Dynamics part 1

In general: Dynamics describes the connection between an objects motion and the forces acting upon it.

When dealing with a manipulator and we want it to move it from its starting position to a target position, we need to know the dynamic properties of the manipulator so that we can predict how the various forces and accelerations will affect the movement.

- If for example too small forces are applied to produce movement, we may have a too slow movement.
- If the forces are too strong, we may risk making the manipulator collide with other objects in the environment or have it swinging excessively with respect to the desired position.

From forward-, inverse-, and velocity Kinematics we have described the motion of the robot <u>without</u> the forces that produce the motions.

Soooo, we need to be able to express the dynamics of the robot, such that we are able to control the motions of the manipulator in a controlled and nice way.

- How much force do each of the respective joints need in order to move?
- How much force is needed to keep the respective joints in place?

A typical task in dynamics is: derive the dynamical model for the manipulator using the Euler-Lagrange equations

Euler-Lagrange:

$$\tau_k = \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k}$$

Dynamic motion of the robotic manipulator:

$$\tau = D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q)$$

- The goal is usually to express the dynamics on this form.

→ We first want to find the Lagrangian term L.

→ Then we use the Euler-Lagrange equation to find τ_K for each joint (if we have 3 joints we have to find τ_1, τ_2, τ_3).

- We use the Lagrangian L for this.
- We have to derivate with respect to time

→ At last we have that

$$\tau = \begin{bmatrix} \tau_1 \\ \vdots \\ \tau_n \end{bmatrix}$$

- n is the number of joints
- $\tau_1 - \tau_n$ are the Euler-Lagrange expressions for each joint.

we use this to write the dynamic equation for the robot:

$$\tau = D(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q)$$

Components of the dynamic model for a manipulator

$$\tau = D(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q)$$

This can be written as following for a 3DOF (3joint) manipulator:

$$\tau = \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{bmatrix} \ddot{q} + \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \dot{q} + \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix}$$

Each term in the dynamical model represents forces that act on the robot (this is described in detail in a book called "Robotics" by Bruno Siciliano (you can find it if you search for it on google ü)).

- $D(q)\ddot{q}$ → the acceleration term
  - $D(q)$ describes the inertia.
    - The diagonal terms of the matrix $D(q)$ represents the moment of inertia at joint i in the manipulator while the other joints are blocked.
    - All other terms represent the effect of acceleration on the joint.

- $C(q, \dot{q})\dot{q}$ → the quadratic velocity term
  - $C(q, \dot{q})$ describes the centrifugal and coriolis forces
    - The diagonal terms describe the centrifugal effect.
    - All other terms describe the coriolis effect.
- $g(q)$ → the gravity term
  - A vector that describes the force exerted on the robot due to gravity.
- $\ddot{q}$ and $\dot{q}$ are vectors
  - $\dot{q}$ is the velocity of the joint variable
    - either $\dot{\theta}$ or $\dot{d}$ based on joint type
  - $\ddot{q}$ is the acceleration of the joint variable
    - either $\ddot{\theta}$ or $\ddot{d}$ based on joint type

Lagrangian intro

- Before we get to the dynamical model of the robot, we have to do some calculations.
- The first step is to calculate the Lagrangian term.

Lagrangian: The difference between Kinetic energy and potential energy.

- $L = K - P$
- We have to make some assumptions about our manipulator in order to be able to calculate the energies.
  - Body composition: a joint and the attached link can be counted as one body/mass or not - some links may count as massless. The task usually presents how many masses there are and what they represent.
  - Body shape: a link with or without its joint can be considered a rectangular/cylindrical solid.

- Mass distribution: mass can be distributed uniformly (or non-uniformly) or concentrated in a single point (point mass).

These assumptions influence how we calculate the dynamical model. Assumptions about body compositions dictate how many masses are present and assumptions about body shape and mass distribution determine how we calculate the energy. ⇒ Read the task carefully.

Finding the Lagrangian - components

$L = K - P$ = kinetic energy - potential energy.

- Kinetic energy:

$$K = \frac{1}{2} m v^T v + \frac{1}{2} \omega^T I \omega$$ For one-link

$$K = \frac{1}{2} \dot{q}^T \left[\sum_{i=1}^{n} m_i J_{vi}(q)^T J_{vi}(q) + J_{\omega i}(q)^T R_i(q) I_i R_i(q)^T J_{\omega i}(q) \right] \dot{q}$$ For n-DOF links

- $v_i = J_v(q)\,\dot{q}$ - $\omega_i = J_\omega(q)\,\dot{q}$

- I_i - inertia tensor ($I_i = R_i I_i R_i^T$ in general)
  → inertia tensor of mass m_i with respect to the base frame.

also called inertia tensor of mass m_i - but it's with respect to the coordinate frame attached to body of mass m_i.

$$I_i = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$$

- gives us an idea about how mass is distributed in a rigid body.

→ I_i is usually given and you have to find I_i yourself.

- R_i - rotation matrices

$$R_i = \begin{bmatrix} R_{xx} & R_{xy} & R_{xz} \\ R_{yx} & R_{yy} & R_{yz} \\ R_{zx} & R_{zy} & R_{zz} \end{bmatrix}$$

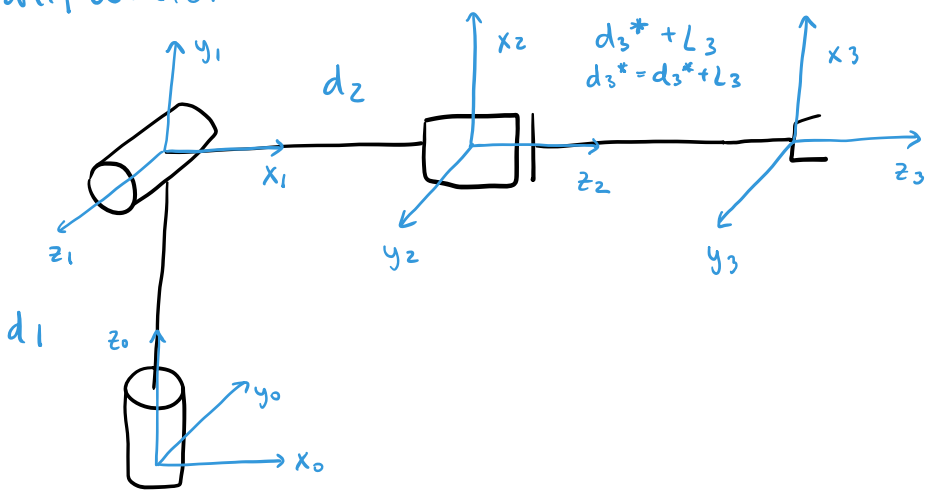
- Potential energy

$$P = mgh$$ For one-link

- $h = r_{ci}$ = height

$$P = \sum_{i=1}^{n} m_i g^T r_{ci}$$ For n-DOF links

Example - how to find the lagrangian for a spherical manipulator:



1) Find the Jacobian for the manipulator

$$T_1^0 = \begin{bmatrix} s_1 & 0 & s_1 & 0 \\ c_1 & 0 & -c_1 & 0 \\ 0 & 1 & 0 & d_1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Transformation matrices

$$T_2^0 = \begin{bmatrix} -c_1c_2 & s_1 & c_1c_2 & 0 \\ -s_1s_2 & -c_1 & c_2s_1 & 0 \\ c_2 & 0 & s_2 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_3^0 = \begin{bmatrix} -c_1c_2 & s_1 & c_1c_2 & c_1c_2(d_3+d_2) \\ -s_1s_2 & -c_1 & c_2s_1 & c_2s_1(d_3+d_2) \\ c_2 & 0 & s_2 & d_1 + s_2(d_3+d_2) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Helpful variables:

$$z_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad z_1 = \begin{bmatrix} s_1 \\ -c_1 \\ 0 \end{bmatrix}, \quad z_2 = \begin{bmatrix} c_1c_2 \\ c_2s_1 \\ s_2 \end{bmatrix}$$

$$O_3 - O_0 = \begin{bmatrix} c_1c_2(d_3+d_2) \\ c_2s_1(d_3+d_2) \\ d_1 + s_2(d_3+d_2) \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} c_1c_2(d_3+d_2) \\ c_2s_1(d_3+d_2) \\ d_1 + s_2(d_3+d_2) \end{bmatrix}$$

$$O_3 - O_1 = \begin{bmatrix} c_1c_2(d_3+d_2) \\ c_2s_1(d_3+d_2) \\ d_1 + s_2(d_3+d_2) \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ d_1 \end{bmatrix} = \begin{bmatrix} c_1c_2(d_3+d_2) \\ c_2s_1(d_3+d_2) \\ s_2(d_3+d_2) \end{bmatrix}$$

Finding J_v and J_w:

$$J_{v_1} = z_0 \times (o_3 - o_0) = \begin{bmatrix} -c_2 s_1 (d_3 + d_2) \\ c_1 c_2 (d_3 + d_2) \\ 0 \end{bmatrix}$$

$$J_{v_2} = z_1 \times (o_3 - o_1) = \begin{bmatrix} -c_1 s_2 (d_3 + d_2) \\ -s_1 s_2 (d_3 + d_2) \\ c_2 (d_3 + d_2) \end{bmatrix}$$

$$J_{v_3} = z_2 = \begin{bmatrix} c_1 c_2 \\ c_2 s_1 \\ s_2 \end{bmatrix}$$

$$J_{w_1} = z_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$J_{w_2} = z_1 = \begin{bmatrix} s_1 \\ -c_1 \\ 0 \end{bmatrix}, \quad J_{w_3} = 0$$

$J_{v_i} = z_{i-1} \times (o_n - o_{i-1})$ rev.
$J_{v_i} = z_{i-1}$ lin.

$J_{w_i} = z_{i-1}$ rev.
$J_{w_i} = 0$ lin.

Putting it together yields the jacobian matrix

$$J = \begin{bmatrix} J_v \\ J_w \end{bmatrix} = \begin{bmatrix} J_{v_1} & J_{v_2} & J_{v_3} \\ J_{w_1} & J_{w_2} & J_{w_3} \end{bmatrix}$$

$$= \begin{bmatrix} -c_2 s_1 (d_3 + d_2) & -c_2 s_1 (d_3 + d_2) & c_1 c_2 \\ c_1 c_2 (d_3 + d_2) & -s_1 s_2 (d_3 + d_2) & c_2 s_1 \\ 0 & c_2 (d_3 + d_2) & s_2 \\ 0 & s_1 & 0 \\ 0 & -c_1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{matrix} \} J_v \\ \\ \\ \} J_w \\ \\ \end{matrix}$$

2) Finding the Kinetic energy

- We find the kinetic energy for each mass separately and add the kinetic energies for each mass to get the total kinetic energy.
  - Since our example has 3 joints and thus 3 separate masses, the total kinetic energy is: $K = K_1 + K_2 + K_3$.
  - we calculate kinetic energy for each mass present and not the joints/links.
- We use the jacobian matrix as a starting point.

- The idea is that we calculate the kinetic energy of each mass separately and completely ignore the next masses.

By pretending that only joints up to the i-th exists.

  - So when calculating K_1 we only look at mass 1 and ignore all variables that deal with masses 2 and 3.
  - when calculating K_2 we look at masses 1 and 2 and ignore variables that deal with mass 3.
  - For mass 3, we look at the whole robot.

Finding K_1: $K_1 = \frac{1}{2} m_1 v_1^T v_1 + \frac{1}{2} \omega_1^T I_1 \omega_1$

→ we set all values that are connected to mass 2 and 3 to 0 (because we only focus on mass 1).

→ we set column 2 and 3 in the jacobian to 0 because they are related to mass 2 and 3.

This gives us:

$$J_v = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad J_w = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

→ we have inertia tensor $I_1 = 0$ (defined usually in the task). Be aware that I_1 is a matrix, so since it's 0, it will be filled with 0s.

→ The linear and angular velocities are:

we need the linear and angular velocity of each mass

$$v_1 = J_{v_1} \dot{q} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\omega_1 = J_{w_1} \dot{q} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dot{q}_1 \end{bmatrix}$$

→ We put it all together to get K_1:

$$K_1 = \frac{1}{2} m_1 v_1^T v_1 + \frac{1}{2} \omega_1^T I_1 \omega_1$$

$$= \frac{1}{2} m_1 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 \\ 0 \\ \dot{q}_1 \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{q}_1 \end{bmatrix} = 0$$

Finding K_2 : $K_2 = \frac{1}{2} m_2 v_2^T v_2 + \frac{1}{2} \omega_2^T I_2 \omega_2$

→ we set all values that are connected to mass 3 to 0 (because we only focus on masses 1 and 2).

→ we set column 3 in the jacobian to 0 because it is related to mass 3.

This gives us:

$$J_v = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad J_w = \begin{bmatrix} 0 & s_1 & 0 \\ 0 & -c_1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

→ we have inertia tensor $I_2 = 0$ here too

→ The linear and angular velocities are:

$$v_2 = J_{v_2} \dot{q} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\omega_2 = J_{\omega 2} \dot{q} = \begin{bmatrix} 0 & s_1 & 0 \\ 0 & -c_1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \begin{bmatrix} \dot{q}_2 s_1 \\ \dot{q}_2 c_1 \\ \dot{q}_1 \end{bmatrix}$$

→ we put it all together to get K_2:

$$K_1 = K_2 = \frac{1}{2} m_2 v_2^T v_2 + \frac{1}{2} \omega_2^T I_2 \omega_2$$

$$= \frac{1}{2} m_2 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} \dot{q}_2 s_1 \\ \dot{q}_2 c_1 \\ \dot{q}_1 \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_2 s_1 \\ \dot{q}_2 c_1 \\ \dot{q}_1 \end{bmatrix} = 0$$

Finding K_3: $K_3 = \frac{1}{2} m_3 v_3^T v_3 + \frac{1}{2} \omega_3^T I_3 \omega_3$

→ we don't set any values to 0 because we are calculating the kinetic energy for the last mass and now all previous masses are included.

This gives us:

$$J_v = \begin{bmatrix} -c_2 s_1 (d_3 + d_2) & -c_2 s_1 (d_3 + d_2) & c_1 c_2 \\ c_1 c_2 (d_3 + d_2) & -s_1 s_2 (d_3 + d_2) & c_2 s_1 \\ 0 & c_2 (d_3 + d_2) & s_2 \end{bmatrix}$$

$$J_w = \begin{bmatrix} 0 & s_1 & 0 \\ 0 & -c_1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

→ we have inertia tensor $I_3 = 0$ here too

→ The linear and angular velocities are:

$$v_3 = J_{v3} \dot{q} = \begin{bmatrix} -c_2 s_1 (d_3 + d_2) & -c_2 s_1 (d_3 + d_2) & c_1 c_2 \\ c_1 c_2 (d_3 + d_2) & -s_1 s_2 (d_3 + d_2) & c_2 s_1 \\ 0 & c_2 (d_3 + d_2) & s_2 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix}$$

$$= \begin{bmatrix} -c_2 s_1 (d_3 + d_2) \dot{q}_1 - c_2 s_1 (d_3 + d_2) \dot{q}_2 + c_1 c_2 \dot{q}_3 \\ c_1 c_2 (d_3 + d_2) \dot{q}_1 - s_1 s_2 (d_3 + d_2) \dot{q}_2 + c_2 s_1 \dot{q}_3 \\ c_2 (d_3 + d_2) \dot{q}_2 + s_2 \dot{q}_3 \end{bmatrix}$$

$$\omega_3 = J_{w3} \dot{q} = \begin{bmatrix} 0 & s_1 & 0 \\ 0 & -c_1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \begin{bmatrix} \dot{q}_2 s_1 \\ \dot{q}_2 c_1 \\ \dot{q}_1 \end{bmatrix}$$

→ we put it all together to get K_3:

$$K_3 = \frac{1}{2} m_3 v_3^T v_3 + \frac{1}{2} \omega_3^T I_3 \omega_3$$

$$= \frac{1}{2} m_3 \begin{bmatrix} -c_2 s_1 (d_3 + d_2) \dot{q}_1 - c_2 s_1 (d_3 + d_2) \dot{q}_2 + c_1 c_2 \dot{q}_3 \\ c_1 c_2 (d_3 + d_2) \dot{q}_1 - s_1 s_2 (d_3 + d_2) \dot{q}_2 + c_2 s_1 \dot{q}_3 \\ c_2 (d_3 + d_2) \dot{q}_2 + s_2 \dot{q}_3 \end{bmatrix}^T$$

$$\begin{bmatrix} -c_2 s_1 (d_3 + d_2) \dot{q}_1 - c_2 s_1 (d_3 + d_2) \dot{q}_2 + c_1 c_2 \dot{q}_3 \\ c_1 c_2 (d_3 + d_2) \dot{q}_1 - s_1 s_2 (d_3 + d_2) \dot{q}_2 + c_2 s_1 \dot{q}_3 \\ c_2 (d_3 + d_2) \dot{q}_2 + s_2 \dot{q}_3 \end{bmatrix}$$

$$+ \frac{1}{2} \begin{bmatrix} \dot{q}_2 s_1 \\ \dot{q}_2 c_1 \\ \dot{q}_1 \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_2 s_1 \\ \dot{q}_2 c_1 \\ \dot{q}_1 \end{bmatrix}$$

$$= \frac{1}{2} m_3 \left((\dot{q}_3 s_2 + \dot{q}_2 c_2 (d_3 + d_2))^2 + \dot{q}_1 c_2 s_1 (d_3 + d_2) - \dot{q}_3 c_1 c_2 \right.$$
$$\left. + (\dot{q}_2 c_1 s_2 (d_3 + d_2))^2 + \dot{q}_3 s_1 c_2 + \dot{q}_1 c_2 c_1 (d_3 + d_2) - \dot{q}_2 s_1 s_2 (d_3 + d_2) \right)$$

The total kinetic energy is given by $K = K_1 + K_2 + K_3$.

3) Finding the potential energy

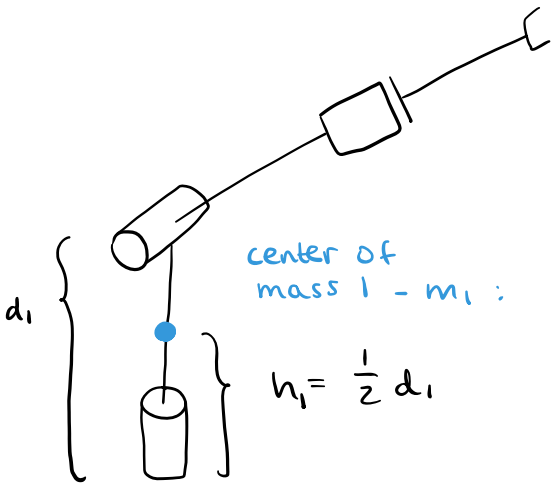
→ we use the formula $P = mgh$ to calculate the potential energy for each mass.

→ We use geometry to calculate the height to the point where the mass is centered (depending on what the task specifies.

→ In this task we have 3 masses, and all are uniformly distributed.

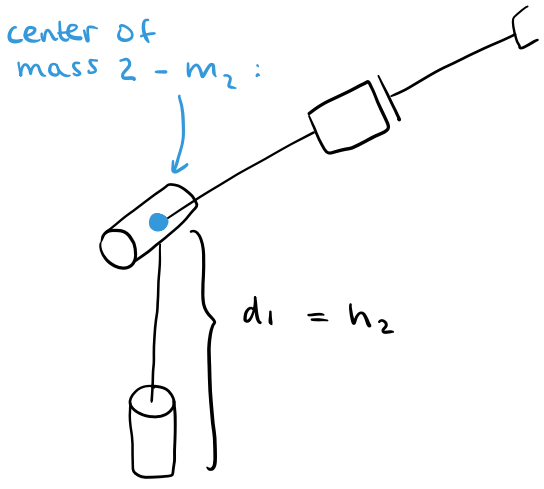
→ g is the gravitational constant

Finding P1: $P_1 = m_1 g h_1$



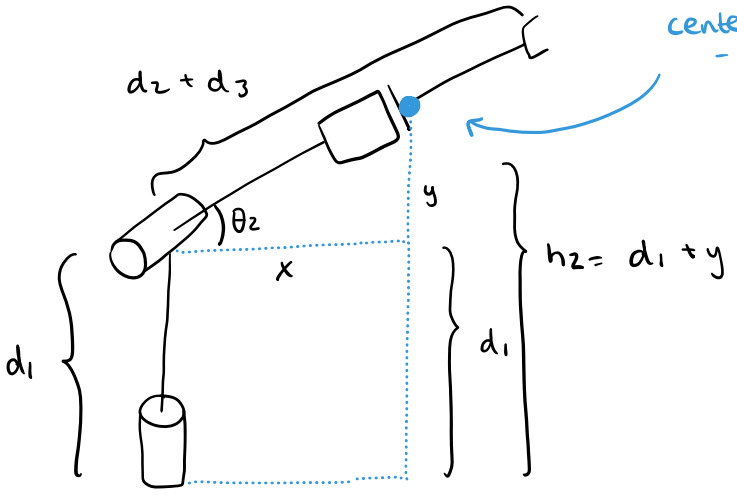
From the drawing we can see that $h_1 = \frac{1}{2} d_1$ so the potential energy is $P_1 = \frac{1}{2} m_1 g d_1$

Finding P2: $P_2 = m_2 g h_2$



From the drawing we can see that $h_2 = d_1$ so the potential energy is $P_2 = m_2 g d_1$

Finding P3: $P_3 = m_3 g h_3$



center of mass 3 - m_3
- middle of the length $d_2 + d_3$

$$\sin\theta_2 = \frac{y}{\frac{1}{2}(d_2+d_3)}$$

$$\Rightarrow y = \frac{1}{2}(d_2+d_3)s_2$$

$$\Rightarrow h_3 = d_1 + \frac{1}{2}(d_2+d_3)s_2$$

$$P_3 = m_3 g h_3 = m_3 g \left(d_1 + \frac{1}{2}(d_2+d_3)s_2\right)$$

Now we can find the total potential energy:

$$P = P_1 + P_2 + P_3 = \frac{1}{2}m_1 g d_1 + m_2 g d_1 + m_3 g \left(d_1 + \frac{1}{2}(d_2+d_3)s_2\right)$$

4) Assembling the Lagrangian term

→ Now that we have found the kinetic and potential energy, we can assemble the Lagrangian:

$$L = K - P = (K_1 + K_2 + K_3) - (P_1 + P_2 + P_3)$$

$$= \frac{1}{2} m_3 \Big((\dot{q}_3 s_2 + \dot{q}_2 c_2 (d_3+d_2))^2 + \dot{q}_1 c_2 s_1 (d_3+d_2) - \dot{q}_3 c_1 c_2$$

$$+ (\dot{q}_2 c_1 s_2 (d_3+d_2))^2 + \dot{q}_3 s_1 c_2 + \dot{q}_1 c_2 c_1 (d_3+d_2) - \dot{q}_2 s_1 s_2 (d_3+d_2) \Big)$$

$$- \frac{1}{2} m_1 g d_1 - m_2 g d_1 - m_3 g \left(d_1 + \frac{1}{2}(d_2+d_3)s_2\right)$$

That's it! :)

→ Now we have found the Lagrangian and we are one step closer to describing the dynamics of our manipulator. This will be the theme of next weeks group session.

Tips for the oblig:

1) start early!!!!

   → There are a lot of calculations and math to do, so you should do yourself a favor and start early!!!

2) The expressions might be long and a "little" bit ugly, so you can use python/matlab in order to check the expressions you got.

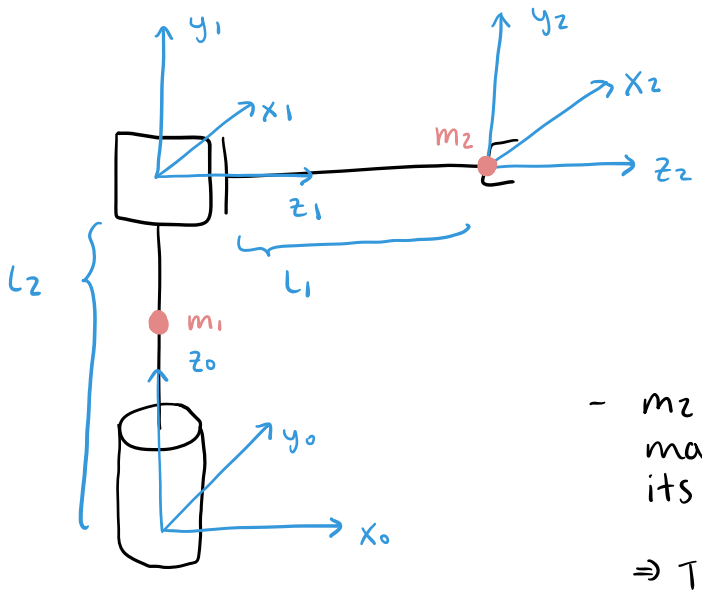
3) Try to do task 1 by hand - this is relevant for the exam as you will most likely have to calculate dynamics for a 2 DOF robot.

4) Don't solve task 2 by hand!! As a previous group teacher said - you will hate yourself beyond recognition :).

   → Use python or matlab here if you want to be happy :)

5) Read the task and the information about the robot (like inertia or mass distribution) CAREFULLY as there are "hints" about what you have to do.

Example - dynamics of a planar 2DOF robot



- L_1 is a rectangular solid with uniformly distributed mass. This is equivalent to having its mass concentrated in the middle of the link. This mass is m_1.

- m_2 is a point mass. A point mass has no inertia, so its inertia tensor is always 0.

⇒ This means that its rotational kinetic term is always 0, so you don't have to calculate it at all. (So, $K_{w_2} = 0$.).

- The first link is a rectangular solid with uniformly distributed mass m_1. The inertia tensor for m_1 is:

$$I_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{1,zz} \end{bmatrix}$$

Task: Find the Lagrangian term.

1) Find T_1^0 and T_2^0.

DH-table:

i	θ_i	d_i	α_i	a_i
1	$\theta_1^* + 90°$	L_1	90°	0
2	0	L_2^*	0	0

$$T_1^0 = A_1 = \begin{bmatrix} -s_1 & 0 & c_1 & 0 \\ c_1 & 0 & s_1 & 0 \\ 0 & 1 & 0 & L_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & L_2^* \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_2^0 = A_1 A_2 = \begin{bmatrix} -s_1 & 0 & c_1 & c_1 L_2^* \\ c_1 & 0 & s_1 & s_1 L_2^* \\ 0 & 1 & 0 & L_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2) Find the Jacobian

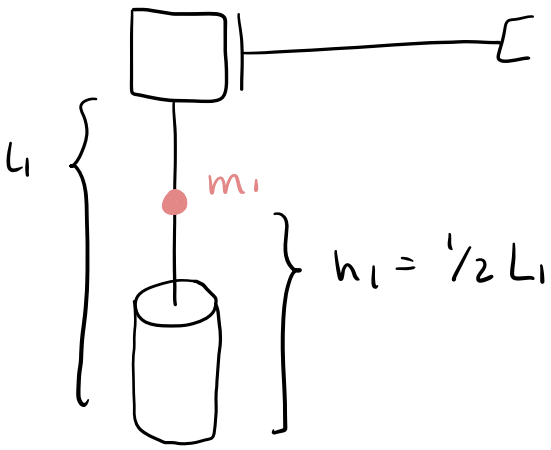
$$J = \begin{bmatrix} J_v \\ J_w \end{bmatrix} \qquad J_v = [\, z_0 \times (o_2 - o_0) \quad z_1 \,] \qquad J_w = [\, z_0 \quad 0 \,]$$

$$J_v = \begin{bmatrix} -s_1 L_2 * & c_1 \\ c_1 L_2 * & s_1 \\ 0 & 0 \end{bmatrix} \qquad J_w = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

3) Calculate the potential and Kinetic energy

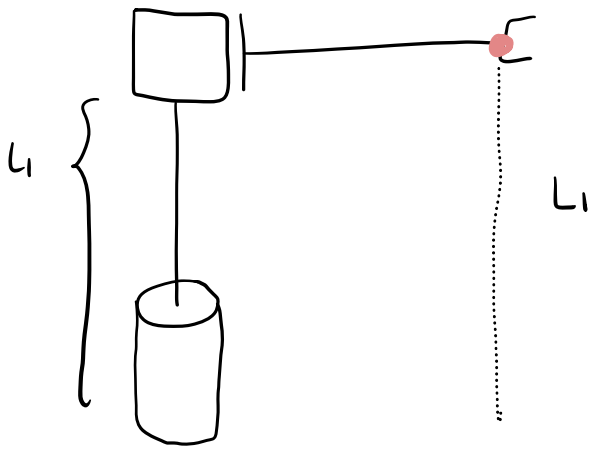
Potential : $P = P_1 + P_2 = m_1 g h_1 + m_2 g h_2$

mass 1:



- m_1 is uniformly distributed, so its at the center of the body/link
- $P_1 = m_1 g h_1 = 1/2\, m_1 g L_1$

mass 2 :



- m_2 is a point mass, so the mass center is in that point.
- $P_2 = m_2 g h_2 = m_2 g L_1$

Total potential energy:

$P = P_1 + P_2 = 1/2 m_1 g L_1 + m_2 g L_1$

Kinetic :

$$K = K_1 + K_2$$
$$= \tfrac{1}{2} m_1 v_1^T v_1 + \tfrac{1}{2} \omega_1^T I_1 \omega_1 + \tfrac{1}{2} m_2 v_2^T v_2 + \tfrac{1}{2} \omega^T I_2 \omega_2$$

Components :

$$v_1 = J_{v_1} \dot{q} = \begin{bmatrix} -s_1 \cdot 0 & 0 \\ c_1 \cdot 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{L}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$v_2 = J_{v_2} \dot{q} = \begin{bmatrix} -s_1 L_2 & c_1 \\ c_1 L_2 & s_1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{L}_2 \end{bmatrix} = \begin{bmatrix} -s_1 L_2 \dot{\theta}_1 + c_1 \dot{L}_2 \\ c_1 L_2 \dot{\theta}_1 + s_1 \dot{L}_2 \\ 0 \end{bmatrix}$$

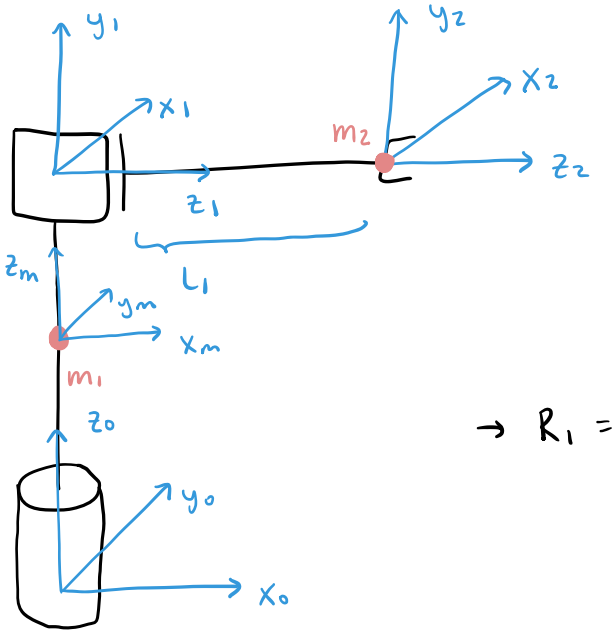
$$\omega_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{L}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix}$$

→ There is no point in calculating ω_2 because m_2 is a point mass and has inertia tensor equal to 0, so its angular kinetic velocity component will be 0.

→ The last component is I_1 – the inertia tensor. As described earlier, we have $I_1 = R_1 I_1 R_1^T$.

To compute this, we need to know R_1, while I_1 was given.

→ R_1 is given by assigning a coordinate frame to the mass and finding the rotation from frame 0 to this frame.



→ The rotation between frame 0 and frame m is given by $R_{(z,\theta_1)}$, so a rotation around z_0 about an arbitrary angle theta θ_1.

$$\rightarrow R_1 = \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

→ Calculating $I_1 = R_1 I_1 R_1^T$

$$I_1 = \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{1,zz} \end{bmatrix} \begin{bmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{1,zz} \end{bmatrix} \begin{bmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{1,zz} \end{bmatrix}$$

Now we have all components, so we can calculate K_1 and K_2.

$$K_1 = \tfrac{1}{2} m_1 v_1^T v_1 + \tfrac{1}{2} \omega_1^T I_1 \omega_1$$

$$= \frac{1}{2} m_1 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{1,zz} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 I_{1,zz} \end{bmatrix}^T \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} = \tfrac{1}{2} \dot{\theta}_1^2 I_{1,zz}$$

$$K_2 = \tfrac{1}{2} m_2 v_2^T v_2 + \tfrac{1}{2} \omega_2^T I_2 \omega_2$$

$$= \frac{1}{2} m_2 \begin{bmatrix} -s_1 L_2 \dot{\theta}_1 + c_1 \dot{L}_2 \\ c_1 L_2 \dot{\theta}_1 + s_1 \dot{L}_2 \\ 0 \end{bmatrix}^T \begin{bmatrix} -s_1 L_2 \dot{\theta}_1 + c_1 \dot{L}_2 \\ c_1 L_2 \dot{\theta}_1 + s_1 \dot{L}_2 \\ 0 \end{bmatrix}$$

$$= \frac{1}{2} m_2 \left(s_1^2 L_2^2 \dot{\theta}_1 - s_1 c_1 L_2 \dot{L}_2 \dot{\theta}_1 - s_1 c_1 L_2 \dot{L}_2 \dot{\theta}_1 + c_1^2 \dot{L}_2^2 \right.$$
$$\left. + c_1^2 L_2^2 \dot{\theta}_1 + s_1 c_1 L_2 \dot{L}_2 \dot{\theta}_1 + s_1 c_1 L_2 \dot{L}_2 \dot{\theta}_1 + s_1^2 \dot{L}_2^2 \right)$$

$$= \frac{1}{2} m_2 \left(L_2^2 \dot{\theta}_1 + \dot{L}_2^2 \right)$$

Total kinetic energy: $K = K_1 + K_2 = \frac{1}{2} \dot{\theta}_1^2 I_{1,zz} + \frac{1}{2} m_2 \left(L_2^2 \dot{\theta}_1 + \dot{L}_2^2 \right)$

The Lagrangian: $L = K - P$

$$\underline{L = \tfrac{1}{2} \dot{\theta}_1^2 I_{1,zz} + \tfrac{1}{2} m_2 \left(L_2^2 \dot{\theta}_1 + \dot{L}_2^2 \right) - \tfrac{1}{2} m_1 g L_1 - m_2 g L_1}$$