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3.1 Six Basic NP-Complete Problems

When seasoned practitioners are confronted with a problem Π to be proved NP-complete, they have the advantage of having a wealth of experience to draw upon. They may well have proved a similar problem Π' NP-complete in the past or have seen such a proof. This will suggest that they try to prove Π NP-complete by mimicking the NP-completeness proof for Π' or by transforming Π' itself to Π. In many cases this may lead rather easily to an NP-completeness proof for Π.

All too often, however, no known NP-complete problem similar to Π can be found (even using the extensive lists at the end of this book). In such cases the practitioner may have no direct intuition as to which of the hundreds of known NP-complete problems is best suited to serve as the basis for the desired proof. Nevertheless, experience can still narrow the choices down to a core of basic problems that have been useful in the past. Even though in theory *any* known NP-complete problem can serve just as well as any other for proving a new problem NP-complete, in practice certain problems do seem to be much better suited for this task. The following six problems are among those that have been used most frequently, and we suggest that these six can serve as a "basic core" of known NP-complete problems for the beginner.

3-SATISFIABILITY (3SAT)

INSTANCE: Collection $C = \{c_1, c_2, \ldots, c_m\}$ of clauses on a finite set U of variables such that $|c_i| = 3$ for $1 \leqslant i \leqslant m$.

QUESTION: Is there a truth assignment for U that satisfies all the clauses in C?

3-DIMENSIONAL MATCHING (3DM)

INSTANCE: A set $M \subseteq W \times X \times Y$, where W, X, and Y are disjoint sets having the same number q of elements.

QUESTION: Does M contain a *matching*, that is, a subset $M' \subseteq M$ such that $|M'| = q$ and no two elements of M' agree in any coordinate?

VERTEX COVER (VC)

INSTANCE: A graph $G = (V,E)$ and a positive integer $K \leqslant |V|$.

QUESTION: Is there a *vertex cover* of size K or less for G, that is, a subset $V' \subseteq V$ such that $|V'| \leqslant K$ and, for each edge $\{u,v\} \in E$, at least one of u and v belongs to V'?

CLIQUE

INSTANCE: A graph $G = (V,E)$ and a positive integer $J \leqslant |V|$.

QUESTION: Does G contain a *clique* of size J or more, that is, a subset $V' \subseteq V$ such that $|V'| \geqslant J$ and every two vertices in V' are joined by an edge in E?

HAMILTONIAN CIRCUIT (HC)

INSTANCE: A graph $G = (V,E)$.

QUESTION: Does G contain a Hamiltonian circuit, that is, an ordering $<v_1, v_2, \ldots, v_n>$ of the vertices of G, where $n = |V|$, such that $\{v_n, v_1\} \in E$ and $\{v_i, v_{i+1}\} \in E$ for all i, $1 \leqslant i < n$?

PARTITION

INSTANCE: A finite set A and a "size" $s(a) \in Z^+$ for each $a \in A$.

QUESTION: Is there a subset $A' \subseteq A$ such that

$$\sum_{a \in A'} s(a) = \sum_{a \in A - A'} s(a) \ ?$$

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Lemma 3.1 For any graph $G=(V,E)$ and subset $V' \subseteq V$, the following statements are equivalent:

(a) V' is a vertex cover for G.
(b) $V-V'$ is an independent set for G.
(c) $V-V'$ is a clique in the *complement* G^c of G, where $G^c=(V,E^c)$ with $E^c = \{\{u,v\}: u,v \in V \text{ and } \{u,v\} \notin E\}$.

Thus we see that, in a rather strong sense, these three problems might be regarded simply as "different versions" of one another. Furthermore, the relationships displayed in the lemma make it a trivial matter to transform any one of the problems to either of the others.

For example, to transform VERTEX COVER to CLIQUE, let $G=(V,E)$ and $K \leqslant |V|$ constitute any instance of VC. The corresponding instance of CLIQUE is provided simply by the graph G^c and the integer $J=|V|-K$.

This implies that the NP-completeness of all three problems will follow as an immediate consequence of proving that any one of them is NP-complete. We choose to prove this for VERTEX COVER.

Theorem 3.3 VERTEX COVER is NP-complete.
Proof: It is easy to see that VC $\in$ NP since a nondeterministic algorithm need only guess a subset of vertices and check in polynomial time whether that subset contains at least one endpoint of every edge and has the appropriate size.

We transform 3SAT to VERTEX COVER. Let $U=\{u_1,u_2, \ldots, u_n\}$ and $C=\{c_1,c_2, \ldots, c_m\}$ be any instance of 3SAT. We must construct a graph $G=(V,E)$ and a positive integer $K \leqslant |V|$ such that G has a vertex cover of size K or less if and only if C is satisfiable.

As in the previous proof, the construction will be made up of several components. In this case, however, we will have only truth-setting components and satisfaction testing components, augmented by some additional edges for communicating between the various components.

For each variable $u_i \in U$, there is a truth-setting component $T_i=(V_i,E_i)$, with $V_i=\{u_i,\overline{u}_i\}$ and $E_i=\{\{u_i,\overline{u}_i\}\}$, that is, two vertices joined by a single edge. Note that any vertex cover will have to contain at least one of u_i and $\overline{u}_i$ in order to cover the single edge in E_i.

For each clause $c_j \in C$, there is a satisfaction testing component $S_j=(V_j',E_j')$, consisting of three vertices and three edges joining them to form a triangle:

$$V_j' = \{a_1[j],a_2[j],a_3[j]\}$$
$$E_j' = \{\{a_1[j],a_2[j]\},\{a_1[j],a_3[j]\},\{a_2[j],a_3[j]\}\}$$

CIRCUIT was shown to be NP-complete by restricting its instances to directed graphs in which each arc (u,v) occurs only in conjunction with the oppositely directed arc (v,u), thereby obtaining a problem identical to the undirected HAMILTONIAN CIRCUIT problem.

Thus proofs by restriction can be seen to embody a different way of looking at things than the standard NP-completeness proofs. Instead of trying to discover a way of transforming a known NP-complete problem to our target problem, we focus on the target problem itself and attempt to restrict away its "inessential" aspects until a known NP-complete problem appears.

We now give a number of additional examples of problems proved NP-complete by restriction, stating each proof with the brevity it deserves.

(1) **MINIMUM COVER**

INSTANCE: Collection C of subsets of a set S, positive integer K.
QUESTION: Does C contain a *cover* for S of size K or less, that is, a subset $C' \subseteq C$ with $|C'| \leqslant K$ and such that $\bigcup_{c \in C'} c = S$?
Proof: Restrict to X3C by allowing only instances having $|c|=3$ for all $c \in C$ and having $K = |S|/3$.

(2) **HITTING SET**

INSTANCE: Collection C of subsets of a set S, positive integer K.
QUESTION: Does S contain a *hitting set* for C of size K or less, that is, a subset $S' \subseteq S$ with $|S'| \leqslant K$ and such that S' contains at least one element from each subset in C?
Proof: Restrict to VC by allowing only instances having $|c|=2$ for all $c \in C$.

(3) **SUBGRAPH ISOMORPHISM**

INSTANCE: Two graphs, $G=(V_1,E_1)$ and $H=(V_2,E_2)$.
QUESTION: Does G contain a subgraph *isomorphic* to H, that is, a subset $V \subseteq V_1$ and a subset $E \subseteq E_1$ such that $|V|=|V_2|, |E|=|E_2|$, and there exists a one-to-one function $f: V_2 \rightarrow V$ satisfying $\{u,v\} \in E_2$ if and only if $\{f(u),f(v)\} \in E$?
Proof: Restrict to CLIQUE by allowing only instances for which H is a complete graph, that is, E_2 contains all possible edges joining two members of V_2.

(4) **BOUNDED DEGREE SPANNING TREE**

INSTANCE: A graph $G=(V,E)$ and a positive integer $K \leqslant |V|-1$.
QUESTION: Is there a *spanning tree* for G in which no vertex has degree exceeding K, that is, a subset $E' \subseteq E$ such that $|E'|=|V|-1$, the graph $G'=(V,E')$ is connected, and no vertex in V is included in more than K edges from E'?
Proof: Restrict to HAMILTONIAN PATH by allowing only instances in which $K=2$.

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A1.2 SUBGRAPHS AND SUPERGRAPHS

[GT19] CLIQUE

INSTANCE: Graph $G=(V,E)$, positive integer $K \leqslant |V|$.
QUESTION: Does G contain a clique of size K or more, i.e., a subset $V' \subseteq V$ with $|V'| \geqslant K$ such that every two vertices in V' are joined by an edge in E?
Reference: [Karp, 1972]. Transformation from VERTEX COVER (see Chapter 3).
Comment: Solvable in polynomial time for graphs obeying any fixed degree bound d, for planar graphs, for edge graphs, for chordal graphs [Gavril, 1972], for comparability graphs [Even, Pnueli, and Lempel, 1972], for circle graphs [Gavril, 1973], and for circular arc graphs (given their representation as families of arcs) [Gavril, 1974a]. The variant in which, for a given r, $0<r<1$, we are asked whether G contains a clique of size $r|V|$ or more is NP-complete for any fixed value of r.

[GT20] INDEPENDENT SET

INSTANCE: Graph $G=(V,E)$, positive integer $K \leqslant |V|$.
QUESTION: Does G contain an independent set of size K or more, i.e., a subset

$V' \subseteq V$ such that $|V'| \geqslant K$ and such that no two vertices in V' are joined by an edge in E?
Reference: Transformation from VERTEX COVER (see Chapter 3).
Comment: Remains NP-complete for cubic planar graphs [Garey, Johnson, and Stockmeyer, 1976], [Garey and Johnson, 1977a], [Maier and Storer, 1977], for edge graphs of directed graphs [Gavril, 1977a], for total graphs of bipartite graphs [Yannakakis and Gavril, 1978], and for graphs containing no triangles [Poljak, 1974]. Solvable in polynomial time for bipartite graphs (by matching, e.g., see [Harary, 1969]), for edge graphs (by matching), for graphs with no vertex degree exceeding 2, for chordal graphs [Gavril, 1972], for circle graphs [Gavril, 1973], for circular arc graphs (given their representation as families of arcs) [Gavril, 1974a], for comparability graphs [Golumbic, 1977}, and for claw-free graphs [Minty, 1977]. match

[GT21] INDUCED SUBGRAPH WITH PROPERTY Π (*)

INSTANCE: Graph $G=(V,E)$, positive integer $K \leqslant |V|$.
QUESTION: Is there a subset $V' \subseteq V$ with $|V'| \geqslant K$ such that the subgraph of G induced by V' has property Π (see comments for possible choices for Π)?
Reference: [Yannakakis, 1978a], [Yannakakis, 1978b], [Lewis, 1978]. Transformation from 3SAT.
Comment: NP-hard for any property Π that holds for arbitrarily large graphs, does not hold for all graphs, and is "hereditary," i.e., holds for all induced subgraphs of G whenever it holds for G. If in addition one can determine in polynomial time whether Π holds for a graph, then the problem is NP-complete. Examples of such properties Π include "G is a clique," "G is an independent set," "G is planar," "G is bipartite," "G is outerplanar," "G is an edge graph," "G is chordal," "G is a comparability graph," and "G is a forest." The same general results hold if G is restricted to planar graphs and Π satisfies the above constraints for planar graphs, or if G is restricted to acyclic directed graphs and Π satisfies the above constraints for such graphs. A weaker result holds when G is restricted to bipartite graphs [Yannakakis, 1978b].

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[SP10] COMPARATIVE CONTAINMENT

INSTANCE: Two collections $R=\{R_1,R_2,\ldots,R_k\}$ and $S=\{S_1,S_2,\ldots,S_l\}$ of subsets of a finite set X, weights $w(R_i) \in Z^+$, $1 \leqslant i \leqslant k$, and $w(S_j) \in Z^+$, $1 \leqslant j \leqslant l$.
QUESTION: Is there a subset $Y \subseteq X$ such that

$$\sum_{Y \subseteq R_i} w(R_i) \geqslant \sum_{Y \subseteq S_j} w(S_j)\ ?$$

Reference: [Plaisted, 1976]. Transformation from VERTEX COVER.
Comment: Remains NP-complete even if all subsets in R and S have weight 1 [Garey and Johnson, ——].

[SP11] 3-MATROID INTERSECTION

INSTANCE: Three matroids $(E,F_1),(E,F_2),(E,F_3)$, positive integer $K \leqslant |E|$. (A matroid (E,F) consists of a set E of elements and a non-empty family F of subsets of E such that (1) $S \in F$ implies all subsets of S are in F and (2) if two sets $S,S' \in F$ satisfy $|S|=|S'|+1$, then there exists an element $e \in S-S'$ such that $(S' \cup \{e\}) \in F$.)

i.e. each. mengdene

QUESTION: Is there a subset $E' \subseteq E$ such that $|E'|=K$ and $E' \in (F_1 \cap F_2 \cap F_3)$?
Reference: Transformation from 3DM.
Comment: The related 2-MATROID INTERSECTION problem can be solved in polynomial time, even if the matroids are described by giving polynomial time algorithms for recognizing their members, and even if each element $e \in E$ has a weight $w(e) \in Z^+$, with the goal being to find an $E' \in (F_1 \cap F_2)$ having maximum total weight (e.g., see [Lawler, 1976a]).

A3.2 WEIGHTED SET PROBLEMS

[SP12] PARTITION

INSTANCE: Finite set A and a size $s(a) \in Z^+$ for each $a \in A$.
QUESTION: Is there a subset $A' \subseteq A$ such that $\sum_{a \in A'} s(a) = \sum_{a \in A-A'} s(a)$?
Reference: [Karp, 1972]. Transformation from 3DM (see Section 3.1.5).
Comment: Remains NP-complete even if we require that $|A'|=|A|/2$, or if the elements in A are ordered as $a_1,a_2,\ldots,a_{2n}$ and we require that A' contain exactly one of a_{2i-1},a_{2i} for $1 \leqslant i \leqslant n$. However, all these problems can be solved in pseudo-polynomial time by dynamic programming (see Section 4.2).

[SP13] SUBSET SUM

INSTANCE: Finite set A, size $s(a) \in Z^+$ for each $a \in A$, positive integer B.
QUESTION: Is there a subset $A' \subseteq A$ such that the sum of the sizes of the elements in A' is exactly B?
Reference: [Karp, 1972]. Transformation from PARTITION.
Comment: Solvable in pseudo-polynomial time (see Section 4.2).