

IN3070/4070 – Logic – Autumn 2025

Lecture 7: Resolution

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25th September 2025



DEPARTMENT OF
INFORMATICS



UNIVERSITY OF
OSLO

Today's Plan

- ▶ Introduction
- ▶ Repetition: Negation Normal Form
- ▶ Conjunctive Normal Form
- ▶ Clausal Form
- ▶ Resolution
- ▶ Soundness of Resolution
- ▶ Completeness of Resolution

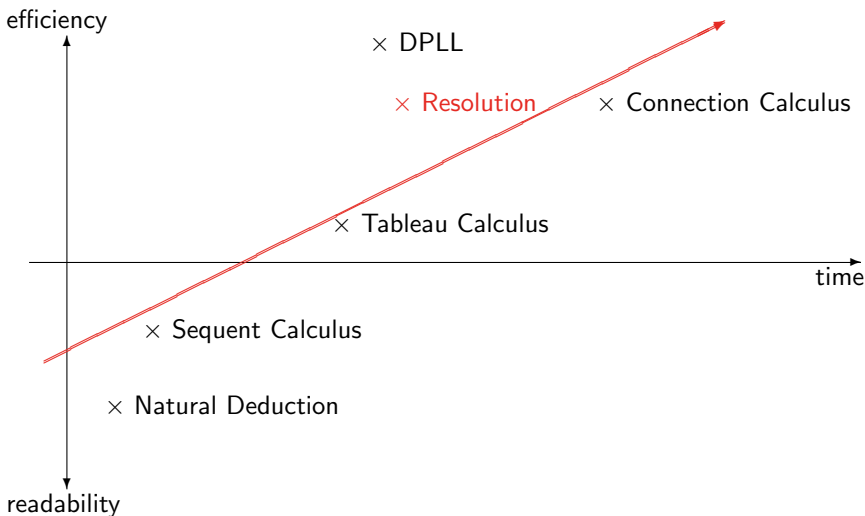
Oblig 2

- ▶ Published on Monday, 29 September
- ▶ Has a question about first order resolution
 - ▶ Start on the rest, do that after the next lecture
- ▶ Deadline for first attempt after two weeks

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Proof Search Calculi



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- ▶ works for first-order formulae in clausal form (e.g. conjunctive or disjunctive normal form)
- ▶ consists of one (two for first-order) **inference rules** and one **axiom**
- ▶ is one of the most popular proof search calculi
- ▶ has been implemented in many automated theorem provers



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Theorem 2.1.

Every formula in first-order logic can be transformed into an equivalent formula in NNF.

Proof.

To convert an arbitrary formula to a formula in NNF, remove implications, and push negations inwards, preserving equivalence, using the following:

$$A \rightarrow B \equiv \neg A \vee B$$

$$\neg(A \wedge B) \equiv \neg A \vee \neg B$$

$$\neg(A \vee B) \equiv \neg A \wedge \neg B$$

$$\neg(\forall x A) \equiv \exists x \neg A$$

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$$\neg(\neg A) \equiv A$$



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To convert an arbitrary propositional formula to a formula in CNF perform the following steps, each of which preserves logical equivalence:

- (1) Convert to negation normal form.
- (2) Use the distributive laws to move conjunctions inside disjunctions to the outside

$$A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$$



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$(p \vee \neg q) \wedge (\neg p \vee q)$ in clausal form: $\{\{p, \neg q\}, \{\neg p, q\}\}$

Transformation to Clausal Form

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This follows from the previous theorem, where we transformed a formula to CNF. Each disjunction is then transformed to a clause (of literals), and the clausal form is the set of these clauses. □

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The Resolution Rule

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The complementary literal $\bar{L}$ of a literal L is A if L is of the form $\neg A$, otherwise it is $\neg L$.

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Definition 5.2 (Resolution Rule).

*Let C_1, C_2 be clauses with $L \in C_1$ and $\bar{L} \in C_2$. The **resolvent** C' of C_1 and C_2 is $(C_1 \setminus \{L\}) \cup (C_2 \setminus \{\bar{L}\})$. C_1 and C_2 are the **parents** of C' .*

The Resolution Rule – Example

Example: Let $C_1 = \{a, b, \neg c\}$ and $C_2 = \{b, c, \neg e\}$.

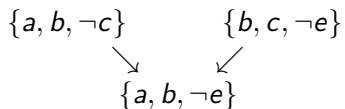
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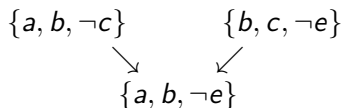
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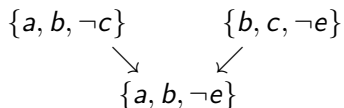
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Observations:

- ▶ if $\{a, b, \neg c\}$ and $\{b, c, \neg e\} \equiv (a \vee b \vee \neg c) \wedge (b \vee c \vee \neg e)$ are **true** in $\mathcal{I}$, then $(a \vee b)$ is true (if c is true) or $(b \vee \neg e)$ is true (if c is false); hence $(a \vee b \vee \neg e)$ is **true**

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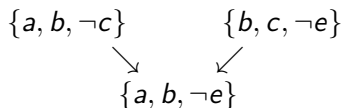
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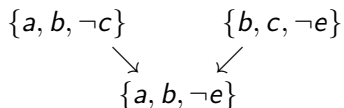
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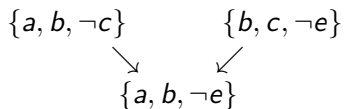
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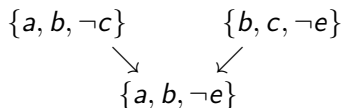
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- ▶ **goal:** derive empty clause $\square$

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Given a set of clauses S .

- 1. apply the resolution rule to a pair of clauses $\{C_1, C_2\} \subseteq S$ that has not been chosen before; let C' be the resolvent*

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- ▶ CNF: $p \wedge q \wedge \neg p$

Resolution Calculus – Example 1

- ▶ Prove validity of: $(p \wedge q) \rightarrow p$
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- ▶ Prove validity of: $(p \wedge q) \rightarrow p$
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- ▶ Show unsatisfiability of: $\neg((p \wedge q) \rightarrow p)$
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- ▶ Clause set: $\{\{p\}, \{q\}, \{\neg p\}\}$
- ▶ Resolve $\{p\}$ with $\{\neg p\}$
- ▶ Resolvent: $\square$

Resolution Calculus – Example 2

- Prove validity of: $p \wedge (p \rightarrow q) \rightarrow q$

Resolution Calculus – Example 2

- ▶ Prove validity of: $p \wedge (p \rightarrow q) \rightarrow q$
- ▶ Show unsatisfiability of: $\neg(p \wedge (p \rightarrow q) \rightarrow q)$

Resolution Calculus – Example 2

- ▶ Prove validity of: $p \wedge (p \rightarrow q) \rightarrow q$
- ▶ Show unsatisfiability of: $\neg(p \wedge (p \rightarrow q) \rightarrow q)$
- ▶ Equivalent to: $p \wedge (p \rightarrow q) \wedge \neg q$

Resolution Calculus – Example 2

- ▶ Prove validity of: $p \wedge (p \rightarrow q) \rightarrow q$
- ▶ Show unsatisfiability of: $\neg(p \wedge (p \rightarrow q) \rightarrow q)$
- ▶ Equivalent to: $p \wedge (p \rightarrow q) \wedge \neg q$
- ▶ CNF: $p \wedge (\neg p \vee q) \wedge \neg q$

Resolution Calculus – Example 2

- ▶ Prove validity of: $p \wedge (p \rightarrow q) \rightarrow q$
- ▶ Show unsatisfiability of: $\neg(p \wedge (p \rightarrow q) \rightarrow q)$
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- ▶ Clause set: $\{\{p\}, \{\neg p, q\}, \{\neg q\}\}$

Resolution Calculus – Example 2

- ▶ Prove validity of: $p \wedge (p \rightarrow q) \rightarrow q$
- ▶ Show unsatisfiability of: $\neg(p \wedge (p \rightarrow q) \rightarrow q)$
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- ▶ CNF: $p \wedge (\neg p \vee q) \wedge \neg q$
- ▶ Clause set: $\{\{p\}, \{\neg p, q\}, \{\neg q\}\}$
- ▶ Resolution step 1: between $\{p\}$ and $\{\neg p, q\}$

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- ▶ Resolvent: $\{q\}$
- ▶ New clause set: $\{\{p\}, \{\neg p, q\}, \{\neg q\}, \{q\}\}$
- ▶ Resolution step 2: between $\{\neg q\}$ and $\{q\}$

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- ▶ Resolvent: $\{q\}$
- ▶ New clause set: $\{\{p\}, \{\neg p, q\}, \{\neg q\}, \{q\}\}$
- ▶ Resolution step 2: between $\{\neg q\}$ and $\{q\}$
- ▶ Resolvent: $\square$
- ▶ Need to use resolution rule on new resolvents

Resolution Calculus – Example 3

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 2. $\{p\}$
 3. $\{q\}$
 4. $\{\neg r\}$
 5. $\{\neg q, r\}$ — resolvent of 1. and 2.

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Resolution Calculus – Example 4

► Prove validity of: $(p \rightarrow q) \rightarrow ((p \rightarrow r) \rightarrow (p \rightarrow (q \wedge r)))$

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Definition 5.4 (Resolution Calculus).

*The **resolution calculus** has one axiom and one (inference) rule.*

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 \\
 \hline
 C_1, \dots, C_i \cup \{L\}, \dots, C_j \cup \{\bar{L}\}, \dots, C_n, C_i \cup C_j \\
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- ▶ **terminates**, if all clause pairs $C_i \cup \{L\}, C_j \cup \{\bar{L}\}$ have been considered

Outline

- ▶ Introduction
- ▶ Repetition: Negation Normal Form
- ▶ Conjunctive Normal Form
- ▶ Clausal Form
- ▶ Resolution
- ▶ Soundness of Resolution
- ▶ Completeness of Resolution

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2. A set of clauses containing the empty clause is unsatisfiable

Resolution Preserves Satisfiability

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$\mathcal{I} \models L$ $\mathcal{I} \models C_2$, and clauses are *disjunctions* of their literals, so $\mathcal{I}$ satisfies one of the literals in C_2 , but not $\bar{L}$. So: $\mathcal{I} \models C_2 \setminus \{\bar{L}\}$.

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But the empty clause $\square$ contains no literals, so that is a contradiction.



Outline

- ▶ Introduction
- ▶ Repetition: Negation Normal Form
- ▶ Conjunctive Normal Form
- ▶ Clausal Form
- ▶ Resolution
- ▶ Soundness of Resolution
- ▶ Completeness of Resolution

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- ▶ We will go through Robinson's original proof

Semantic Trees

The completeness proof uses the following concept:

Definition 7.1 (Semantic Trees).

A *semantic tree* is a binary tree where:

- ▶ The root is labelled by the symbol $\perp$,
- ▶ Every node has either no children or two children,
- ▶ For every node that has children, there is some atom A such that one child is labeled with A and the other with $\neg A$
- ▶ There are not two complementary literals A and $\neg A$ on any path starting at the root.

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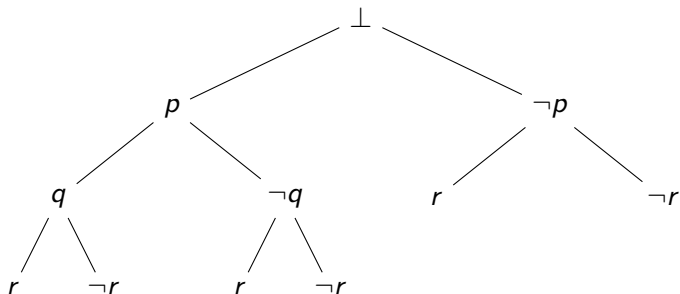
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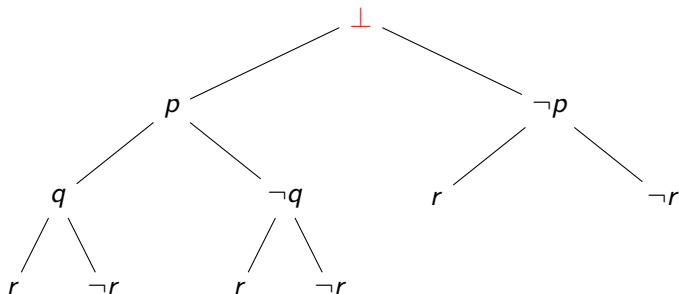
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Not a data structure, just needed for the completeness proof

Semantic Trees — Example

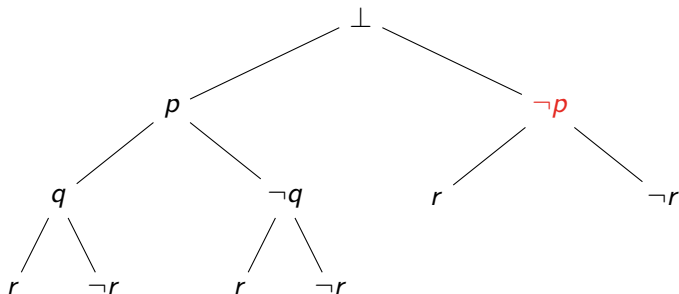


Semantic Trees — Example



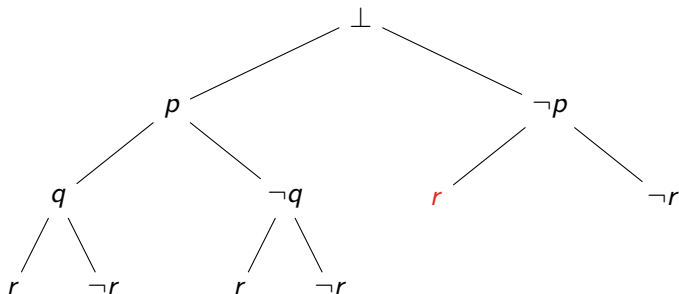
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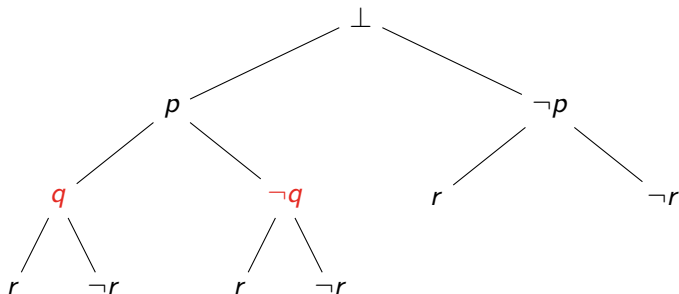
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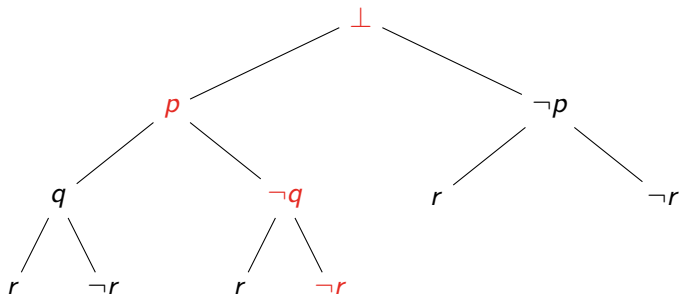
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Semantic Trees — Example



- ▶ Root labelled with $\perp$
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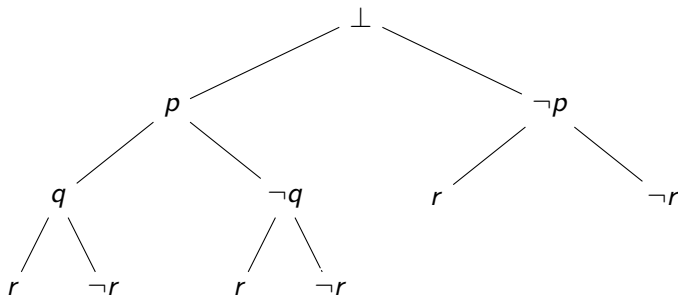
Semantic Trees — Example



- ▶ Root labelled with $\perp$
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- ▶ Complementary siblings
- ▶ No complementary pairs on a path

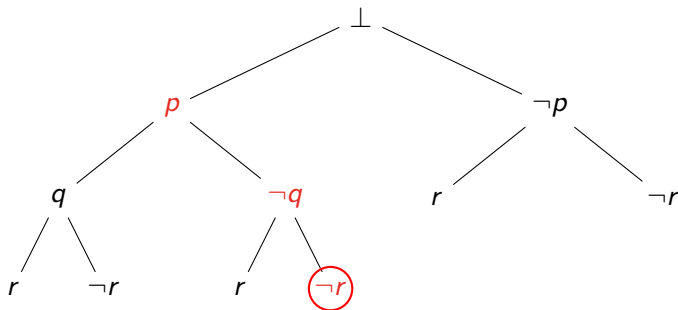
Partial Interpretations

The path to every node n in a semantic tree gives a 'partial interpretation' $\mathcal{I}_n$:



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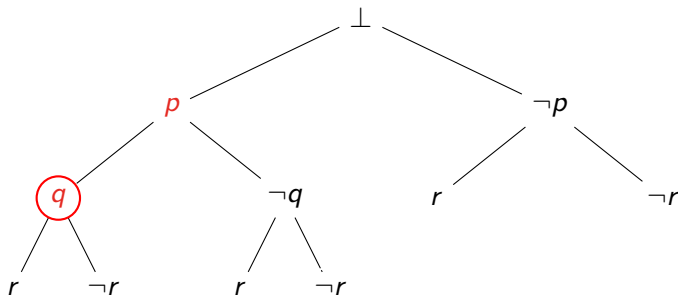
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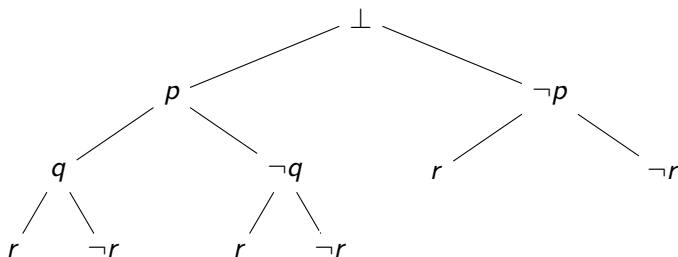
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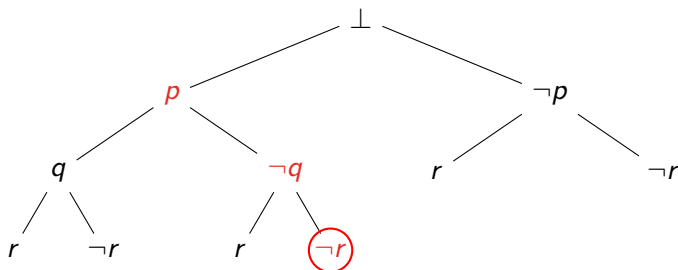
Failure Nodes – Motivation

Sometimes, such a ‘partial interpretation’ is enough to falsify a clause:



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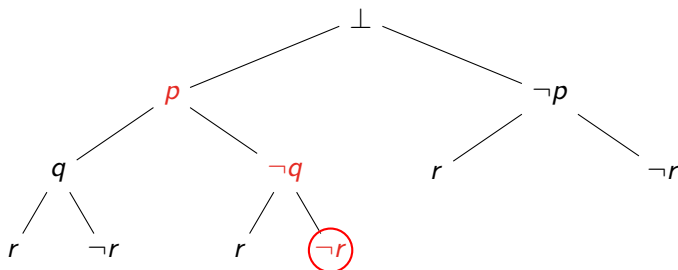
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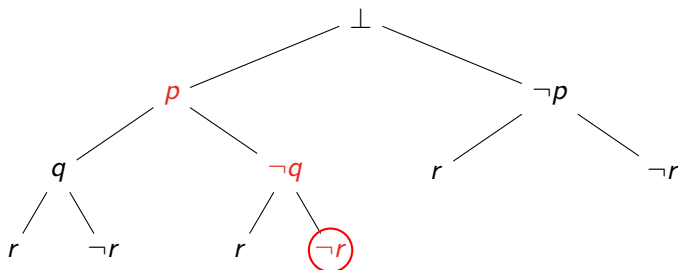
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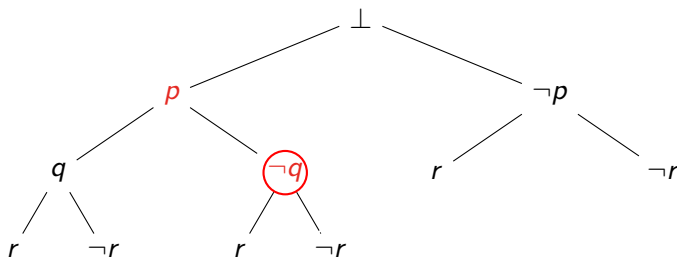
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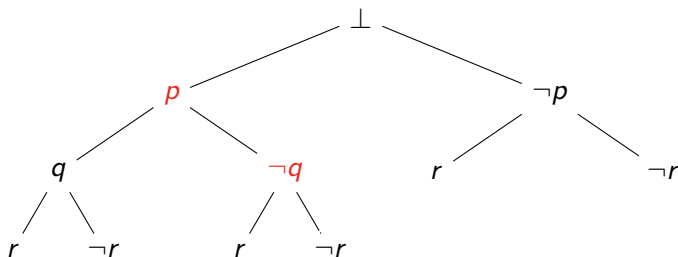
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- ▶ It remains false further down.

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Note: S has the root as a failure node iff $\square \in S$.

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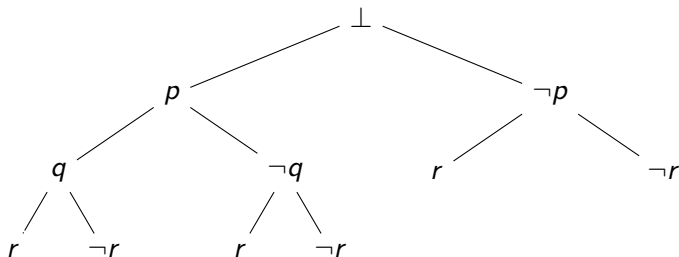
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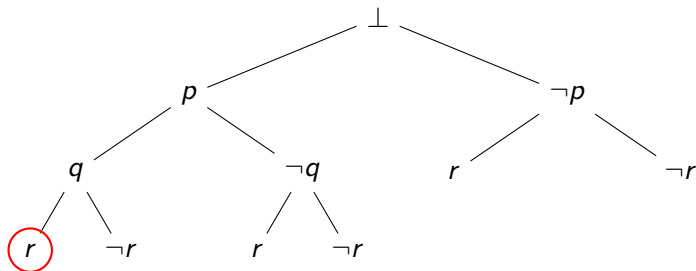
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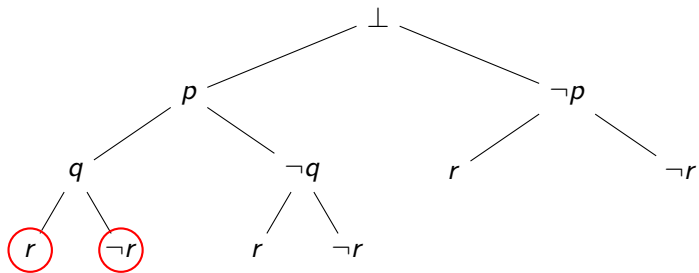
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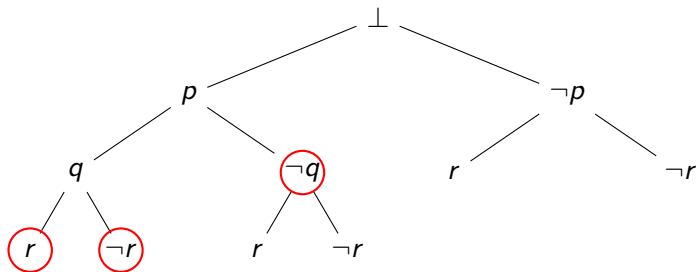
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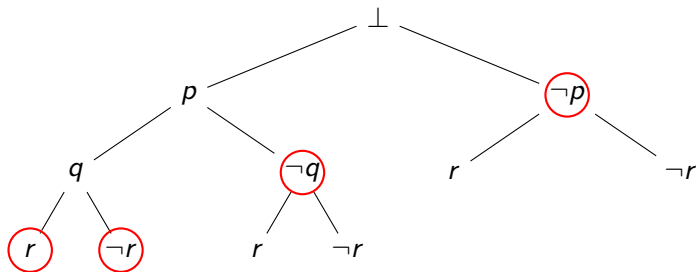
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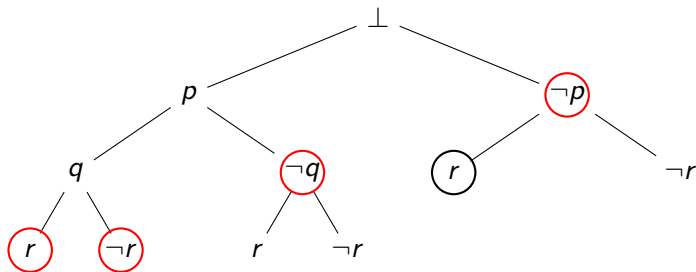
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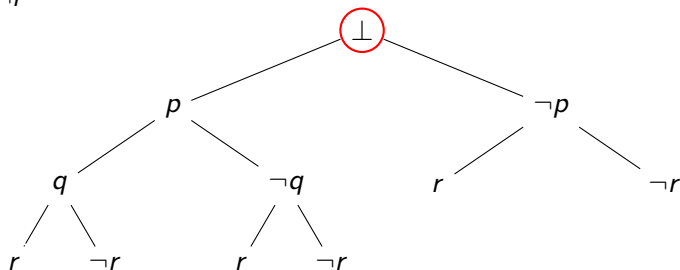
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Not a failure node: parent node falsifies clause 4.

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1. $\neg p \vee \neg q \vee \neg r$
2. $\neg q \vee r$
3. $\neg p \vee q$
4. p
5. $p \vee \neg r$
6. $\square$



The empty clause is falsified by the root node

Closed Semantic Trees

Definition 7.4.

*Given a semantic tree and a clause set S , a branch of the tree is **closed** if it contains a failure node.*

Closed Semantic Trees

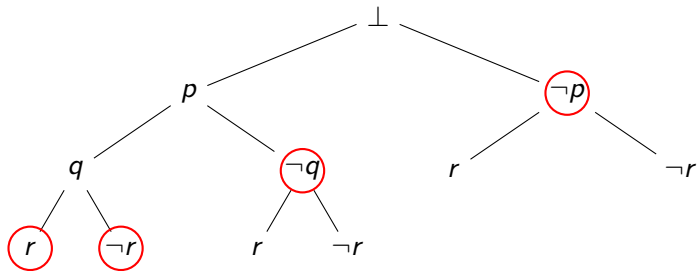
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The semantic tree is closed if all branches contain failure nodes.

Closed Semantic Tree – Example

1. $\neg p \vee \neg q \vee \neg r$
2. $\neg q \vee r$
3. $\neg p \vee q$
4. p
5. $p \vee \neg r$



The semantic tree is closed for these 5 clauses.

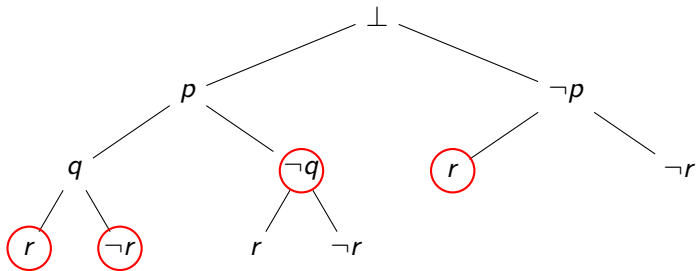
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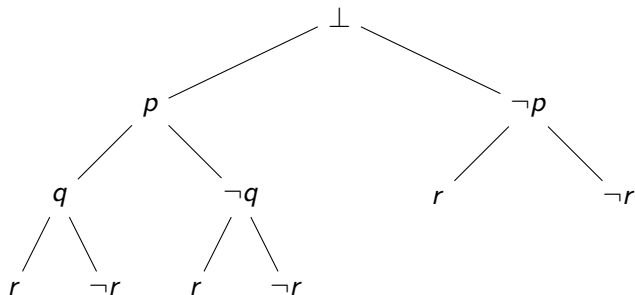
Without p , it is not closed.

Complete Semantic Trees

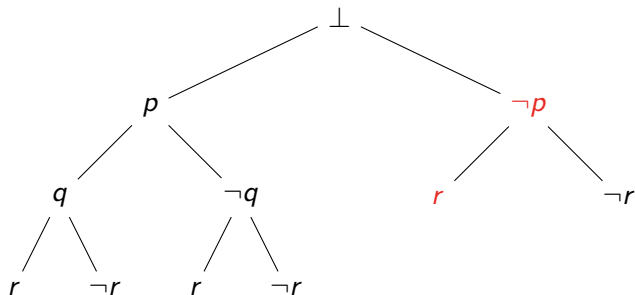
Definition 7.5.

*A semantic tree is **complete** if for every atomic formula A and every branch (from root to leaf) either A or $\neg A$ occurs*

Example: Complete Semantic Tree

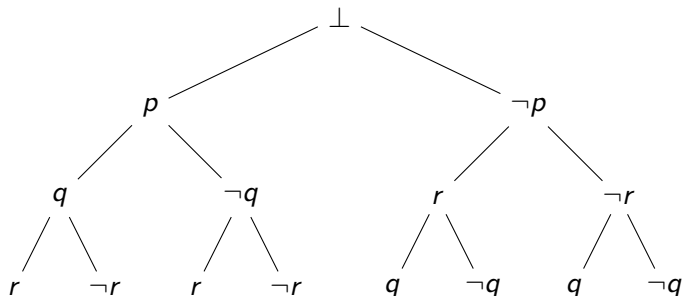


Example: Complete Semantic Tree



Not complete, since neither q nor $\neg q$ on branch

Example: Complete Semantic Tree



Complete for vocabulary $\{p, q, r\}$

Complete Semantic Trees

Definition 7.5.

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Every branch $\mathcal{B}$ in a *complete* semantic tree corresponds to an interpretation $\mathcal{I}_{\mathcal{B}}$ with $\mathcal{I} \models A$ iff A is on the branch.

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A complete semantic tree 'enumerates' all possible interpretations.

Unsatisfiable Clause Sets close Semantic Trees

Theorem 7.1.

A clause set is unsatisfiable iff there is a closed semantic tree for it.

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Proof.

⇒ Let S be an unsatisfiable clause set.

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Proof.

$\Rightarrow$ Let S be an unsatisfiable clause set. Construct a complete semantic tree. For each branch $\mathcal{B}$, $\mathcal{I}_{\mathcal{B}} \not\models S$

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⇒ Let S be an unsatisfiable clause set. Construct a complete semantic tree. For each branch $\mathcal{B}$, $\mathcal{I}_{\mathcal{B}} \not\models S$, so $\mathcal{I}_{\mathcal{B}} \not\models C$ for some clause $C \in S$,

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The falsifying nodes highest up on each branch are failure nodes.

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The falsifying nodes highest up on each branch are failure nodes. So the semantic tree is closed.
- ⇐ Let S be a clause set and let a closed semantic tree be given.

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- ⇐ Let S be a clause set and let a closed semantic tree be given. For any interpretation $\mathcal{I}$, there is a branch in the tree such that $\mathcal{I} \models L$ for all literals L on that branch.

Unsatisfiable Clause Sets close Semantic Trees

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 The falsifying nodes highest up on each branch are failure nodes. So the semantic tree is closed.
- ⇐ Let S be a clause set and let a closed semantic tree be given. For any interpretation $\mathcal{I}$, there is a branch in the tree such that $\mathcal{I} \models L$ for all literals L on that branch. Since there is a failure node for some clause $C \in S$ on that branch,

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- ⇐ Let S be a clause set and let a closed semantic tree be given. For any interpretation $\mathcal{I}$, there is a branch in the tree such that $\mathcal{I} \models L$ for all literals L on that branch. Since there is a failure node for some clause $C \in S$ on that branch, the atoms on the branch entail $\neg C$,

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- ⇐ Let S be a clause set and let a closed semantic tree be given. For any interpretation $\mathcal{I}$, there is a branch in the tree such that $\mathcal{I} \models L$ for all literals L on that branch. Since there is a failure node for some clause $C \in S$ on that branch, the atoms on the branch entail $\neg C$, so $\mathcal{I} \not\models C$, and thus $\mathcal{I} \not\models S$.

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The falsifying nodes highest up on each branch are failure nodes. So the semantic tree is closed.

⇐ Let S be a clause set and let a closed semantic tree be given. For any interpretation $\mathcal{I}$, there is a branch in the tree such that $\mathcal{I} \models L$ for all literals L on that branch. Since there is a failure node for some clause $C \in S$ on that branch, the atoms on the branch entail $\neg C$, so $\mathcal{I} \not\models C$, and thus $\mathcal{I} \not\models S$.

This holds for arbitrary interpretations $\mathcal{I}$, so S is unsatisfiable. □

Resolution Steps from Closed Semantic Trees

Lemma 7.2.

Let S be an unsatisfiable clause set, with a closed semantic tree

Resolution Steps from Closed Semantic Trees

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Resolution Steps from Closed Semantic Trees

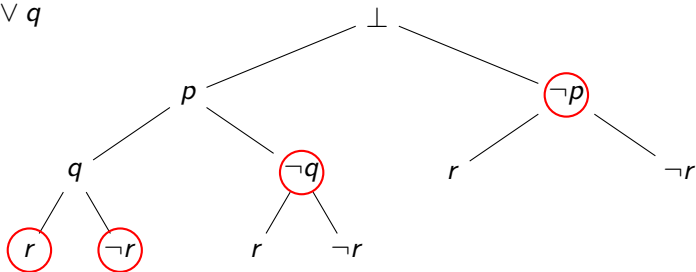
Lemma 7.2.

Let S be an unsatisfiable clause set, with a closed semantic tree, and $\square \notin S$. Then

- ▶ *a resolution step is possible from S ,*
- ▶ *and the resulting clause set S' has a smaller closed semantic tree*

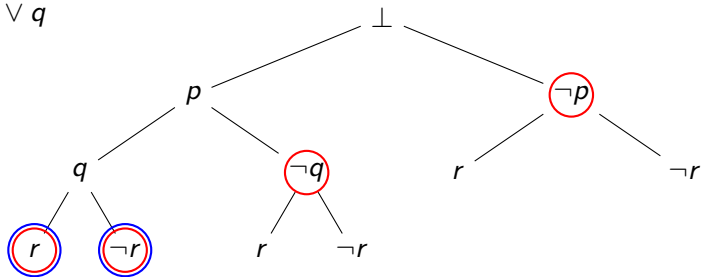
Idea of proof

1. $\neg p \vee \neg q \vee \neg r$
2. $\neg q \vee r$
3. $\neg p \vee q$
4. p



Idea of proof

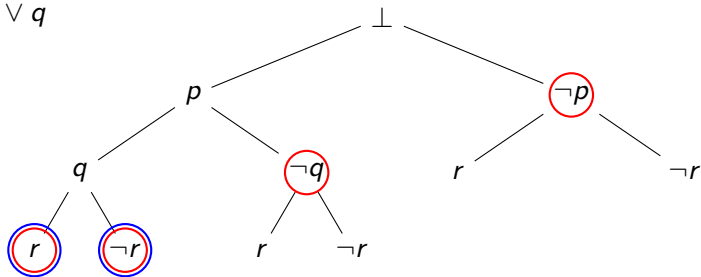
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- There are two sibling failure nodes

Idea of proof

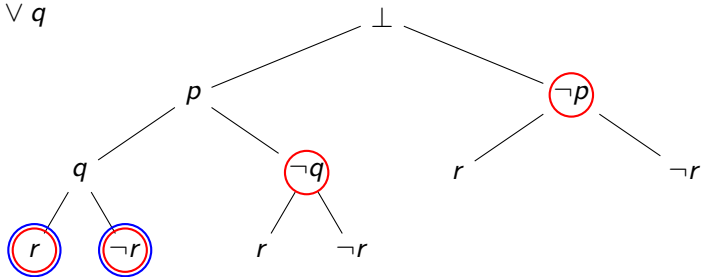
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- There are two sibling failure nodes
- They falsify two clauses

Idea of proof

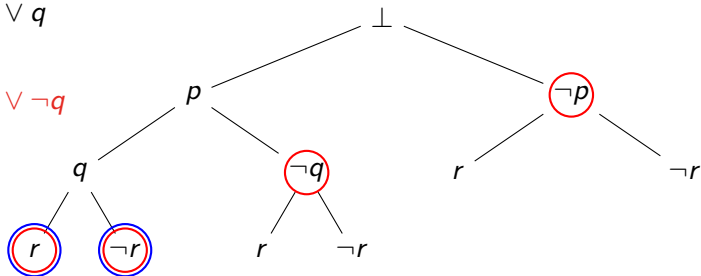
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- There are two sibling failure nodes
- They falsify two clauses with complementary literals

Idea of proof

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3. $\neg p \vee q$
4. p
5. $\neg p \vee \neg q$



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- They can be resolved to a new clause $\neg p \vee \neg q$

Idea of proof

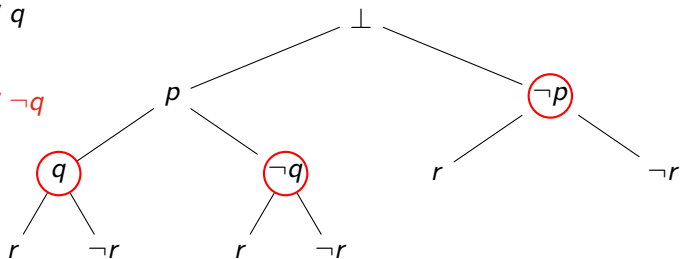
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3. $\neg p \vee q$

4. p

5. $\neg p \vee \neg q$



- ▶ There are two sibling failure nodes
- ▶ They falsify two clauses with complementary literals
- ▶ They can be resolved to a new clause $\neg p \vee \neg q$
- ▶ Which is falsified by the parent node

There are two sibling failure nodes

- ▶ Let n_0 be the root.

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- ▶ ...
- ▶ This either finds sibling failure nodes...
- ▶ or it constructs a path in the tree without a failure node, but that is not possible.

Sibling Failure Nodes give Resolution Opportunities

- Let n_1 and n_2 be sibling failure nodes

Sibling Failure Nodes give Resolution Opportunities

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- ▶ Either n or a node between n and the root becomes a failure node.
- ▶ This gives a closed semantic tree with ≥ 2 nodes less than before.

Completeness of Resolution

Theorem 7.2.

If S is an unsatisfiable clause set, then there is a resolution derivation of the empty clause from S .

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 - ▶ Leading to a clause set with a smaller closed semantic tree
- ▶ Since the tree is finite, this cannot go on forever.



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- ▶ There exists a closed semantic tree for S
- ▶ As long as S does not contain the empty clause,
 - ▶ It is possible to apply a resolution step to S
 - ▶ Leading to a clause set with a smaller closed semantic tree
- ▶ Since the tree is finite, this cannot go on forever.
- ▶ Therefore, eventually the semantic tree must consist of only the root. . .



Completeness of Resolution

Theorem 7.2.

If S is an unsatisfiable clause set, then there is a resolution derivation of the empty clause from S .

Proof.

- ▶ There exists a closed semantic tree for S
- ▶ As long as S does not contain the empty clause,
 - ▶ It is possible to apply a resolution step to S
 - ▶ Leading to a clause set with a smaller closed semantic tree
- ▶ Since the tree is finite, this cannot go on forever.
- ▶ Therefore, eventually the semantic tree must consist of only the root. . .
- ▶ . . . and S contain the empty clause $\square$.

