String Search

26th August 2020

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Search Problems have become increasingly important

- Vast ammounts of information is available
  - Google and similar search engines search for given strings (or sets of strings) on all registered web-pages.
  - The amount of stored digital information grows steadily (rapidly?)
    - 3 zettabytes (10^{21} = 1 000 000 000 000 000 000 000 = trilliard) in 2012
    - 4.4 zettabytes in 2013
    - 44 zettabytes in 2020 (estimated)
    - 175 zettabytes in 2025 (estimated)
- Search for a given pattern in DNA strings (about 3 giga-letters (10^9) in human DNA).
- Searching for similar patterns is also relevant
  - The genetic sequences in organisms are changing over time because of mutations.
  - Searches for similar patterns are treated in Ch. 20.5. We will look at that in connection with **Dynamic Programming**

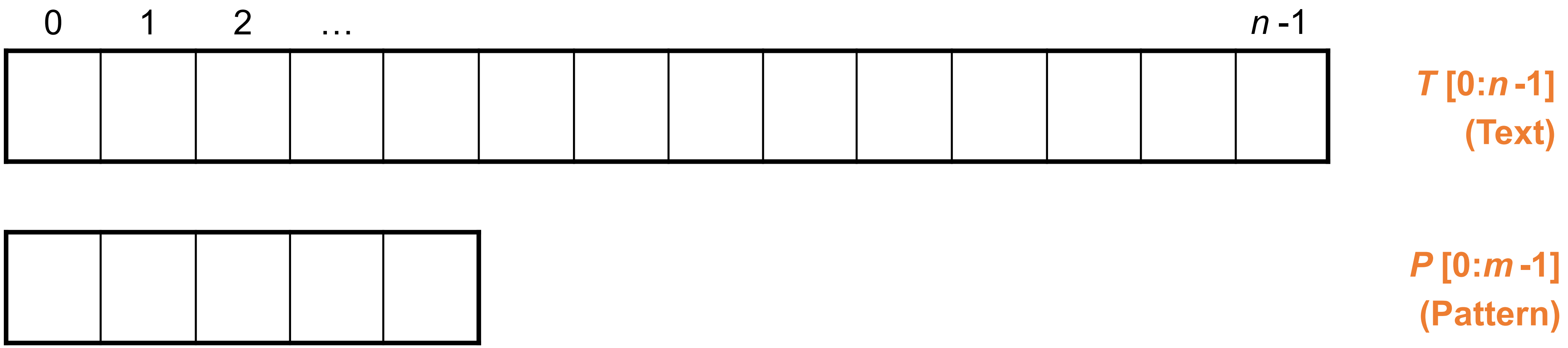
Definitions

- An **alphabet** is a finite set of «symbols» $A = \{a_1 , a_2 , \ldots, a_k\}$.
- A **string** $S = S\,[0\!: n\,\text{-}1]$ or $S = < s_0\, s_1 \ldots\, s_{n\text{-}1} >$ of length n is a sequence of n symbols from A.

String Search:

Given two strings T (= Text) and P (= Pattern), P is usually much shorter than T.

Decide whether P occurs as a (continuous) substring in T, and if so, find where it occurs.



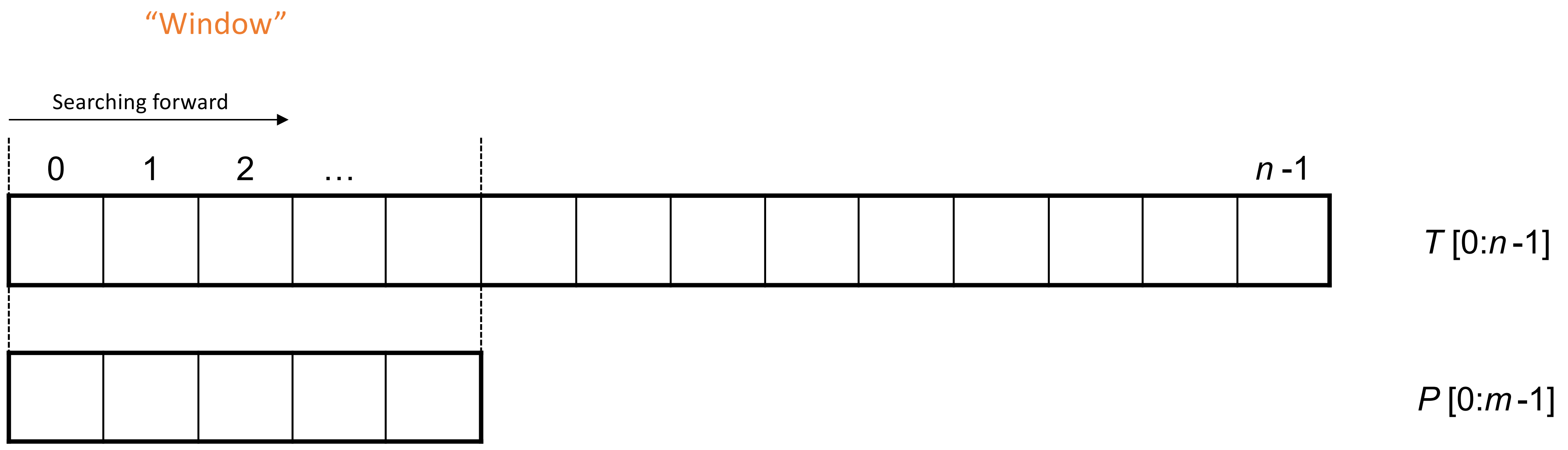
Variants of String Search

- Naive algorithm, no preprocessing of *T* or *P*
  - Assume that the length of *T* and *P* are *n* and *m* respectively
  - The naive algorithm is already a polynomial-time algorithm, with worst case execution time *O*($n*m$)*, which is also O*(n^2).

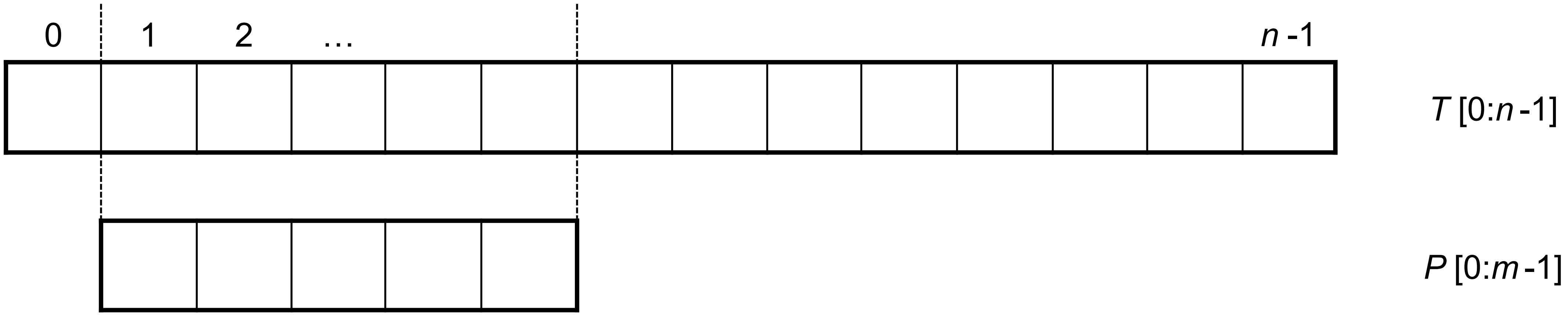
- Preprocessing of *P* (the pattern) for each new *P*
  - Prefix-search: The Knuth-Morris-Pratt algorithm
  - Suffix-search: The Boyer-Moore algorithm
  - Hash-based: The Karp-Rabin algorithm

- Preprocess the text *T*
  (Used when we search the same text a lot of times (with different patterns), done to an extreme degree in search engines.)
  - Suffix trees: Data structure that relies on a structure called a Trie.

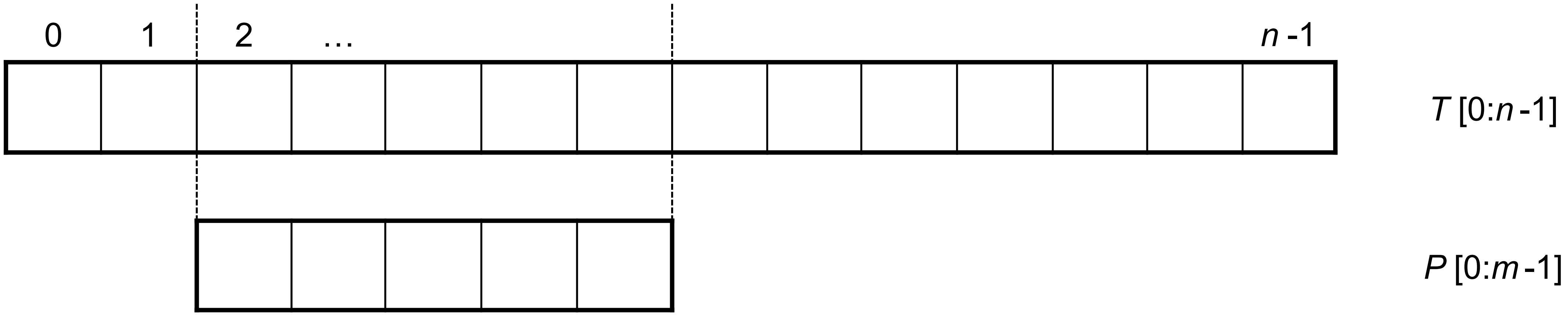
The naive algorithm (Prefix based)



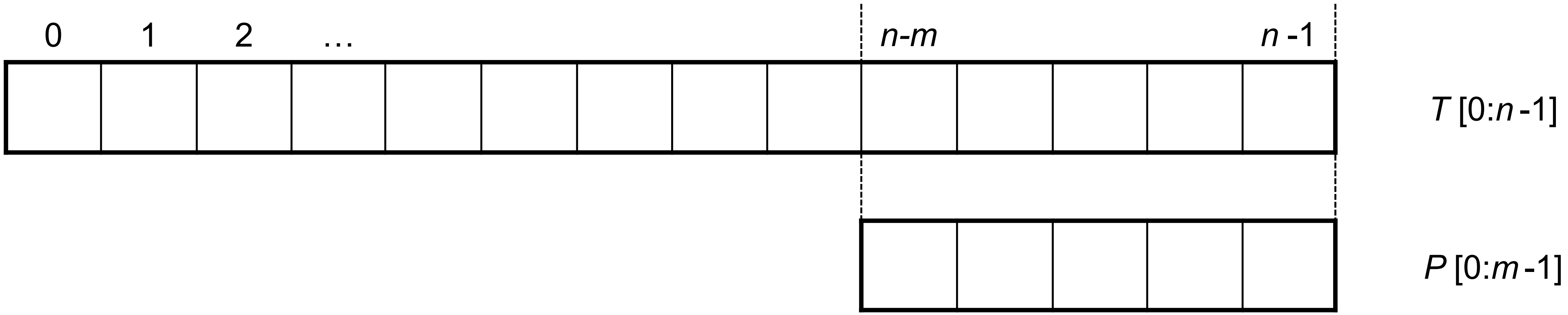
The naive algorithm



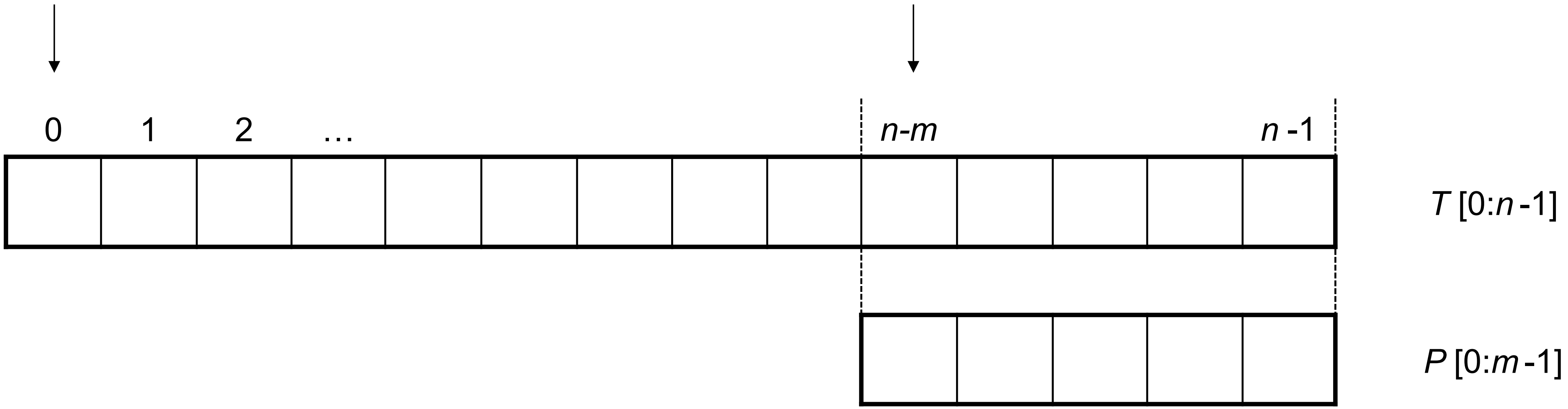
The naive algorithm



The naive algorithm



The naive algorithm



function *NaiveStringMatcher* (*P* [0:*m* -1], *T* [0:*n* -1])

    for *s* ← 0 **to** *n* - *m* **do**

        if *T* [*s* :*s* + *m* - 1] = *P* **then**   // is window = P?

            return(*s*)

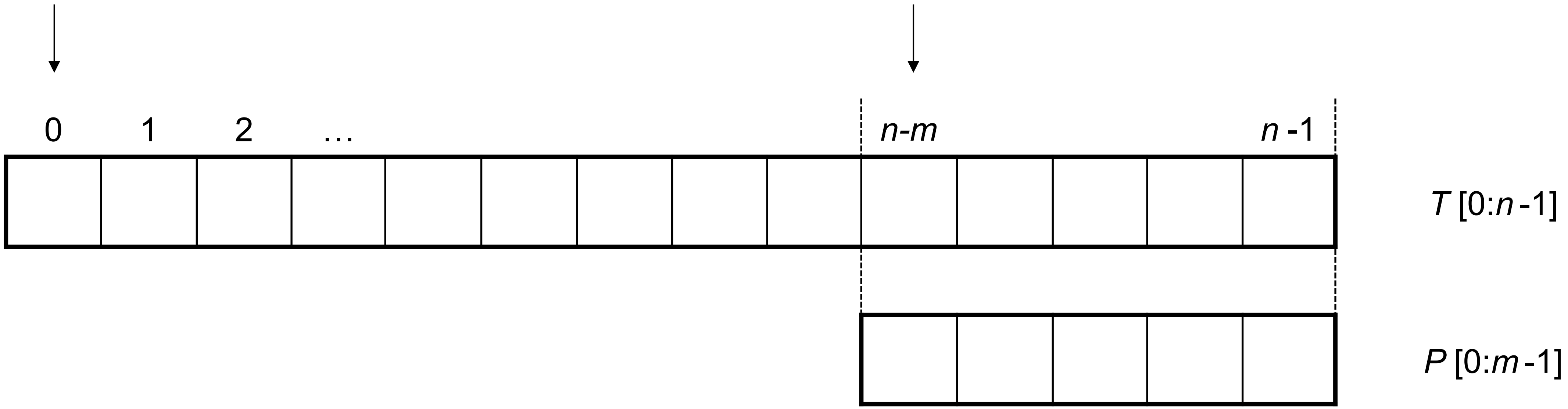
        endif

    endfor

    return(-1)

end *NaiveStringMatcher*

The naive algorithm




```
function NaiveStringMatcher (P [0:m -1], T [0:n -1])
   for s ← 0 to n - m do
         if T [s :s + m - 1] = P  then           // is window = P?
              return(s)
         endif
   endfor
   return(-1)
end NaiveStringMatcher
```

The for-loop is executed $n - m + 1$ times.

Each string test has up to m symbol comparisons

$O(nm)$ execution time (worst case)

The Knuth-Morris-Pratt algorithm (Prefix based)

- There is room for improvement in the naive algorithm
  - The naive algorithm moves the window (pattern) only one character at a time.
  - But we can move it farther, based on what we know from earlier comparisons.

Search forward

0	0	1	0	0	1	0	0	2	0	0	0	1	0	0	2	0	1	2	…

0	0	1	0	0	2	0	1

The Knuth-Morris-Pratt algorithm

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Search forward

0	0	1	0	0	1	0	0	2	0	0	0	1	0	0	2	0	1	2	…

0	0	1	0	0	2	0	1

The Knuth-Morris-Pratt algorithm

0	**0**	**1**	**0**	**0**	1	0	0	2	0	0	0	1	0	0	2	0	1	2	...

0	0	1	0	0	2	0	1

The Knuth-Morris-Pratt algorithm

0	0	1	0	0	1	0	0	2	0	0	0	1	0	0	2	0	1	2	…
	0	0	1	0	0	2	0	1											

We move the pattern one step: Mismatch

The Knuth-Morris-Pratt algorithm

0	0	1	0	0	1	0	0	2	0	0	0	1	0	0	2	0	1	2	…
		0	0	1	0	0	2	0	1										

We move the pattern two steps: Mismatch

The Knuth-Morris-Pratt algorithm

0	**0**	**1**	**0**	**0**	1	0	0	2	0	0	0	1	0	0	2	0	1	2	…

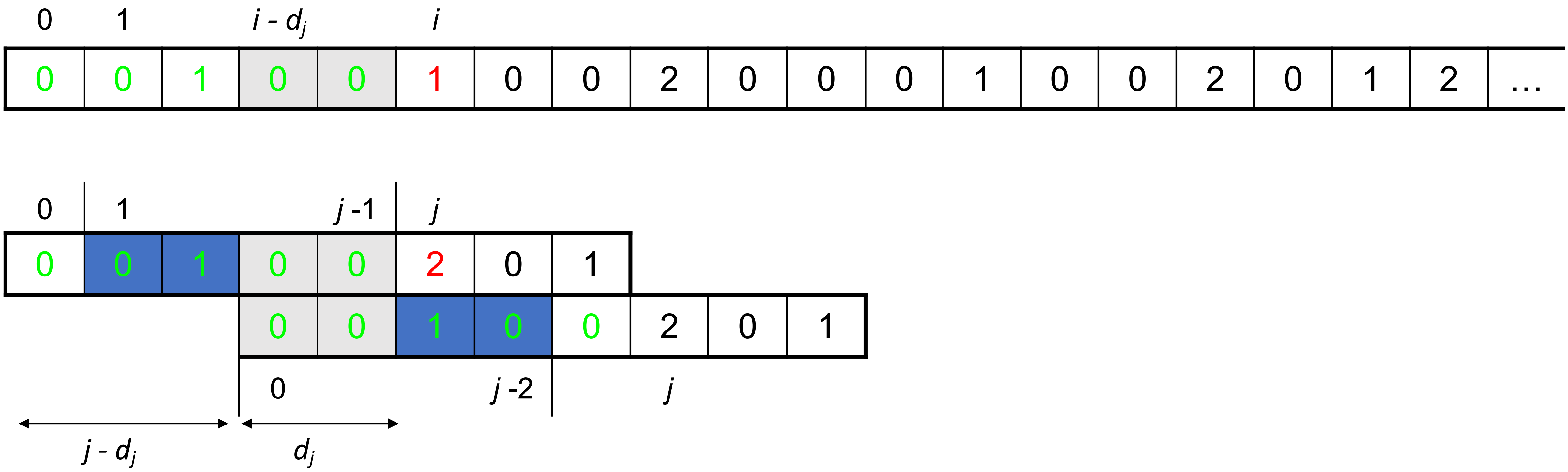
0	0	1	0	0	2	0	1

3

We move the pattern three steps: Now, there is at least a match in the part of *T* where we had a match previously

- We can skip a number of tests and move the pattern more than one step before we start comparing characters again. (3 in the above situation.)
- The key is that we know what the characters of *T* and *P* are up to the point where *P* and *T* got different. (*T* and *P* are equal up to this point.)
- For each possible index *j* in *P,* we assume that the first difference between *P* and *T* occurs at *j*, and from that compute how far we can move *P* before the next string-comparison.
- It may well be that we never get an overlap like the one above, and we can then move *P* all the way to the point in *T* where we found an inequality. This is the best case for the efficiency of the algorithm.

The Knuth-Morris-Pratt algorithm



d_j is the longest suffix of $P\,[1 : j-1]$ that is also prefix of $P\,[0 : j-2]$

We know that if we move P less than $j - d_j$ steps, there can be no (full) match.

And we know that, after this move, $P\,[0 : d_j -1]$ will match the corresponding part of T.

Thus we can start the comparison at d_j in P and compare $P\,[d_j : m-1]$ with the symbols from index i in T.

Idea behind the Knuth-Morris-Pratt algorithm

- We will produce a table *Next* [0: *m*-1] that shows how far we can move *P* when we get a (first) mismatch at index *j* in *P*, *j* = 0,1,2, … , *m*-1

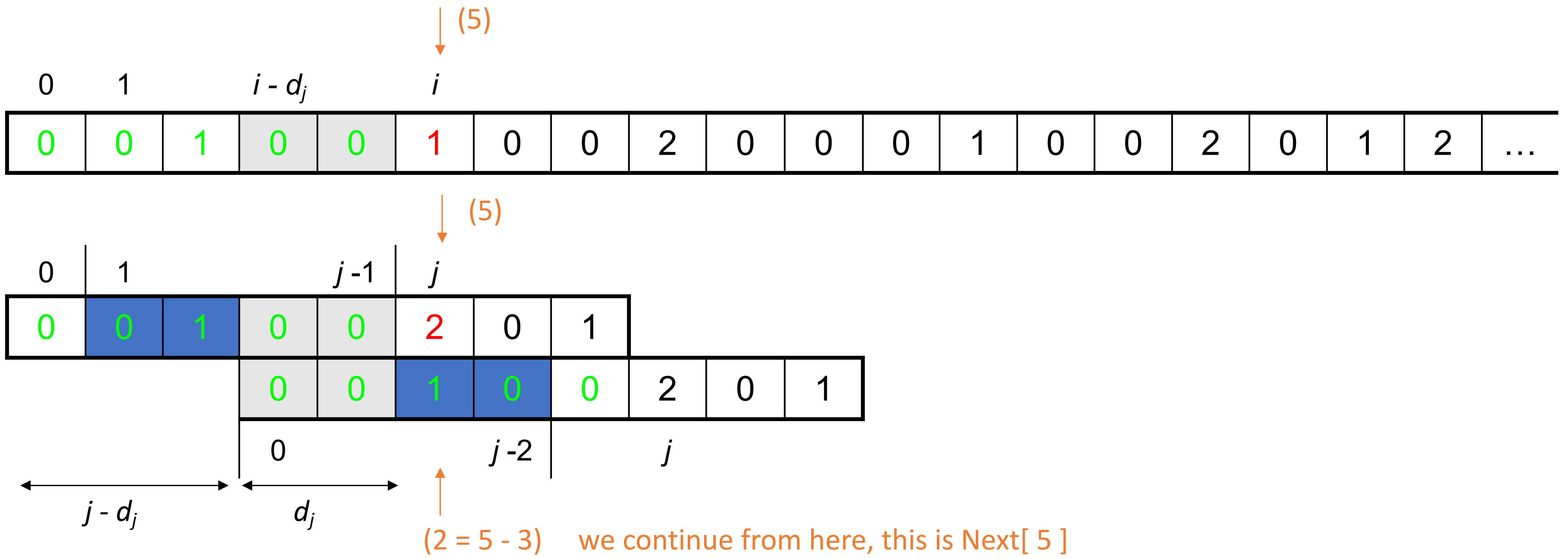
- But the array *Next* will not give this number directly. Instead, *Next* [*j*] will contain the new (and smaller value) that *j* should have when we resume the search after a mismatch at *j* in *P* (see below)
  - That is: *Next* [*j*] = *j* – <number of steps that *P* should be moved>,
  - or: *Next* [*j*] *is the value that is named* d_j on the previous slide

- After *P* is moved, we know that the first d_j symbols of *P* are equal to the corresponding symbols in T (that's how we chose d_j).

- So, the search can continue from index *i* in *T* and *Next* [*j*] in *P*.

- **The array *Next*[] can be computed from *P* alone!**

The Knuth-Morris-Pratt algorithm



```
function KMPStringMatcher (P [0:m -1], T [0:n -1])
    i ← 0   // indeks i T
    j ← 0   // indeks i P
    CreateNext(P [0:m -1], Next [n -1])
    while i < n do
            if P [ j ] = T [ i ] then
                    if j = m –1 then                    // check full match
                            return(i – m + 1)
                    endif
                    i ← i +1
                    j ← j +1
            else
                    j ← Next [ j ]
                    if j = 0 then
                            if T [ i ] ≠ P [0] then
                                    i ← i +1
                            endif
                    endif
            endif
    endwhile
    return(-1)
end KMPStringMatcher
```

$O(n)$

Calculating the array Next[] from P

function *CreateNext* (*P* [0:*m* -1], *Next* [0:*m* -1])

…

end *CreateNext*

- This can be written straight-ahead with simple searches, and will then use time $O(m^2)$.
- A more clever approach finds the array *Next* in time $O(m)$.
- We will look at the procedure in an exercise next week.

The Knuth-Morris-Pratt algorithm, example

0	0	1	0	0	1	0	0	2	0	0	0	1	0	0	2	0	1	2	…

0	0	1	0	0	2	0	1

The array *Next* for the string *P* above:

j =	0	1	2	3	4	5	6	7
Next[*j*] =	0	0	1	1	1	2	0	1

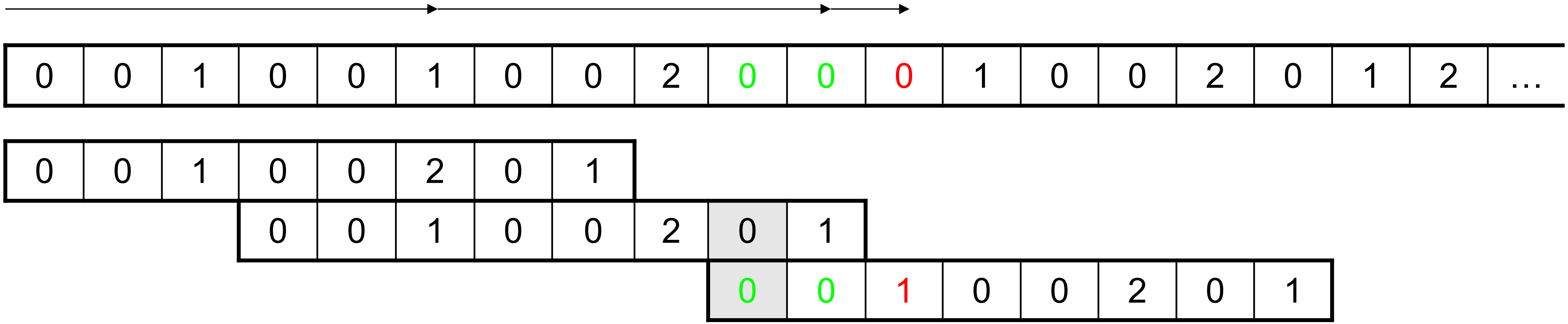
The Knuth-Morris-Pratt algorithm, example

0	0	1	0	0	1	0	0	2	0	0	0	1	0	0	2	0	1	2	…
0	0	1	0	0	2	0	1												
			0	0	1	0	0	2	0	1									

The array *Next* for the string *P* above:

j =	0	1	2	3	4	5	6	7
Next[*j*] =	0	0	1	1	1	2	0	1

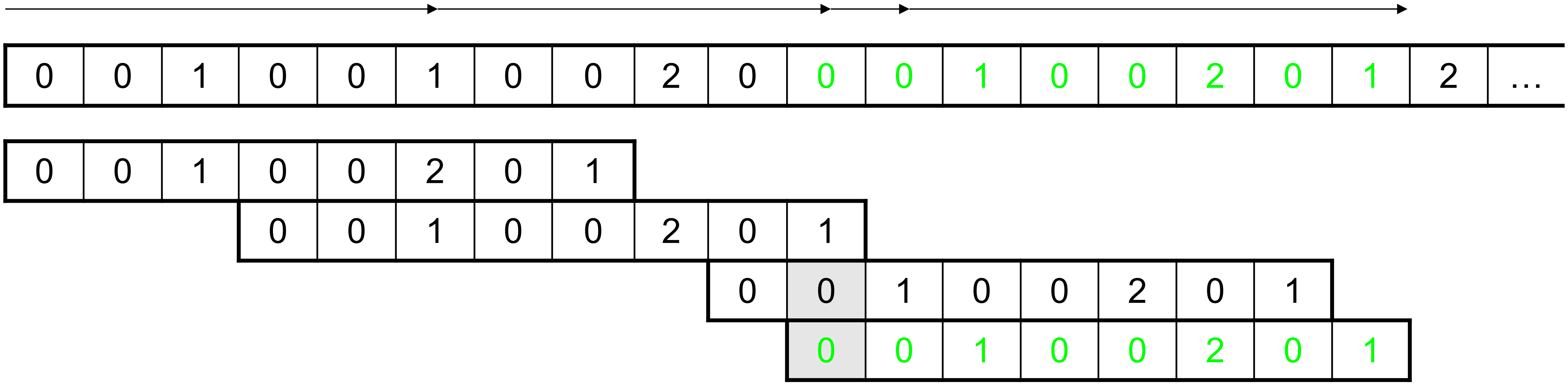
The Knuth-Morris-Pratt algorithm, example



The array *Next* for the string *P* above:

j =	0	1	2	3	4	5	6	7
Next[j] =	0	0	1	1	1	2	0	1

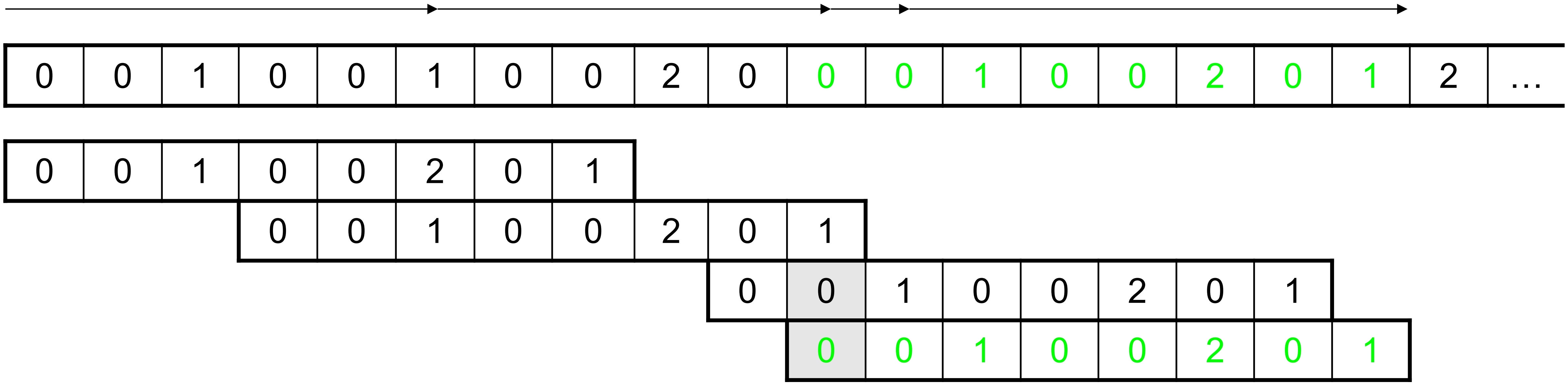
The Knuth-Morris-Pratt algorithm, example



The array *Next* for the string *P* above:

j =	0	1	2	3	4	5	6	7
Next[*j*] =	0	0	1	1	1	2	0	1

The Knuth-Morris-Pratt algorithm, example



The array *Next* for the string *P* above:

j =	0	1	2	3	4	5	6	7
Next[*j*] =	0	0	1	1	1	2	0	1

This is a linear algorithm: worst case runtime *O*(*n*).

The Boyer-Moore algorithm (Suffix based)

- The naive algorithm, and Knuth-Morris-Pratt is prefix-based (from left to right through *P*)
- The Boyer-Moore algorithm (and variants of it) is suffix-based (from right to left in *P*)
- Horspool proposed a simplification of Boyer-Moore, and we will look at the resulting algorithm here.

B	M	m	a	t	c	h	e	r	_	s	h	i	f	t	_	c	h	a	r	a	c	t	e	r	_	e	x	…

c	h	a	r	a	c	t	e	r

The Boyer-Moore algorithm (Horspool)

Comparing from the end of *P*

B	M	m	a	t	c	h	e	**r**	_	s	h	i	f	t	_	c	h	a	r	a	c	t	e	r	_	e	x	…

c	h	a	**r**	a	c	t	e	r

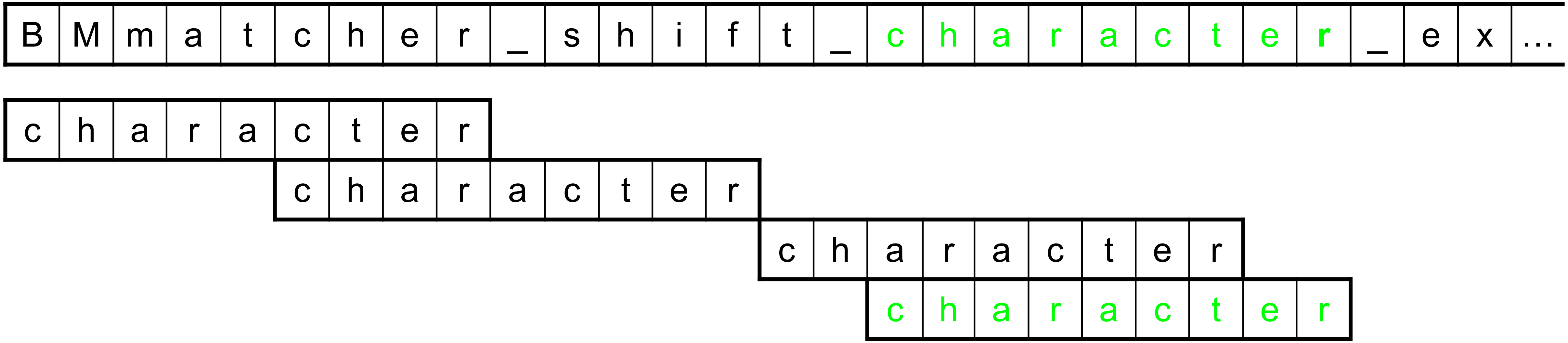
The Boyer-Moore algorithm (Horspool)

B	M	m	a	t	c	h	e	r	_	s	h	i	**f**	t	_	c	h	a	r	a	c	t	e	r	_	e	x	…
c	h	a	r	a	c	t	e	r																				
					c	h	a	r	a	c	t	e	r															

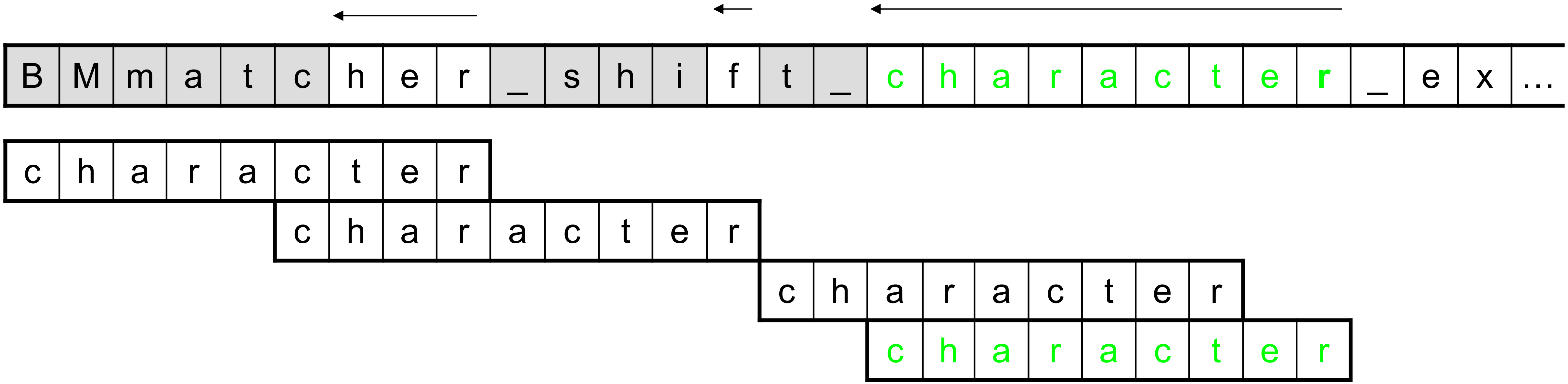
The Boyer-Moore algorithm (Horspool)

B	M	m	a	t	c	h	e	r	_	s	h	i	f	t	_	c	h	a	r	a	c	**t**	e	r	_	e	x	…
c	h	a	r	a	c	t	e	r																				
					c	h	a	r	a	c	t	e	r															
														c	h	a	r	a	c	**t**	e	r						

The Boyer-Moore algorithm (Horspool)



The Boyer-Moore algorithm (Horspool)



Worst case execution time $O(mn)$, same as for the naive algorithm!

However: Sub-linear ($\leq n$), as the average execution time is $O(n\ (\log_{|A|} m) / m)$.

```
function HorspoolStringMatcher (P [0:m -1], T [0:n -1])
   i ← 0
   CreateShift(P [0:m -1], Shift [0:|A| - 1])
   while i < n – m do
         j ← m – 1
         while j ≥ 0 and T [ i + j ] = P [ j ] do
               j ← j -1
         endwhile
         if j = 0 then
               return( i )
         endif
         i ← i + Shift[ T[ i + m -1] ]
   endwhile
   return(-1)
end HorspoolStringMatcher
```

Calculating the array Shift[] from P

function *CreateShift* (*P* [0:*m* -1], Shift [0:|A| - 1])

…

end *CreateShift*

- We must preprocess *P* to find the array *Shift*.
- The size of Shift[ ] is the number of symbols in the alphabet.
- We search from the end of *P* (minus the last symbol), and calculate the distance from the end for every first occurence of a symbol.
- For the symbols not occuring in *P*, we know:

Shift [*t*] = <the length of *P*> (*m*)

This will give a “full shift”.

The Karp-Rabin algorithm (hash based)

- We assume that the alphabet for our strings is $A = \{0, 1, 2, \ldots, k-1\}$.
- Each symbol in A can be seen as a digit in a number system with base k
- Thus each string in A^* can be seen as number in this system (and we assume that the most significant digit comes first, as usual)

Example:
k = 10, and A = {0,1, 2, …, 9} we get the traditional decimal number system
The string ”6832355” can then be seen as the number 6 832 355.

- Given a string $P[0: m-1]$. We can then calculate the corresponding number P' using $m - 1$ multiplications and $m - 1$ additions (Horners rule, computed from the innermost right expression and outwards):

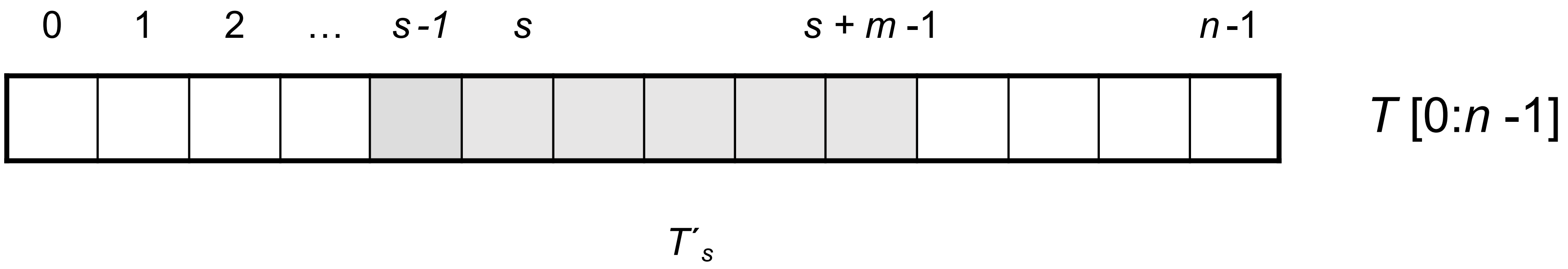
$$P' = P[m-1] + k\,(P[m-2] + \ldots + k\,(P[1] + k\,(P[0])\ldots))$$

Example (written as it computed from left to right):
1234 = (((1*10) + 2)*10 + 3)*10 + 4

The Karp-Rabin algorithm

- Given a string $T\,[0{:}\ n-1]$, and an integer s (start-index), and a pattern of length m. We then refer to the substring $T\,[s{:}\ s+m-1]$ as T_s, and its value is referred to as T'_s
- The algorithm:
  - We first compute the value P' for the pattern P.
  - Based on Horners rule, we compute T'_0, T'_1, T'_2, …, and successively compare these numbers to P'.
- This is very much like the naive algorithm.
- However: Given T'_{s-1} and k^{m-1}, we can compute T'_s in constant time:

!



The Karp-Rabin algorithm

This constant time computation can be done as follows (where $T'_{s\,-1}$ is defined as on the previous slide, and $k^{\,m-1}$ is pre-computed):

$$T'_s = k * (T'_{s\,-1} - k^{\,m-1} * T\,[s]) + T\,[s{+}m] \qquad\qquad s = 1, \ldots, n - m$$

Example:

k = 10, A = {0,1, 2, …, 9} (the usual decimal number system) and m = 7.

$T'_{s\,-1}$ = 7937245

T'_s = 9372458

T'_s = 10 * (7937245 – (1000000 * 7)) + 8 = 9372458

The Karp-Rabin algorithm

- We can compute T'_s in constant time when we know T'_{s-1} and k^{m-1}.
- We can therefore compute
  - P' and
  - T'_s, $s = 0, 1, \ldots, n - m$ ($n - m + 1$ numbers)

  in time $O(n)$.
- We can threfore "theoretically" implement the search algorithm in time $O(n)$.
- However, the numbers T'_s and P' will be so large that storing and comparing them will take too long time (in fact $O(m)$ time – back to the naive algorithm again).
- The Karp-Rabin trick is to instead use modular arithmetic:
  - We do all computations modulo a value q.
- The value q should be chosen as a prime, so that kq just fits in a register (of e.g. 64 bits).
- A prime number is chosen as this will distribute the values well.

The Karp-Rabin algorithm

- We compute $T'^{(q)}_s$ and $P'^{(q)}$, where

  $T'^{(q)}_s = T'_s \bmod q,$

  $P'^{(q)} = P' \bmod q,$ *(only once)*

  and compare.

  x mod *y* is the remainder when deviding *x* with *y*, this is always in the interval $\{0, 1, \ldots, y-1\}$.

- We can get $T'^{(q)}_s = P'^{(q)}$ even if $T'_s \neq P'$. This is called a spurious match.

- So, if we have $T'^{(q)}_s = P'^{(q)}$, we have to fully check whether $T_s = P$.

- With large enough *q*, the probability for getting spurious matches is low (see next slides)

function *KarpRabinStringMatcher* (P [0:m -1], T [0:n -1], k, q)

$c \leftarrow k^{m-1} \bmod q$

$P'^{(q)} \leftarrow 0$

$T'^{(q)}_s \leftarrow 0$

for $i \leftarrow 1$ **to** m **do**

$P'^{(q)} \leftarrow (k * P'^{(q)} + P[\,i\,]) \bmod q$

$T'^{(q)}_0 \leftarrow (\mathrm{k} * T'^{(q)}_0 + T[\,i\,]) \bmod q$

endfor

for $s \leftarrow 0$ **to** $n - m$ **do**

if $s > 0$ **then**

$T'^{(q)}_s \leftarrow (\mathrm{k} * (\,T'^{(q)}_{s-1} - T[\,s\,] * c) + T[\,s + m\,]) \bmod q$

endif

if $T'^{(q)}_s = P'^{(q)}$ **then**

if $T_s = P$ **then**

return(s)

endif

endif

endfor

return(-1)

end *KarpRabinStringMatcher*

The Karp-Rabin algorithm , time considerations

- The worst case running time occurs when the pattern *P* is found at the end of the string *T.*
- If we assume that the strings are distributed uniformally, the probability that $T'^{(q)}_s$ is equal to $P'^{(q)}$ (which is in the interval {0, 1, …, q-1}) is 1/q
- Thus $T'^{(q)}_s$, for s = 0, 1, …, n-m-1 will for each s lead to a spurious match with probability 1/q.
- With the real match at the end of *T*, we will on average get (n - m) / q spurious matches during the search
- Each of these will lead to *m* symbol comparisons. In addition, we have to check whether$T'^{(q)}_{n-m}$ equals *P* when we finally find the correct match at the end.
- Thus the number of comparisons of single symbols and computations of new values $T'^{(q)}_s$ will be:

$$\left(\frac{n-m}{q}+1\right)m+(n-m+1)$$

- We can choose values so that $q >> m$. Thus the runing time will be $O(n)$.

Multiple searches in a fixed string T (structure)

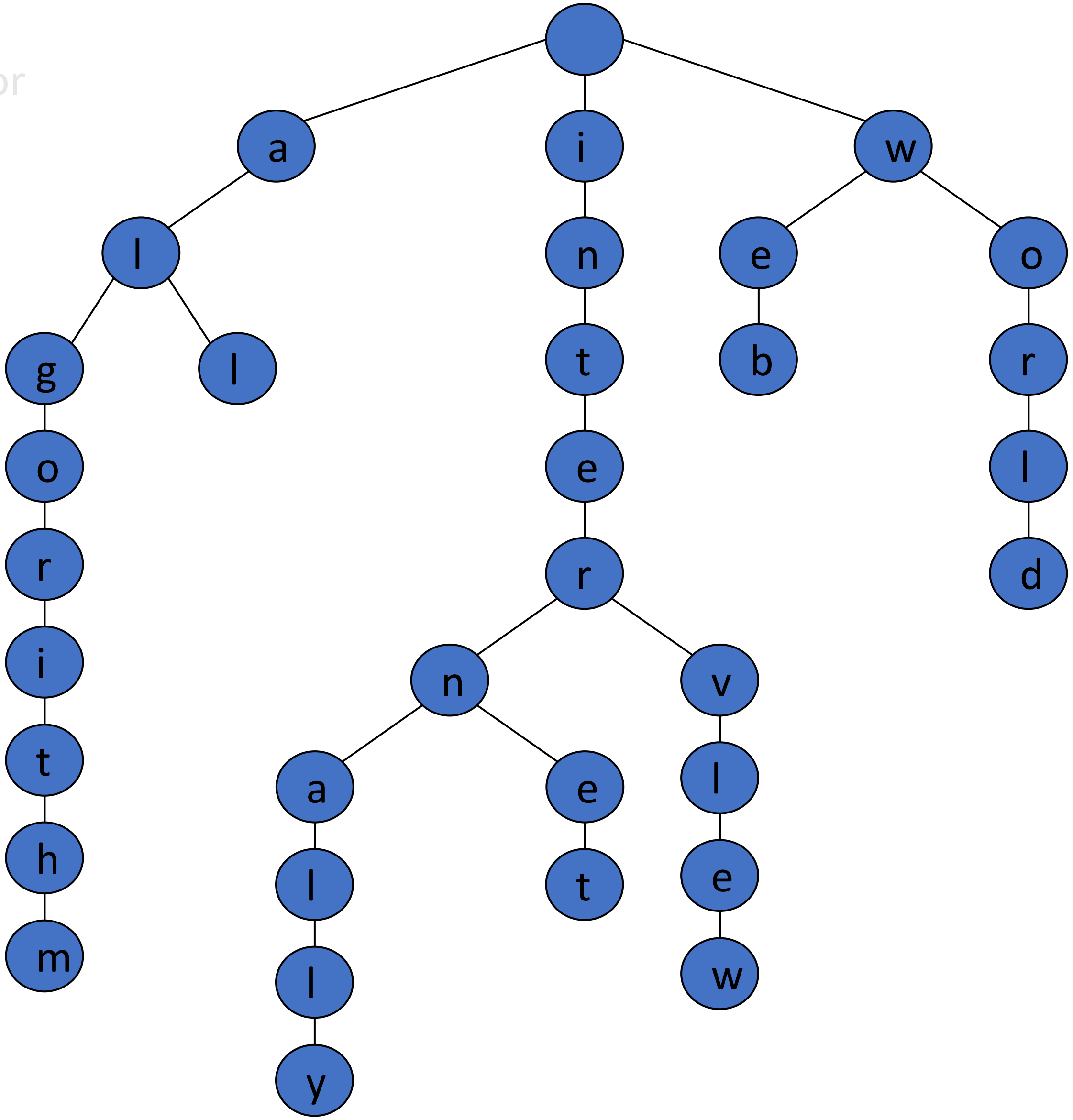
- It is then usually smart to *preprocess T*, so that later searches in *T* for different patterns *P* will be fast.
  - Search engines (like Google or Bing) do this in a very clever way, so that searches in huge number of web-pages can be done extremely fast.

- We often refer to this as *indexing* the text (or data set), and this can be done in a number of ways. We will look at the following technique:
  - Suffix trees, which relies on “Tries” trees.
  - So we first look at Tries.

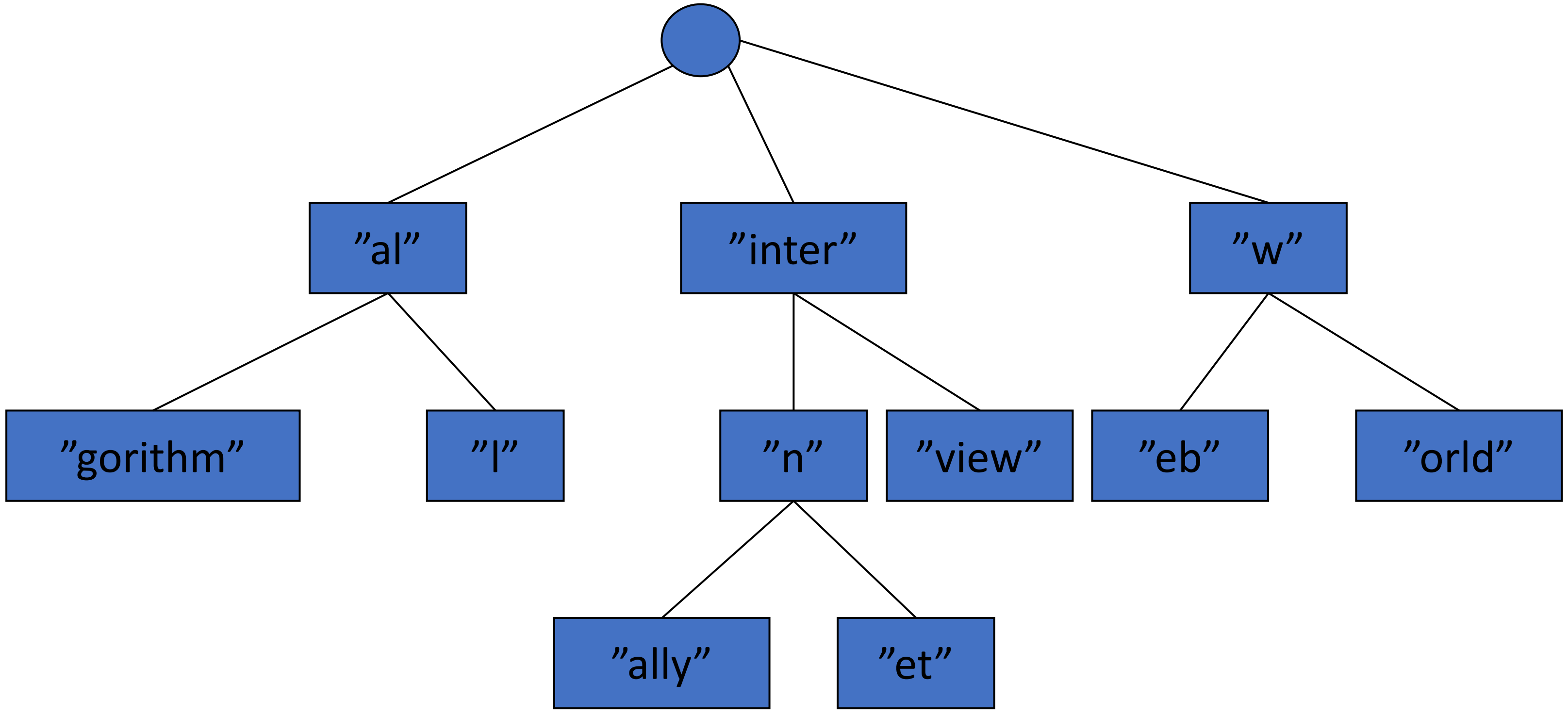
- T may also gradually change over time. We then have to update the index for each such change.
  - The index of a search engine is updated when the crawler finds a new web page.

Tries (word play on Tree / Retrieval)

In the textbook there is an error here

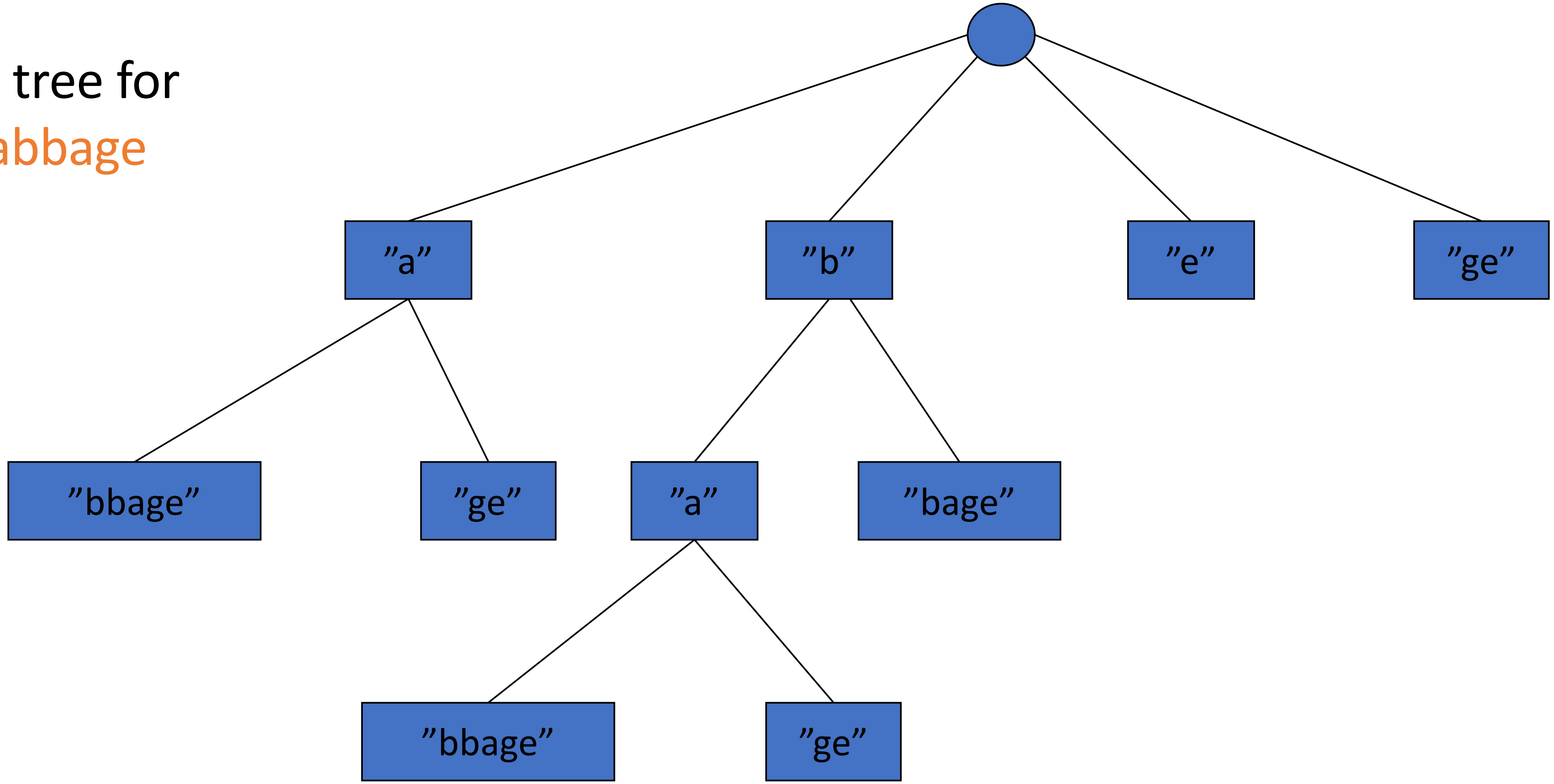


Compressed trie

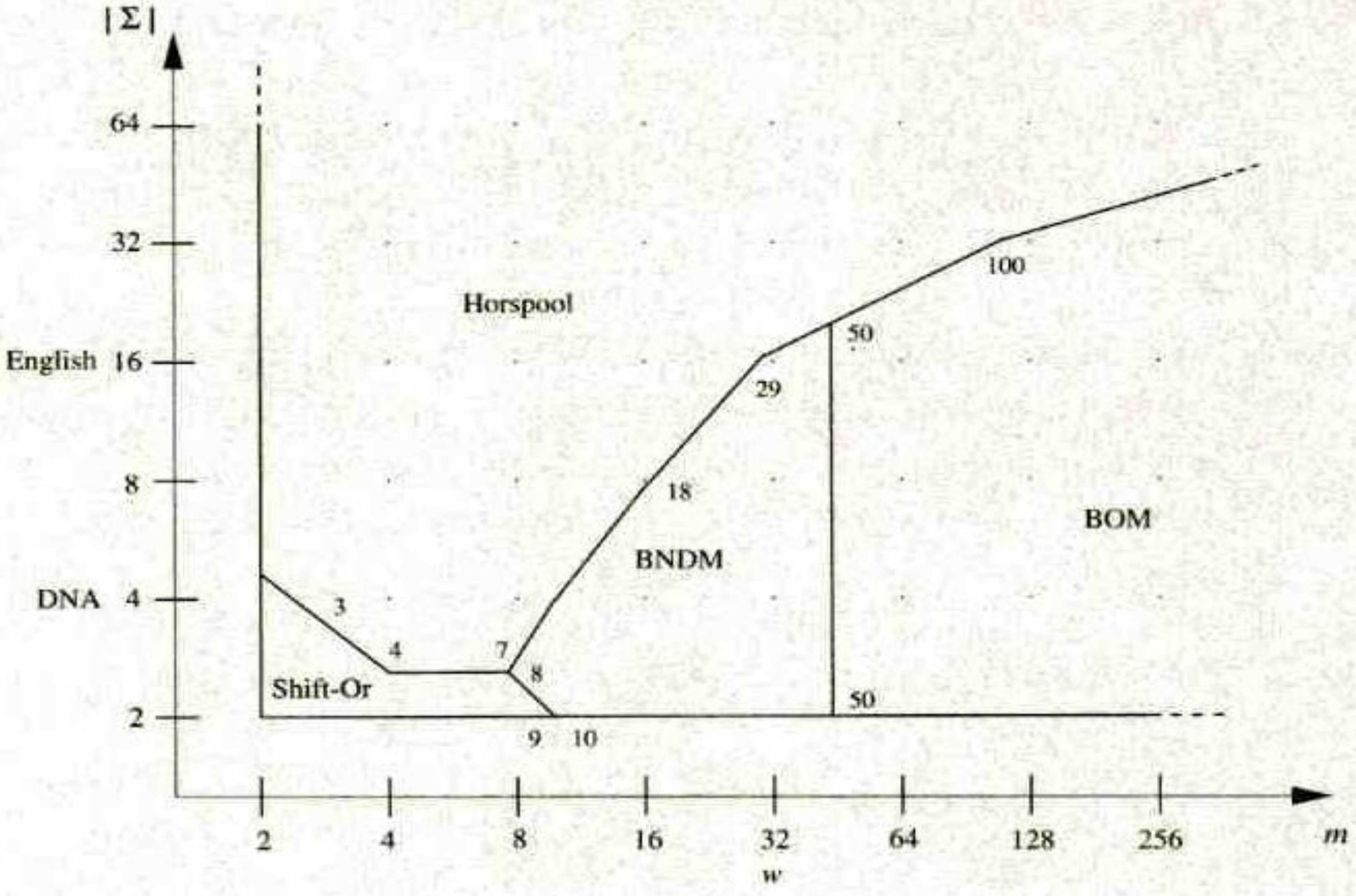


Suffix trees (compressed)

Suffix tree for
T = babbage



- Looking for *P* in this trie will decide whether *P* occurs as a substring of *T*, all substrings have a path strting in the root.


|Σ|
64
32
English 16
8
DNA 4
2
Horspool
BNDM
BOM
Shift-Or
3
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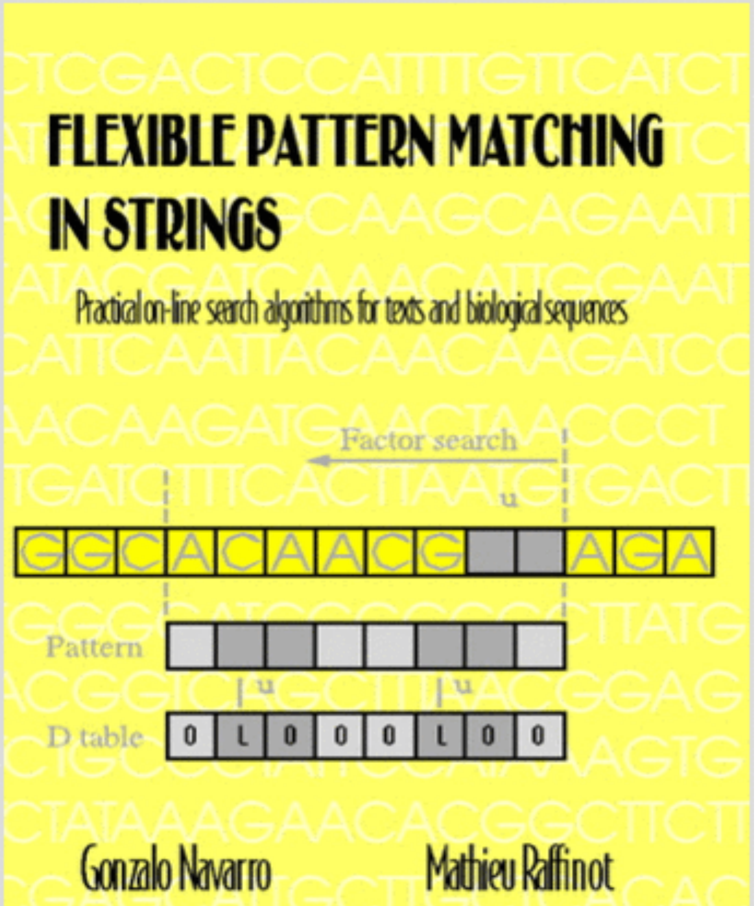
FLEXIBLE PATTERN MATCHING IN STRINGS
Practical on-line search algorithms for texts and biological sequences
Factor search
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Pattern
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