

IN3070/4070 – Logic – Autumn 2026

Lecture 5: Soundness & Completeness for 1st-order LK

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DEPARTMENT OF
INFORMATICS



UNIVERSITY OF
OSLO

Next Week

- ▶ **Two lectures next week:**
- ▶ Monday 10:15–11:00, Lille Aud
- ▶ Thursday 10:15–12:00, Lille Aud

Today's Plan

- ▶ Preliminaries and Reminders
- ▶ Soundness Proof
- ▶ Completeness: Preliminaries
- ▶ Proof of Completeness
- ▶ Examples of Counter-model Construction

Outline

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Theorem 1.1.

The sequent calculus LK is sound.

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- ▶ Some rules require “fresh” constants, so we assume that the set of constant symbols $\mathcal{A}$ is (countably) infinite.
- ▶ A root sequent $\Gamma \Longrightarrow \Delta$ consists of *closed* formulae.
- ▶ We show that if $\Gamma \Longrightarrow \Delta$ is provable, then $\Gamma \Longrightarrow \Delta$ is valid

Reminer: Semantics for Sequents

Definition 1.2 (Valid sequent).

A sequent $\Gamma \implies \Delta$ is **valid** if all interpretations that satisfy all formulae in Γ satisfy at least one formula in Δ .

Definition 1.3 (Countermodel/falsifiable sequent).

- ▶ An interpretation $\mathcal{I}$ is a **countermodel** for the sequent $\Gamma \implies \Delta$ if $v_{\mathcal{I}}(A) = T$ for all formulae $A \in \Gamma$ and $v_{\mathcal{I}}(B) = F$ for all formulae $B \in \Delta$

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- ▶ We say that a countermodel for a sequent *falsifies* the sequent.
- ▶ A sequent is *falsifiable* if it has a countermodel.

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Reminder: Substitution Lemma

Theorem 1.2 (Substitution Lemma for Formulae).

*Given an interpretation $\mathcal{I} = (D, \iota)$ and a variable assignment α for $\mathcal{I}$.
Given also a variable $y \in \mathcal{V}$, a formula A and a term $s \in \mathcal{T}$, such that
 $\{y \setminus s\}$ is capture-free for A .*

$$v_{\mathcal{I}}(\alpha, A[y \setminus s]) = v_{\mathcal{I}}(\alpha\{y \leftarrow v_{\mathcal{I}}(\alpha, s)\}, A)$$

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Definition 1.4 (Capture-free substitution).

A substitution σ is *capture-free* for a formula A if for every free variable x in A , none of the variables in $\sigma(x)$ is bound in A .

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Note: if $t \in \mathcal{T}$ is a closed term, then $\{y \setminus t\}$ is capture-free for any A .

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Preservation of Falsifiability

Definition 2.1.

An LK-rule θ *preserves falsifiability* (upwards) if whenever the conclusion w of an instance $\frac{w_1 \cdots w_n}{w}$ of θ is falsifiable, then also at least one of the premises w_i is falsifiable

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Lemma 2.1.

All LK-rules preserve falsifiability.

- ▶ We have shown that the rules for propositional connectives ($\wedge$, $\vee$, $\rightarrow$, $\neg$) have this property.
- ▶ It remains to show that also the $\forall$ and $\exists$ rules preserve falsifiability.

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$$\frac{\Gamma, \forall x A, A[x \setminus t] \implies \Delta}{\Gamma, \forall x A \implies \Delta} \forall\text{-left} \quad t \text{ is a closed term}$$

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- ▶ And therefore: $\mathcal{I} \models A[x \setminus t]$.

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- ▶ Then $\mathcal{I}'$ falsifies the premiss:
 - ▶ Since a does not occur in the conclusion, $\mathcal{I}'$ and $\mathcal{I}$ interpret the fml. in Γ og Δ equally.

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- ▶ Based on $\mathcal{I}$ and d , we define a **new** model $\mathcal{I}'$ as follows:
 - ▶ $\mathcal{I}' = (D, \iota')$ is identical to $\mathcal{I}$ except for the interpretation of the constant a .
 - ▶ a is interpreted as d , ie. $a^{\iota'} = d$.
- ▶ Then $\mathcal{I}'$ falsifies the premiss:
 - ▶ Since a does not occur in the conclusion, $\mathcal{I}'$ and $\mathcal{I}$ interpret the fml. in Γ og Δ equally. $\mathcal{I}'$ makes all fml. in Γ true and all fml. in Δ false.

Proof: $\exists$ -left preserves falsifiability

$$\frac{\Gamma, A[x \backslash a] \implies \Delta}{\Gamma, \exists x A \implies \Delta} \exists\text{-left} \quad a \text{ is a constant that does not occur in the conclusion}$$

- ▶ Assume that $\mathcal{I} = (D, \iota)$ falsifies the conclusion $\Gamma, \exists x A \implies \Delta$.
- ▶ $\mathcal{I}$ makes all formulae in $\Gamma \cup \{\exists x A\}$ true and all formulae in Δ false.
- ▶ We have to find an interpretation that falsifies the premiss.
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 - ▶ $v_{\mathcal{I}'}(\alpha\{x \leftarrow d\}, A) = v_{\mathcal{I}}(\alpha\{x \leftarrow d\}, A) = T$, and $d = v_{\mathcal{I}'}(\alpha, a)$, so $\mathcal{I}' \models A[x \backslash a]$, by the Substitution Lemma.

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Proof: $\exists$ -right and $\forall$ -right preserve satisfiability

- ▶ The proof for $\forall$ -right is dual to that for $\exists$ -left
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How to show the Soundness Theorem?

As for propositional logic, we show the following lemmas:

1. All LK-rules preserve falsifiability upwards.
2. An LK-derivation with a falsifiable root sequent has at least one falsifiable leaf sequent
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Finally, we use these lemmas to show the soundness theorem.

Existence of a falsifiable leaf sequent

Lemma 2.2.

If the root sequent $\mathcal{I}$ of an LK-derivation is falsifiable, then at least one of the leaf sequents is falsifiable.

- ▶ As for propositional logic, the proof is by structural induction on the LK-derivation.
- ▶ The base case (one sequent $\Gamma \Rightarrow \Delta$) is trivial since $\Gamma \Rightarrow \Delta$ is both root and leaf sequent.
- ▶ Two induction steps, for one-premise and two-premise rules
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- ▶ Therefore, the same formula $p(t_1, \dots, t_n)$ is satisfied in the succedent.

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- ▶ Then $\mathcal{P}$ has a leaf sequent that is not an axiom, since axioms are not falsifiable.
- ▶ So $\mathcal{P}$ cannot be an LK-proof.



Outline

- ▶ Preliminaries and Reminders
- ▶ Soundness Proof
- ▶ **Completeness: Preliminaries**
- ▶ Proof of Completeness
- ▶ Examples of Counter-model Construction

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- ▶ Intuitively, the Herbrand universe of T is the set of all *closed* terms that can be constructed from the constant and function symbols in T .

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- ▶ If all branches in a derivation can be closed, then the derivation is finite. I.e. proofs are finite.

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3. If a $\exists$ formula occurs in an antecedent, or a $\forall$ formula in a succedent, then the $\exists$ -left, resp. $\forall$ -right rules are applied to the formula on that branch.
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Corollary 3.1.

If T is a finitely branching tree, where all branches are finitely long, then T is finite.

Outline

- ▶ Preliminaries and Reminders
- ▶ Soundness Proof
- ▶ Completeness: Preliminaries
- ▶ **Proof of Completeness**
- ▶ Examples of Counter-model Construction

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- ▶ Such an interpretation is often called a **Herbrand model** or a **term model**.

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Proof of Completeness (Propositional connectives)

Induction step: From the assumption (induction hypothesis) that our statement holds for all smaller formulae, we have to show that it holds for $\neg A$, $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$, $\forall x A$, and $\exists x A$.

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We only need to cover quantified formulae

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- ▶ I.e. $\mathcal{I} \models \exists x A$.

Proof of Completeness ($\exists$ in Succedent)

Assume that $\exists x A \in \mathcal{B}^\perp$.

- We have to show that $\mathcal{I} \not\models \exists x A$. Assume that this does not hold.

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Assume that $\exists x A \in \mathcal{B}^\perp$.

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- ▶ The idea of the completeness proof is important: we construct an interpretation from something purely syntactic.

Outline

- ▶ Preliminaries and Reminders
- ▶ Soundness Proof
- ▶ Completeness: Preliminaries
- ▶ Proof of Completeness
- ▶ Examples of Counter-model Construction

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- Abbreviate px for $p(x)$, qb for $q(b)$, etc.

Counter-model Construction, Ex. 1

$$\frac{
 \frac{
 \frac{
 \overline{A, pa \Rightarrow \forall x qx, pa}
 }{
 A, pa \rightarrow qa, pa \Rightarrow \forall x qx
 }
 }{
 \underbrace{\forall x (px \rightarrow qx)}_A, pa \Rightarrow \forall x qx
 }
 }{
 \frac{
 qa, A, pa \Rightarrow qb
 }{
 qa, A, pa \Rightarrow \forall x qx
 }
 }$$

- Abbreviate px for $p(x)$, qb for $q(b)$, etc.

Counter-model Construction, Ex. 1

$$\frac{
 \frac{
 \frac{
 \frac{
 \frac{
 qa, A, pa \Rightarrow qb
 }{
 qa, A, pa \Rightarrow \forall x qx
 }
 }{
 A, pa \rightarrow qa, pa \Rightarrow \forall x qx
 }
 }{
 \underbrace{\forall x (px \rightarrow qx)}_A, pa \Rightarrow \forall x qx
 }
 }{
 A, pa \Rightarrow \forall x qx, pa
 }
 }{
 }$$

- Abbreviate px for $p(x)$, qb for $q(b)$, etc.

Counter-model Construction, Ex. 1

$$\begin{array}{c}
 \frac{A, pa \Rightarrow \forall x qx, pa}{\quad} \qquad \frac{\frac{qa, A, pb \rightarrow qb, pa \Rightarrow qb}{qa, A, pa \Rightarrow qb}}{qa, A, pa \Rightarrow \forall x qx} \\
 \hline
 \frac{A, pa \rightarrow qa, pa \Rightarrow \forall x qx}{\underbrace{\forall x (px \rightarrow qx), pa \Rightarrow \forall x qx}_A}
 \end{array}$$

- Abbreviate px for $p(x)$, qb for $q(b)$, etc.

Counter-model Construction, Ex. 1

$$\begin{array}{c}
 \frac{A, pa \Rightarrow \forall x qx, pa}{\quad} \qquad \frac{\frac{qa, A, pb \rightarrow qb, pa \Rightarrow qb}{qa, A, pa \Rightarrow qb}}{qa, A, pa \Rightarrow \forall x qx} \\
 \hline
 \frac{A, pa \rightarrow qa, pa \Rightarrow \forall x qx}{\underbrace{\forall x (px \rightarrow qx), pa \Rightarrow \forall x qx}_A}
 \end{array}$$

- Abbreviate px for $p(x)$, qb for $q(b)$, etc.

Counter-model Construction, Ex. 1

$$\begin{array}{c}
 \frac{qa, A, pa \Rightarrow qb, pb \quad qb, qa, A, pa \Rightarrow qb}{qa, A, pb \rightarrow qb, pa \Rightarrow qb} \\
 \frac{qa, A, pa \Rightarrow qb}{qa, A, pa \Rightarrow \forall x qx} \\
 \frac{A, pa \Rightarrow \forall x qx, pa}{A, pa \rightarrow qa, pa \Rightarrow \forall x qx} \\
 \frac{\forall x (px \rightarrow qx), pa \Rightarrow \forall x qx}{\underbrace{\hspace{10em}}_A}
 \end{array}$$

- Abbreviate px for $p(x)$, qb for $q(b)$, etc.

Counter-model Construction, Ex. 1

$$\begin{array}{c}
 \frac{qa, A, pa \Rightarrow qb, pb \quad qb, qa, A, pa \Rightarrow qb}{qa, A, pb \rightarrow qb, pa \Rightarrow qb} \\
 \frac{qa, A, pa \Rightarrow qb}{qa, A, pa \Rightarrow \forall x qx} \\
 \frac{A, pa \Rightarrow \forall x qx, pa}{A, pa \rightarrow qa, pa \Rightarrow \forall x qx} \\
 \frac{\forall x (px \rightarrow qx), pa \Rightarrow \forall x qx}{\underbrace{\hspace{10em}}_A}
 \end{array}$$

- Abbreviate px for $p(x)$, qb for $q(b)$, etc.

Counter-model Construction, Ex. 1

$$\begin{array}{c}
 \frac{qa, A, pa \Rightarrow qb, pb \quad \overline{qb, qa, A, pa \Rightarrow qb}}{qa, A, pb \rightarrow qb, pa \Rightarrow qb} \\
 \frac{qa, A, pa \Rightarrow qb}{qa, A, pa \Rightarrow \forall x qx} \\
 \frac{A, pa \Rightarrow \forall x qx, pa}{A, pa \rightarrow qa, pa \Rightarrow \forall x qx} \\
 \frac{\forall x (px \rightarrow qx), pa \Rightarrow \forall x qx}{\underbrace{\hspace{10em}}_A}
 \end{array}$$

- Abbreviate px for $p(x)$, qb for $q(b)$, etc.

Counter-model Construction, Ex. 1

$$\begin{array}{c}
 \mathcal{B} \\
 \hline
 qa, A, pa \Rightarrow qb, pb \quad \quad \quad \overline{qb, qa, A, pa \Rightarrow qb} \\
 \hline
 qa, A, pb \rightarrow qb, pa \Rightarrow qb \\
 \hline
 \quad \quad \quad \overline{qa, A, pa \Rightarrow qb} \\
 \quad \quad \quad \hline
 \quad \quad \quad qa, A, pa \Rightarrow \forall x qx \\
 \hline
 \overline{A, pa \Rightarrow \forall x qx, pa} \quad \quad \quad \overline{A, pa \rightarrow qa, pa \Rightarrow \forall x qx} \\
 \hline
 \underbrace{\forall x (px \rightarrow qx), pa \Rightarrow \forall x qx}_A
 \end{array}$$

- Abbreviate px for $p(x)$, qb for $q(b)$, etc.

Counter-model Construction, Ex. 1

$$\begin{array}{c}
 \mathcal{B} \\
 \hline
 qa, A, pa \Rightarrow qb, pb \quad \overline{qb, qa, A, pa \Rightarrow qb} \\
 \hline
 qa, A, pb \rightarrow qb, pa \Rightarrow qb \\
 \hline
 \frac{qa, A, pa \Rightarrow qb}{qa, A, pa \Rightarrow \forall x qx} \\
 \hline
 \frac{A, pa \Rightarrow \forall x qx, pa}{A, pa \rightarrow qa, pa \Rightarrow \forall x qx} \\
 \hline
 \underbrace{\forall x (px \rightarrow qx), pa \Rightarrow \forall x qx}_A
 \end{array}$$

- ▶ Abbreviate px for $p(x)$, qb for $q(b)$, etc.
- ▶ The Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.

Counter-model Construction, Ex. 1

$$\begin{array}{c}
 \mathcal{B} \\
 \hline
 qa, A, pa \Rightarrow qb, pb \quad \overline{qb, qa, A, pa \Rightarrow qb} \\
 \hline
 qa, A, pb \rightarrow qb, pa \Rightarrow qb \\
 \hline
 qa, A, pa \Rightarrow qb \\
 \hline
 qa, A, pa \Rightarrow \forall x qx \\
 \hline
 \overline{A, pa \Rightarrow \forall x qx, pa} \quad \overline{A, pa \rightarrow qa, pa \Rightarrow \forall x qx} \\
 \hline
 \underbrace{\forall x (px \rightarrow qx), pa \Rightarrow \forall x qx}_A
 \end{array}$$

- ▶ Abbreviate px for $p(x)$, qb for $q(b)$, etc.
- ▶ The Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- ▶ Since $pa \in \mathcal{B}^\top$, define $a \in p^\iota$, so $\mathcal{I} \models pa$.

Counter-model Construction, Ex. 1

$$\begin{array}{c}
 \mathcal{B} \\
 \hline
 qa, A, pa \Rightarrow qb, pb \quad \quad \quad \overline{qb, qa, A, pa \Rightarrow qb} \\
 \hline
 qa, A, pb \rightarrow qb, pa \Rightarrow qb \\
 \hline
 \quad \quad \quad \overline{qa, A, pa \Rightarrow qb} \\
 \quad \quad \quad \hline
 \quad \quad \quad qa, A, pa \Rightarrow \forall x qx \\
 \hline
 \overline{A, pa \Rightarrow \forall x qx, pa} \quad \quad \quad \overline{A, pa \rightarrow qa, pa \Rightarrow \forall x qx} \\
 \hline
 \underbrace{\forall x (px \rightarrow qx), pa \Rightarrow \forall x qx}_A
 \end{array}$$

- ▶ Abbreviate px for $p(x)$, qb for $q(b)$, etc.
- ▶ The Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- ▶ Since $pa \in \mathcal{B}^\top$, define $a \in p^\iota$, so $\mathcal{I} \models pa$.
- ▶ Since $qa \in \mathcal{B}^\top$, define $a \in q^\iota$, so $\mathcal{I} \models qa$ and thus $\mathcal{I} \models pa \rightarrow qa$.

Counter-model Construction, Ex. 1

$$\begin{array}{c}
 \mathcal{B} \\
 \hline
 qa, A, pa \Rightarrow qb, pb \quad \quad \quad \overline{qb, qa, A, pa \Rightarrow qb} \\
 \hline
 qa, A, pb \rightarrow qb, pa \Rightarrow qb \\
 \hline
 \quad \quad \quad \overline{qa, A, pa \Rightarrow qb} \\
 \quad \quad \quad \hline
 \quad \quad \quad qa, A, pa \Rightarrow \forall x qx \\
 \hline
 \overline{A, pa \Rightarrow \forall x qx, pa} \quad \quad \quad \overline{A, pa \rightarrow qa, pa \Rightarrow \forall x qx} \\
 \hline
 \underbrace{\forall x (px \rightarrow qx), pa \Rightarrow \forall x qx}_A
 \end{array}$$

- ▶ Abbreviate px for $p(x)$, qb for $q(b)$, etc.
- ▶ The Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- ▶ Since $pa \in \mathcal{B}^\top$, define $a \in p^\iota$, so $\mathcal{I} \models pa$.
- ▶ Since $qa \in \mathcal{B}^\top$, define $a \in q^\iota$, so $\mathcal{I} \models qa$ and thus $\mathcal{I} \models pa \rightarrow qa$.
- ▶ Since $qb \in \mathcal{B}^\perp$, define $b \notin q^\iota$, so $\mathcal{I} \not\models qb$ and thus $\mathcal{I} \not\models \forall x qx$.

Counter-model Construction, Ex. 1

$$\begin{array}{c}
 \mathcal{B} \\
 \hline
 qa, A, pa \Rightarrow qb, pb \quad \quad \quad \overline{qb, qa, A, pa \Rightarrow qb} \\
 \hline
 qa, A, pb \rightarrow qb, pa \Rightarrow qb \\
 \hline
 \quad \quad \quad \overline{qa, A, pa \Rightarrow qb} \\
 \quad \quad \quad \hline
 \quad \quad \quad qa, A, pa \Rightarrow \forall x qx \\
 \hline
 \overline{A, pa \Rightarrow \forall x qx, pa} \quad \quad \quad \overline{A, pa \rightarrow qa, pa \Rightarrow \forall x qx} \\
 \hline
 \underbrace{\forall x (px \rightarrow qx), pa \Rightarrow \forall x qx}_A
 \end{array}$$

- ▶ Abbreviate px for $p(x)$, qb for $q(b)$, etc.
- ▶ The Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- ▶ Since $pa \in \mathcal{B}^\top$, define $a \in p^\iota$, so $\mathcal{I} \models pa$.
- ▶ Since $qa \in \mathcal{B}^\top$, define $a \in q^\iota$, so $\mathcal{I} \models qa$ and thus $\mathcal{I} \models pa \rightarrow qa$.
- ▶ Since $qb \in \mathcal{B}^\perp$, define $b \notin q^\iota$, so $\mathcal{I} \not\models qb$ and thus $\mathcal{I} \not\models \forall x qx$.
- ▶ Since $pb \in \mathcal{B}^\perp$, define $b \notin p^\iota$, so $\mathcal{I} \not\models pb$ and thus $\mathcal{I} \models pb \rightarrow qb$.

Counter-model Construction, Ex. 1

$$\begin{array}{c}
 \mathcal{B} \\
 \hline
 qa, A, pa \Rightarrow qb, pb \quad \quad \quad \overline{qb, qa, A, pa \Rightarrow qb} \\
 \hline
 qa, A, pb \rightarrow qb, pa \Rightarrow qb \\
 \hline
 \quad \quad \quad \overline{qa, A, pa \Rightarrow qb} \\
 \quad \quad \quad \hline
 \quad \quad \quad qa, A, pa \Rightarrow \forall x qx \\
 \hline
 \overline{A, pa \Rightarrow \forall x qx, pa} \quad \quad \quad \overline{A, pa \rightarrow qa, pa \Rightarrow \forall x qx} \\
 \hline
 \underbrace{\forall x (px \rightarrow qx), pa \Rightarrow \forall x qx}_A
 \end{array}$$

- ▶ Abbreviate px for $p(x)$, qb for $q(b)$, etc.
- ▶ The Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- ▶ Since $pa \in \mathcal{B}^\top$, define $a \in p^\iota$, so $\mathcal{I} \models pa$.
- ▶ Since $qa \in \mathcal{B}^\top$, define $a \in q^\iota$, so $\mathcal{I} \models qa$ and thus $\mathcal{I} \models pa \rightarrow qa$.
- ▶ Since $qb \in \mathcal{B}^\perp$, define $b \notin q^\iota$, so $\mathcal{I} \not\models qb$ and thus $\mathcal{I} \not\models \forall x qx$.
- ▶ Since $pb \in \mathcal{B}^\perp$, define $b \notin p^\iota$, so $\mathcal{I} \not\models pb$ and thus $\mathcal{I} \models pb \rightarrow qb$.
- ▶ Therefore also $\mathcal{I} \models \forall x (px \rightarrow qx)$.

Counter-model Construction, Ex. 1

$$\begin{array}{c}
 \mathcal{B} \\
 \hline
 qa, A, pa \Rightarrow qb, pb \quad \quad \quad \overline{qb, qa, A, pa \Rightarrow qb} \\
 \hline
 qa, A, pb \rightarrow qb, pa \Rightarrow qb \\
 \hline
 qa, A, pa \Rightarrow qb \\
 \hline
 qa, A, pa \Rightarrow \forall x qx \\
 \hline
 A, pa \Rightarrow \forall x qx, pa \\
 \hline
 A, pa \rightarrow qa, pa \Rightarrow \forall x qx \\
 \hline
 \underbrace{\forall x (px \rightarrow qx), pa \Rightarrow \forall x qx}_A
 \end{array}$$

- ▶ Abbreviate px for $p(x)$, qb for $q(b)$, etc.
- ▶ The Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- ▶ Since $pa \in \mathcal{B}^\top$, define $a \in p^\iota$, so $\mathcal{I} \models pa$.
- ▶ Since $qa \in \mathcal{B}^\top$, define $a \in q^\iota$, so $\mathcal{I} \models qa$ and thus $\mathcal{I} \models pa \rightarrow qa$.
- ▶ Since $qb \in \mathcal{B}^\perp$, define $b \notin q^\iota$, so $\mathcal{I} \not\models qb$ and thus $\mathcal{I} \not\models \forall x qx$.
- ▶ Since $pb \in \mathcal{B}^\perp$, define $b \notin p^\iota$, so $\mathcal{I} \not\models pb$ and thus $\mathcal{I} \models pb \rightarrow qb$.
- ▶ Therefore also $\mathcal{I} \models \forall x (px \rightarrow qx)$.
- ▶ $\mathcal{I}$ makes all of $\mathcal{B}^\top$ true and all of $\mathcal{B}^\perp$ false.

Counter-model Construction, Ex. 2

$$\underbrace{\forall x (p_x a \rightarrow p_x b)}_A, p_a a \vee p_b a \Rightarrow p_a b$$

Counter-model Construction, Ex. 2

$$\underbrace{\forall x (pxa \rightarrow pxb), paa \vee pba \Rightarrow pab}_A$$

- Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.

Counter-model Construction, Ex. 2

$$\underbrace{\forall x (pxa \rightarrow pxb), paa \vee pba}_{\mathcal{A}} \Rightarrow \textcolor{red}{pab}$$

- Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.

Counter-model Construction, Ex. 2

$$\underbrace{\forall x (pxa \rightarrow pxb), paa \vee pba}_{\mathcal{A}} \Rightarrow \textcolor{red}{pab}$$

- ▶ Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- ▶ Since $pab \in \mathcal{B}^\perp$, define $\langle a, b \rangle \notin p^\iota$, so $\mathcal{I} \not\models pab$.

Counter-model Construction, Ex. 2

$$\underbrace{\forall x (pxa \rightarrow pxb)}_A, paa \vee pba \Rightarrow pab$$

- Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- Since $pab \in \mathcal{B}^\perp$, define $\langle a, b \rangle \notin p^\iota$, so $\mathcal{I} \not\models pab$.

Counter-model Construction, Ex. 2

$$\frac{A, paa \Rightarrow pab \qquad A, pba \Rightarrow pab}{\underbrace{\forall x (pxa \rightarrow pxb), paa \vee pba \Rightarrow pab}_A}$$

- ▶ Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- ▶ Since $pab \in \mathcal{B}^\perp$, define $\langle a, b \rangle \notin p^\iota$, so $\mathcal{I} \not\models pab$.

Counter-model Construction, Ex. 2

$$\frac{A, paa \Rightarrow pab \qquad A, pba \Rightarrow pab}{\underbrace{\forall x (pxa \rightarrow pxb), paa \vee pba \Rightarrow pab}_A}$$

- ▶ Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- ▶ Since $pab \in \mathcal{B}^\perp$, define $\langle a, b \rangle \notin p^\iota$, so $\mathcal{I} \not\models pab$.

Counter-model Construction, Ex. 2

$$\frac{
 \frac{A, paa \rightarrow pab, paa \Rightarrow pab}{A, paa \Rightarrow pab}
 \quad
 \frac{A, pba \Rightarrow pab}{\underbrace{\forall x (pxa \rightarrow pxb), paa \vee pba \Rightarrow pab}_A}
 }{}$$

- ▶ Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- ▶ Since $pab \in \mathcal{B}^\perp$, define $\langle a, b \rangle \notin p^\iota$, so $\mathcal{I} \not\models pab$.

Counter-model Construction, Ex. 2

$$\frac{
 \frac{A, paa \rightarrow pab, paa \Rightarrow pab}{A, paa \Rightarrow pab}
 \quad
 A, pba \Rightarrow pab
 }{
 \underbrace{\forall x (pxa \rightarrow pxb), paa \vee pba \Rightarrow pab}_A
 }$$

- Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- Since $pab \in \mathcal{B}^\perp$, define $\langle a, b \rangle \notin p^\iota$, so $\mathcal{I} \not\models pab$.

Counter-model Construction, Ex. 2

(Both branches closeable)

$A, paa \rightarrow pab, paa \Rightarrow pab$

$A, paa \Rightarrow pab$

$A, pba \Rightarrow pab$

$\forall x (pxa \rightarrow pxb), paa \vee pba \Rightarrow pab$

A

- Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- Since $pab \in \mathcal{B}^\perp$, define $\langle a, b \rangle \notin p^\iota$, so $\mathcal{I} \not\models pab$.

Counter-model Construction, Ex. 2

(Both branches closeable)

 $A, paa \rightarrow pab, paa \Rightarrow pab$ $A, paa \Rightarrow pab$ $A, pba \Rightarrow pab$ $\forall x (pxa \rightarrow pxb), paa \vee pba \Rightarrow pab$ A

- Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- Since $pab \in \mathcal{B}^\perp$, define $\langle a, b \rangle \notin p^\iota$, so $\mathcal{I} \not\models pab$.

Counter-model Construction, Ex. 2

(Both branches closeable)

 $A, paa \rightarrow pab, paa \Rightarrow pab$ $A, paa \Rightarrow pab$ $A, pba \Rightarrow pab$ $\forall x (pxa \rightarrow pxb), paa \vee pba \Rightarrow pab$ A

- Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- Since $pab \in \mathcal{B}^\perp$, define $\langle a, b \rangle \notin p^\iota$, so $\mathcal{I} \not\models pab$.
- Since $pba \in \mathcal{B}^\top$, def. $\langle b, a \rangle \in p^\iota$, so $\mathcal{I} \models pba$ and $\mathcal{I} \models paa \vee pba$.

Counter-model Construction, Ex. 2

(Both branches closeable)

 $A, paa \rightarrow pab, paa \Rightarrow pab$ $A, paa \Rightarrow pab$ $A, pba \Rightarrow pab$ $\forall x (pxa \rightarrow pxb), paa \vee pba \Rightarrow pab$ A

- Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- Since $pab \in \mathcal{B}^\perp$, define $\langle a, b \rangle \notin p^\iota$, so $\mathcal{I} \not\models pab$.
- Since $pba \in \mathcal{B}^\top$, def. $\langle b, a \rangle \in p^\iota$, so $\mathcal{I} \models pba$ and $\mathcal{I} \models paa \vee pba$.

Counter-model Construction, Ex. 2

(Both branches closeable)

$$\frac{
 \frac{A, paa \rightarrow pab, paa \Rightarrow pab}{A, paa \Rightarrow pab}
 \quad
 \frac{A, paa \rightarrow pab, pba \Rightarrow pab}{A, pba \Rightarrow pab}
 }{
 \underbrace{\forall x (pxa \rightarrow pxb), paa \vee pba \Rightarrow pab}_A
 }$$

- ▶ Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- ▶ Since $pab \in \mathcal{B}^\perp$, define $\langle a, b \rangle \notin p^\iota$, so $\mathcal{I} \not\models pab$.
- ▶ Since $pba \in \mathcal{B}^\top$, def. $\langle b, a \rangle \in p^\iota$, so $\mathcal{I} \models pba$ and $\mathcal{I} \models paa \vee pba$.

Counter-model Construction, Ex. 2

(Both branches closeable)

$$\frac{
 \frac{A, paa \rightarrow pab, paa \Rightarrow pab}{A, paa \Rightarrow pab}
 \quad
 \frac{A, paa \rightarrow pab, pba \Rightarrow pab}{A, pba \Rightarrow pab}
 }{
 \underbrace{\forall x (pxa \rightarrow pxb), paa \vee pba \Rightarrow pab}_A
 }$$

- ▶ Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- ▶ Since $pab \in \mathcal{B}^\perp$, define $\langle a, b \rangle \notin p^\iota$, so $\mathcal{I} \not\models pab$.
- ▶ Since $pba \in \mathcal{B}^\top$, def. $\langle b, a \rangle \in p^\iota$, so $\mathcal{I} \models pba$ and $\mathcal{I} \models paa \vee pba$.

Counter-model Construction, Ex. 2

$$\begin{array}{c}
 \text{(Both branches closeable)} \\
 \frac{A, paa \rightarrow pab, paa \Rightarrow pab}{A, paa \Rightarrow pab} \qquad \frac{\frac{A, pba \Rightarrow pab, paa}{A, paa \rightarrow pab, pba \Rightarrow pab} \quad pab, A, pba \Rightarrow pab}{A, pba \Rightarrow pab} \\
 \hline
 \underbrace{\forall x (pxa \rightarrow pxb), paa \vee pba \Rightarrow pab}_A
 \end{array}$$

- Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- Since $pab \in \mathcal{B}^\perp$, define $\langle a, b \rangle \notin p^\iota$, so $\mathcal{I} \not\models pab$.
- Since $pba \in \mathcal{B}^\top$, def. $\langle b, a \rangle \in p^\iota$, so $\mathcal{I} \models pba$ and $\mathcal{I} \models paa \vee pba$.

Counter-model Construction, Ex. 2

(Both branches closeable)

$$\frac{
 \frac{
 \frac{A, paa \rightarrow pab, paa \Rightarrow pab}{A, paa \Rightarrow pab}
 \quad
 \frac{
 \frac{A, pba \Rightarrow pab, paa}{A, paa \rightarrow pab, pba \Rightarrow pab}
 \quad
 \frac{pab, A, pba \Rightarrow pab}{A, pba \Rightarrow pab}
 }{
 \frac{A, pba \Rightarrow pab, paa}{A, paa \rightarrow pab, pba \Rightarrow pab}
 }
 }{
 \underbrace{\forall x (pxa \rightarrow pxb), paa \vee pba \Rightarrow pab}_A
 }$$

- Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- Since $pab \in \mathcal{B}^\perp$, define $\langle a, b \rangle \notin p^\iota$, so $\mathcal{I} \not\models pab$.
- Since $pba \in \mathcal{B}^\top$, def. $\langle b, a \rangle \in p^\iota$, so $\mathcal{I} \models pba$ and $\mathcal{I} \models paa \vee pba$.

Counter-model Construction, Ex. 2

(Both branches closeable)

$$\begin{array}{c}
 \frac{A, paa \rightarrow pab, paa \Rightarrow pab}{A, paa \Rightarrow pab} \quad \frac{\frac{A, pba \Rightarrow pab, paa}{A, paa \rightarrow pab, pba \Rightarrow pab} \quad \frac{pab, A, pba \Rightarrow pab}{A, pba \Rightarrow pab}}{\frac{\forall x (pxa \rightarrow pxb), paa \vee pba \Rightarrow pab}{A}}
 \end{array}$$

- Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- Since $pab \in \mathcal{B}^\perp$, define $\langle a, b \rangle \notin p^\iota$, so $\mathcal{I} \not\models pab$.
- Since $pba \in \mathcal{B}^\top$, def. $\langle b, a \rangle \in p^\iota$, so $\mathcal{I} \models pba$ and $\mathcal{I} \models paa \vee pba$.

Counter-model Construction, Ex. 2

$$\begin{array}{c}
 \text{(Both branches closeable)} \\
 \frac{A, paa \rightarrow pab, paa \Rightarrow pab}{A, paa \Rightarrow pab} \quad \frac{\frac{A, pba \Rightarrow pab, \textcolor{red}{paa}}{A, paa \rightarrow pab, pba \Rightarrow pab} \quad \frac{pab, A, pba \Rightarrow pab}{pab, A, pba \Rightarrow pab}}{A, pba \Rightarrow pab} \\
 \hline
 \underbrace{\forall x (pxa \rightarrow pxb), paa \vee pba \Rightarrow pab}_A
 \end{array}$$

- Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- Since $pab \in \mathcal{B}^\perp$, define $\langle a, b \rangle \notin p^\iota$, so $\mathcal{I} \not\models pab$.
- Since $pba \in \mathcal{B}^\top$, def. $\langle b, a \rangle \in p^\iota$, so $\mathcal{I} \models pba$ and $\mathcal{I} \models paa \vee pba$.

Counter-model Construction, Ex. 2

(Both branches closeable)

$$\begin{array}{c}
 \frac{A, paa \rightarrow pab, paa \Rightarrow pab}{A, paa \Rightarrow pab} \quad \frac{\frac{A, pba \Rightarrow pab, \textcolor{red}{paa}}{A, paa \rightarrow pab, pba \Rightarrow pab} \quad \frac{pab, A, pba \Rightarrow pab}{pab, A, pba \Rightarrow pab}}{A, pba \Rightarrow pab} \\
 \hline
 \underbrace{\forall x (pxa \rightarrow pxb), paa \vee pba \Rightarrow pab}_A
 \end{array}$$

- Herbrand universe of branch $\mathcal{B}$, and domain of $\mathcal{I}$, is $\{a, b\}$.
- Since $pab \in \mathcal{B}^\perp$, define $\langle a, b \rangle \notin p^\iota$, so $\mathcal{I} \not\models pab$.
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 \text{(Both branches closeable)} \\
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- ▶ $\mathcal{I}$ makes all of $\mathcal{B}^\top$ true and all of $\mathcal{B}^\perp$ false.

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Next Week

- ▶ **Two lectures next week:**
- ▶ Monday 10:15–12:00, Lille Aud
- ▶ Thursday 10:15–12:00, Lille Aud