x p(x)} \rightarrow\text{-right}$$

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$$\frac{\frac{\frac{}{\exists x p(x) \Rightarrow p(a)} \exists\text{-left}^*}}{\Rightarrow \exists x p(x) \rightarrow p(a)} \rightarrow\text{-right}$$

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rule $\exists\text{-left}^*$ with $p(x)[x \setminus a]$ **cannot** be applied as a occurs in the premise (Eigenvariable condition!)

Outline

- ▶ Motivation
- ▶ Syntax
- ▶ Variables
- ▶ Semantics
- ▶ The Substitution Lemma
- ▶ Satisfiability & Validity
- ▶ LK for First-order Logic
- ▶ **Summary**

Summary

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- ▶ the **semantics** consists of a **domain** D and an **interpretation** ι
- ▶ the **interpretation** ι relates constants to elements of the domain, function symbols to actual functions, and predicates to relations
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- ▶ Next week: Soundness and completeness