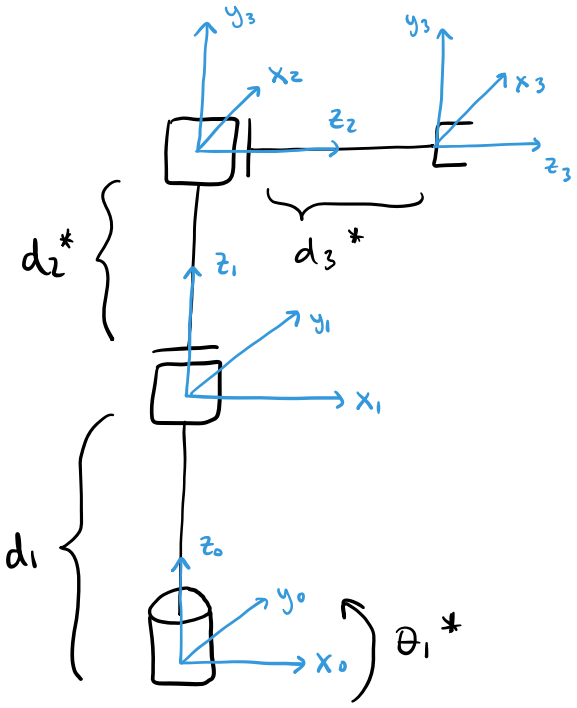
Inverse Kinematics part 1

Repetition forward kinematics

Forward kinematics: You have the angles or length (variables) of the joints and you want to know where the tooltip ends in space.

Example - calculating forward kinematics of a robot:



→ cylindrical manipulator

1) Add coordinate frames to the drawing

Rules:
- z in the direction of action
- x_i 90° to z_{i-1}
- x_i needs to intersect z_{i-1}
  → if not, we move the origin o_i, so they intersect.
- y from the right hand rule

2) Fill in the DH-table

		θ_i	d_i	a_i	α_i
links	1	θ_1^*	d_1	0	0
	2	90°	d_2^*	0	90°
	3	0	d_3^*	0	0

→ θ_i: angle of rotation for a revolute joint
- angle from x_{i-1} to x_i about z_{i-1}

→ d_i: distance moved for a prismatic joint
- distance from o_{i-1} to intersection of x_i and z_{i-1} about z_{i-1}

→ variables for the joints (*)

→ a_i: distance from intersection of x_i and z_{i-1} to o_i

→ α_i: angle from z_{i-1} to z_i measured about x_i

→ essentially we want to find the transformations between these 4 frames.

3) For each link 1, 2, 3 we use the "special matrix" and fill in all the values θ_i, d_i, a_i, α_i that we found:

$$A_i = \begin{bmatrix} C\theta_i & -S\theta_i C\alpha_i & S\theta_i S\alpha_i & a_i C\theta_i \\ S\theta_i & C\theta_i C\alpha_i & -C\theta_i S\alpha_i & a_i S\theta_i \\ 0 & S\alpha_i & C\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow \text{the special matrix}$$

$$A_1 = H_1^0 = \begin{bmatrix} C\theta_1 & -S\theta_1 & 0 & 0 \\ S\theta_1 & C\theta_1 & 0 & 0 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = H_2^1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_3 = H_3^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

4) Find transformation from base frame to tool tip frame

- In forward kinematics we essentially want to find the transformation from base frame to tooltip frame, because we want to know where the tooltip is in space.
- we want to find H_3^0 - remember $H_y^x = H_z^x H_y^z$.

$$H_3^0 = H_1^0 H_2^1 H_3^2$$

$$= \begin{bmatrix} C\theta_1 & -S\theta_1 & 0 & 0 \\ S\theta_1 & C\theta_1 & 0 & 0 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -S\theta_1 & 0 & C\theta_1 & 0 \\ C\theta_1 & 0 & S\theta_1 & 0 \\ 0 & 1 & 0 & d_1 + d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -S\theta_1 & 0 & C\theta_1 & C\theta_1 d_3 \\ C\theta_1 & 0 & S\theta_1 & S\theta_1 d_3 \\ 0 & 1 & 0 & d_1 + d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow \text{the forward kinematics}$$

Lets say we want to find the point where the tooltip is in our world frame. We have the info:

$\theta_1 = 90^\circ$, $d_1 = 100\ mm$, $d_2 = 230\ mm$, $d_3 = 84\ mm$

$$H_3^0 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 84 \\ 0 & 1 & 0 & 330 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

this is where the tooltip is in the world frame

degrees	radians	sin θ	cos θ	tan θ	csc θ	sec θ	cot θ
0^0	0	0	1	0	Not defined	1	Not defined
30^0	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
45^0	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
60^0	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$
90^0	$\frac{\pi}{2}$	1	0	Not defined	1	Not defined	0

Inverse Kinematics

Inverse Kinematics: Given the position of the tooltip, what are the joint variables θ_i and d_i?

→ more useful than forward kinematics, because we often know the point we want to reach (where we want the tool tip to be), but not how to get there (what joint angles and extentions we have to use in order to get there.

→ more difficult cause it requires thinking outside the box.

Method 1 – the analytic approach → recommended :)

→ The idea is to make expressions and solve them algebraically.

→ we want to use the expression for the position of the tooltip, which corresponds to the last column of our forward kinematics matrix (it is a translation vector (x, y, z)).

→ we set up a set of equations on the form:

$x = a$
$y = b$
$z = c$

translation vector from forward kinematics matrix: $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$

→ the goal is to put the variables on the left hand side of the equation

→ we do this by manipulating the expressions algebraically

Some useful formulas to use:

Double Angle Identities

$\sin(2\theta) = 2\sin\theta\cos\theta$

$\cos(2\theta) = \cos^2\theta - \sin^2\theta$

$\cos(2\theta) = 2\cos^2\theta - 1$

$\cos(2\theta) = 1 - 2\sin^2\theta$

Half Angle Indentities

$$\sin^2\theta = \frac{1 - \cos(2\theta)}{2}$$

$$\cos^2\theta = \frac{1 + \cos(2\theta)}{2}$$

Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1$$

Quotient Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

Sum to Product of Two Angles

$$\sin\theta + \sin\phi = 2\sin\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right)$$

$$\sin\theta - \sin\phi = 2\cos\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right)$$

$$\cos\theta + \cos\phi = 2\cos\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right)$$

$$\cos\theta - \cos\phi = -2\sin\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right)$$

Product to Sum of Two Angles

$$\sin\theta\sin\phi = \frac{[\cos(\theta-\phi) - \cos(\theta+\phi)]}{2}$$

$$\cos\theta\cos\phi = \frac{[\cos(\theta-\phi) + \cos(\theta+\phi)]}{2}$$

$$\sin\theta\cos\phi = \frac{[\sin(\theta+\phi) + \sin(\theta-\phi)]}{2}$$

$$\cos\theta\sin\phi = \frac{[\sin(\theta+\phi) - \sin(\theta-\phi)]}{2}$$

Sum/Difference Indentities

$$\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \cos\theta\sin\phi$$

$$\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi$$

Example - calculating the inverse kinematics with analytical appr.

From the previous example, we have the forward kinematics:

$$H_3^0 = \begin{bmatrix} -s\theta_1 & 0 & c\theta_1 & c\theta_1 d_3 \\ c\theta_1 & 0 & s\theta_1 & s\theta_1 d_3 \\ 0 & 1 & 0 & d_1 + d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

From this, we get the equations

$$\left.\begin{aligned} x &= c\theta_1^* d_3^* \\ y &= s\theta_1^* d_3^* \\ z &= d_1 + d_2^* \end{aligned}\right\}$$ To solve these, we use the trigonometry formulas and some creativity

→ our variables are θ_1, d_2 and d_3.

1) Finding d_2^*:

$z = d_1 + d_2^* \Rightarrow \underline{d_2^* = z - d_1}$ → both z and d_1 are known

2) Finding θ_1^*:

$$\left.\begin{aligned} x &= c\theta_1^* d_3^* \\ y &= s\theta_1^* d_3^* \end{aligned}\right\} \Rightarrow \frac{y}{x} = \frac{s\theta_1^* d_3^*}{c\theta_1^* d_3^*} = \frac{s\theta_1^*}{c\theta_1^*} = \tan\theta_1^*$$

$$\tan\theta_1^* = \frac{y}{x} \Rightarrow \underline{\theta_1^* = \arctan\left(\frac{y}{x}\right) = \operatorname{atan2}(y,x)}$$

→ Note: we use the atan2 function (instead of the normal arctangent function).

$$\operatorname{atan2}(y,x) = \begin{cases} \arctan\left(\frac{y}{x}\right) & \text{if } x > 0, \\ \arctan\left(\frac{y}{x}\right) + \pi & \text{if } x < 0 \text{ and } y \geq 0, \\ \arctan\left(\frac{y}{x}\right) - \pi & \text{if } x < 0 \text{ and } y < 0, \\ +\frac{\pi}{2} & \text{if } x = 0 \text{ and } y > 0, \\ -\frac{\pi}{2} & \text{if } x = 0 \text{ and } y < 0, \\ \text{undefined} & \text{if } x = 0 \text{ and } y = 0. \end{cases}$$

For Python:

$$\text{atan2}(x,y) = \arctan\left(\frac{x}{y}\right)$$

→ different programming languages have different implementations

3) Finding d_3^*

$$y = s\theta_1^* \, d_3^* \Rightarrow d_3^* = \frac{y}{s_{\theta_1^*}}$$

we know θ_1 from step 2, so now we can just fill it in

OR

$$x^2 + y^2 = (c_1 d_3^*)^2 + (s_1 d_3^*) = c_1^2 d_3^{*2} + s_1^2 d_3^{*2}$$

$$= d_3^{*2}(c_1^2 + s_1^2) = d_3^{*2}$$

$$\Rightarrow d_3^* = \sqrt{x^2 + y^2}$$

So, we have the inverse kinematic:

$$\theta_1^* = \arctan\left(\frac{y}{x}\right) = \text{atan2}(y,x)$$
$$d_2^* = z - d_1$$
$$d_3^* = \sqrt{x^2+y^2}$$

If we want our tool tip to be in point (0, 84, 330), we can use the equations above to calculate the joint variables:

$x = 0$, $y = 84$, $z = 330$, $d_1 = 100\,mm$

$\theta_1 = \arctan(84/0) = \text{atan2}(84, 0) = \pi/2 = 90°$

$d_2^* = 330 - 100 = 230\,mm$

$d_3^* = \sqrt{0^2 + 84^2} = 84\,mm$ or $d_3 = 0$

→ We can see that the forward and inverse kinematics match :D

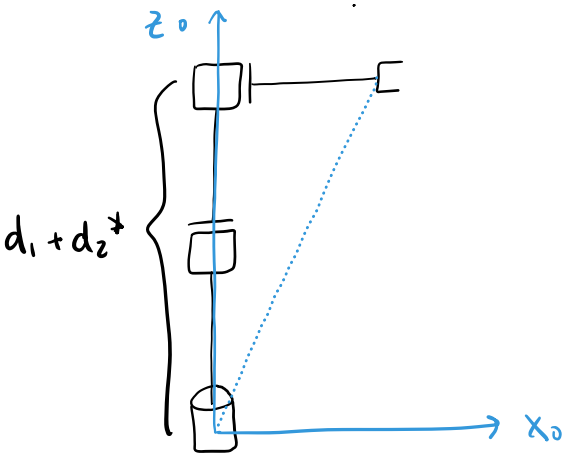
Method 2 - geometric approach

→ the idea is to project the manipulator in various planes and pretend it's in 2D.

→ we look at the manipulator in side view
→ we look at the manipulator from top view
→ and then identify as many equations as possible using trigonometry.

Example - finding the inverse kinematic equations using the geometric approach.

1) Projecting onto the side view plane (x_0, z_0-plane)

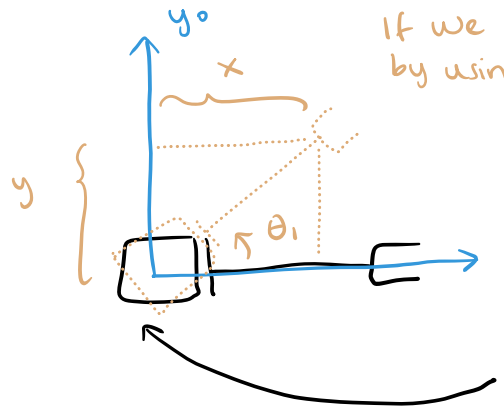


→ The height of the tooltip (which is in the z-direction) is given by the length of link 1 (d_1) and the variable length of link 2 (d_2^*) so:

$$z = d_1 + d_2^* \quad \Rightarrow \quad d_2^* = z - d_1$$

- Note: We cannot find θ_1 from this drawing, cause rotating θ_1 would cause the tooltip to move in and out of the plane, but that's not possible since we're in 2D.

2) Projecting onto the top view plane (x_0, y_0-plane)

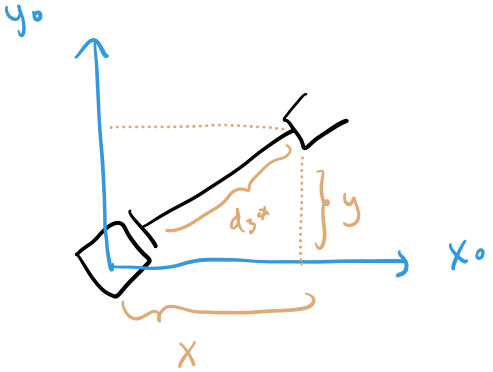


If we draw it like this we can see that we can find θ_1 by using the tangent.

$$\rightarrow \tan\theta_1 = \frac{\sin\theta_1}{\cos\theta_1} = \frac{y}{x}$$

$$\Rightarrow \theta_1 = \arctan\left(y/x\right) = \text{atan2}(y, x)$$

the joints are stacked over eachother



→ Using pythagoras: $c^2 = a^2 + c^2$

$$x^2 + y^2 = d_3^{*2} \quad \Rightarrow \quad d_3^* = \sqrt{x^2 + y^2}$$