UKE 5 - homogene transforma

In robotics we use **frames** (coordinate systems) to describe a robots position and movement. The frames are important for the robot to know where it is and how it should move in order to complete a task. We usually use 3 frames:

- **Base frame**: where the robot is placed - its starting position
- **Task frame**: where the robot is supposed to do a task
- **World frame**: the whole surrounding of the robot.

Homogeneous transforms let us go from one frame to another. They represent the position and orientation of a rigid body (a body that has a solid shape). This means that we can use homogeneous transforms to translate any point represented in one frame to another frame.

A homogeneous transform is described by this matrix

$$H = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix}$$

→ R is a 3x3 rotation matrix
→ d is a 1x3 traslation vector

We have 3 axis on our coordinate systems, so we have 3 rotation matrices: one for rotation around the x-axis, one for y-axis and one for z-axis. They are:

$$R_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

→ rotating the x axis by an angle of θ.

$$R_{y,\theta} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

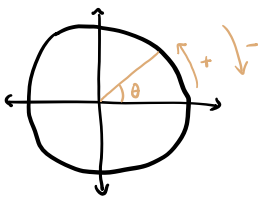
→ rotating the y axis by an angle of θ.

$$R_{z,\theta} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

→ rotating the z axis by an angle of θ.

We often write cos and sin in a shorter way, to take up less space. We have $\cos\theta_1 = c\theta_1$ and $\sin\theta_2 = s\theta_2$.

The axis can either be rotated clockwise or anti clockwise. The **angle is positive when we rotate anti clockwise** and **negative when we rotate clockwise**. Think about the unit circle:



d is the translation vector and it is on the form:

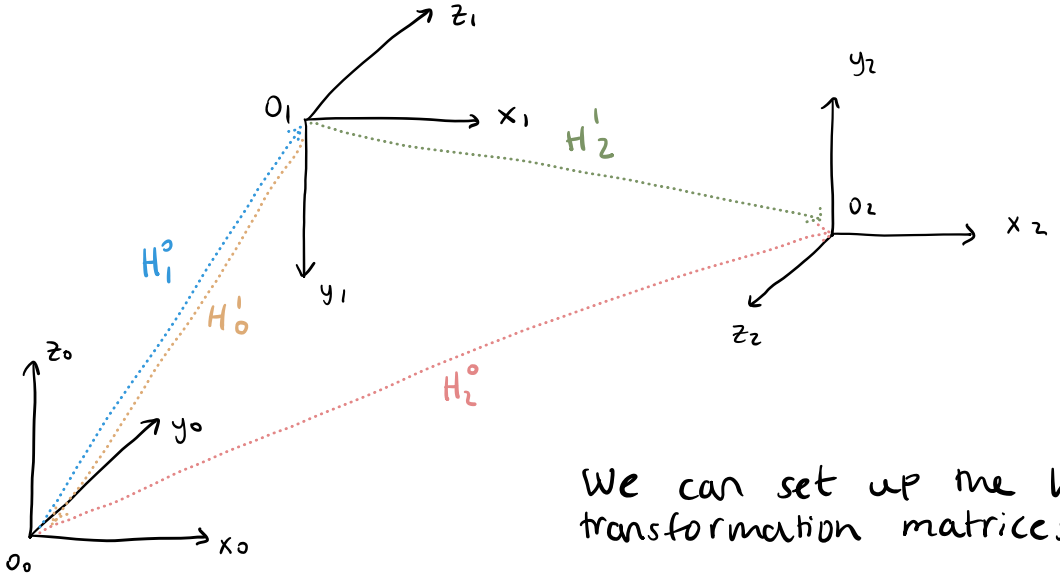
$$d = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

→ translation in x direction
→ translation in y direction
→ translation in z-direction.

How to do calculations with homogeneous transforms

So let's see how we can use it: We have 3 frames and we want to find H_1^0, H_2^0, H_2^1, and H_0^1.

The notation H_y^x means the homogeneous transform from frame x to frame y.



We can set up the homogeneous transformation matrices like this:

$$H_1^0 = \begin{bmatrix} R_1^0 & d_1^0 \\ 0 & 1 \end{bmatrix} \quad H_2^1 = \begin{bmatrix} R_2^1 & d_2^1 \\ 0 & 1 \end{bmatrix} \quad H_2^0 = \begin{bmatrix} R_2^0 & d_2^0 \\ 0 & 1 \end{bmatrix}$$

To find H_0^1 we can use the inverse of H_1^0. This is given by the formula:

$$H_0^1 = (H_1^0)^{-1} = \begin{bmatrix} R_1^{0\,T} & -R_1^{0\,T} d_1^0 \\ 0 & 1 \end{bmatrix}$$

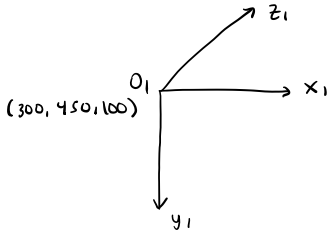
→ T means transposed (the rows become columns and columns become rows).

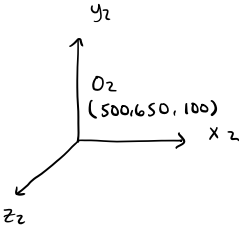
- R_1^0 is the rotation matrix from frame 0 to frame 1. This means how frame 0 rotates to match frame 1.
- d_1^0 is a vector that describes how the frame moves in space so the origin o_0 matches O_1.
- The same applies to R_2^1, R_2^0, d_2^1, d_2^0.

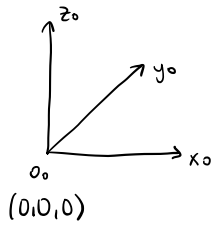
If we say that $o_0 = (0,0,0)$, $o_1 = (300,450,100)$, $o_2 = (500,650,100)$. what are the homogeneous transforms that we defined earlier?

The technique is to find the rotation matrix and then the translation vector for each homogeneous transform.

$$H_1^0 = \begin{bmatrix} R_1^0 & d_1^0 \\ 0 & 1 \end{bmatrix}$$

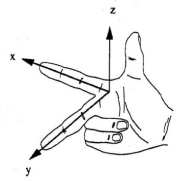


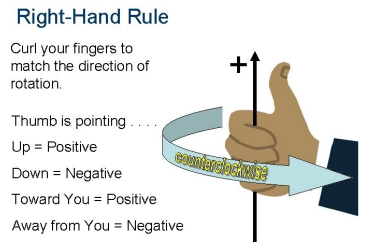




From frames 0 and 1 we can see that x_0 and x_1 are parallel, which means that x_0 is not rotated, so the rotation must be around the x-axis.

Right hand rule:





We use the right hand rules to determine the angle and if its negative or positive.

The angle of rotation between frame 0 and 1 is $-90°$ $(-\pi/2)$. We put this in the rotation matrix for the x-axis:

$$R_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \rightarrow R_1^0 = R_{x,-\pi/2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

The vector d is found by $o_1 - o_0$ which just gives o_1:

$$d = \begin{bmatrix} 300 \\ 450 \\ 100 \end{bmatrix}$$

We insert these into the homogeneous matrix formula to get:

$$H = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix} \rightarrow H_1^0 = \begin{bmatrix} 1 & 0 & 0 & 300 \\ 0 & 0 & 1 & 450 \\ 0 & -1 & 0 & 100 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation matrix

Translation vector

a convention – always like this

We do the similar calculations for H_2^0.

$$H_2^0 = \begin{bmatrix} R_2^0 & d_2^0 \\ 0 & 1 \end{bmatrix}$$

Frame the x-axis is rotated 90° ($\pi/2$) to match frame 1, so $\theta = \pi/2$.

$$R_2^0 = R_{x,\pi/2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

d_2^0 is found by $O_2 - O_0$ which gives $d_2^0 = (500, 650, 100)$.

$$H_2^0 = \begin{bmatrix} 1 & 0 & 0 & 500 \\ 0 & 0 & -1 & 650 \\ 0 & 1 & 0 & 100 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

To find H_0^1 we use the formula for the inverse:

$$H_0^1 = (H_1^0)^{-1} = \begin{bmatrix} R_1^{0\,T} & -R_1^{0\,T} d_1^0 \\ 0 & 1 \end{bmatrix}$$

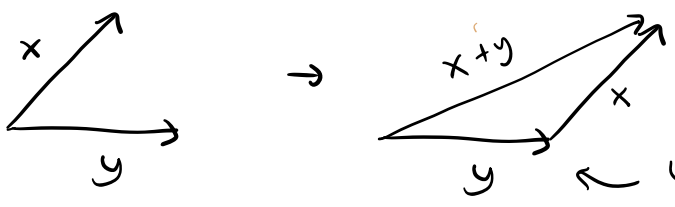
$$R_1^{0\,T} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad -R_1^{0\,T} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$-R_1^{0\,T} d_1^0 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 300 \\ 450 \\ 100 \end{bmatrix} = \begin{bmatrix} -300 \\ 100 \\ -450 \end{bmatrix}$$

$$H_0^1 = \begin{bmatrix} 1 & 0 & 0 & -300 \\ 0 & 0 & -1 & 100 \\ 0 & 1 & 0 & -450 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

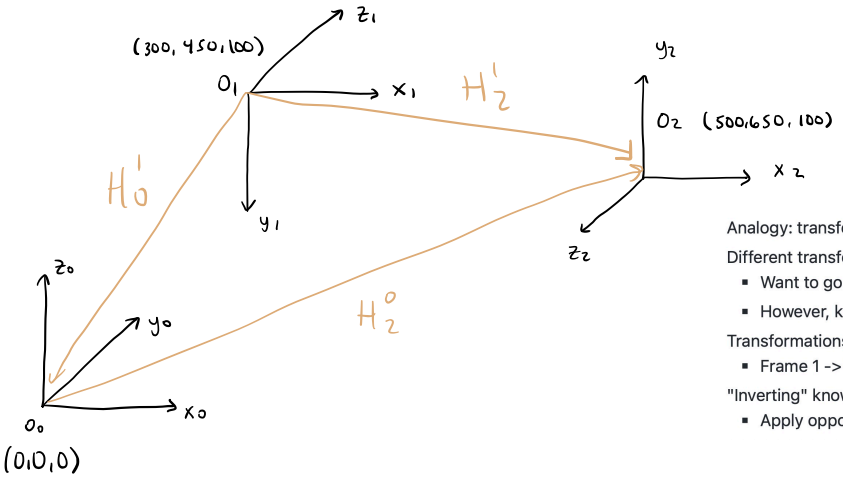
$$H_2^1 = \begin{bmatrix} R_2^1 & d_2^1 \\ 0 & 1 \end{bmatrix}$$

To find H_2^1, then we use vector addition, because that is much easier than looking at the diagrams and trying to figure it out intuitively. We use this concept:



← we turn y around because of the rules for vector addition.

We use the same concept for our coordinate systems:



So to find H_2^1, we have to use H_0^1 and H_2^0.

Analogy: transformation from one frame to another - going between two places (example: home -> school)

Different transformation paths possible. Example:

- Want to go from school to the store, but direct path is unknown.
- However, know these paths: school -> home **and** home -> store

Transformations between frames are similar, all paths are valid:

- Frame 1 -> Frame 3 **OR** Frame 1 -> Frame 2 -> Frame 3

"Inverting" known transformation is possible:

- Apply opposite rotation, then opposite translation

We use a smart trick that helps us combine H_0^1 and H_2^0. The generalization of the rule is:

$H_y^x = H_z^x H_y^z$ → we simply multiply H_z^x and H_y^z (in this order).

→ if we make a diagonal line (always in this direction, not ↘), z and z cancel out so we end up with H_y^x).

Thus, $H_2^1 = H_0^1 H_2^0$. We do some matrix multiplication:

$$H_2^1 = \begin{bmatrix} 1 & 0 & 0 & -300 \\ 0 & 0 & -1 & 100 \\ 0 & 1 & 0 & -450 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 500 \\ 0 & 0 & -1 & 650 \\ 0 & 1 & 0 & 100 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 200 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 200 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

PS: If you have to solve a task where the frames are rotated around more than one axis, you can multiply the rotation matrices together:

$R = R_{y,\phi} R_{z,\theta}$ → here a frame is rotated an angle ϕ around the y-axis and then around z-axis at an angle θ.

→ the order of multiplication must be the same as order of rotation
→ see "compositions of rotations" below.

Compositions of rotations

→ consecutive rotations of a frame with respect to different axes.

A frame can be rotated with respect to more than one axis, so when we want to find the rotation matrix, we have to separate the rotations to find the rotation matrix with respect to one axis and then the other... Then all these are multiplied, but the order matters.

We can rotate the axis with respect to a current frame or fixed frame and this affects how we multiply the rotations together.

- The meaning of current frame is that when you do a first rotation then you have a new position for the frame and when you do another rotation, it is with respect to this current position.

  → Then you **post multiply** → multiply the new rotation matrices after eachother.

  → This is what we would do if our frame rotations were composed of more than one rotation with respect to each other.

- The meaning of fixed frame is that you do all rotations with respect to the base frame.

  → Then you **pre multiply** → multiply the new rotation matrices before eachother (we get them in reverse order)

  → We don't use this soo much, but it's good to know.

For more info about this, read 2.4 in the book and do the tasks for this week.

Parametrizations of rotations

we can represent any rotation between frames in 3 different ways:

1) **Euler angle representation**: Any set of rotation can be described by

- a rotation by ϕ about the z-axis.
- a rotation by θ about the current y-axis
- a rotation ψ about the current z-axis.

This is given by the formula:

$$R_{ZYZ} = R_{z,\phi} R_{y,\theta} R_{z,\psi} = \begin{bmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \begin{bmatrix} c_\psi & -s_\psi & 0 \\ s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ (postmultiplying)

$$= \begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix}$$

2) **Roll, pitch, yaw angles**: Any set of rotations can be described by 3 rotations around the fixed x, y, z-axes.

- yaw - x_0 - ψ
- pitch - y_0 - θ
- roll - z_0 - ϕ

This is given by the formula:

$$R_{XYZ} = R_{z,\phi} R_{y,\theta} R_{x,\psi}$$

$$= \begin{bmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\psi & -s_\psi \\ 0 & s_\psi & c_\psi \end{bmatrix}$$ (pre-multiplying)

$$= \begin{bmatrix} c_\phi c_\theta & -s_\phi c_\psi + c_\phi s_\theta s_\psi & s_\phi s_\psi + c_\phi s_\theta c_\psi \\ s_\phi c_\theta & c_\phi c_\psi + s_\phi s_\theta s_\psi & -c_\phi s_\psi + s_\phi s_\theta c_\psi \\ -s_\theta & c_\theta s_\psi & c_\theta c_\psi \end{bmatrix}$$

3) **Axis/angle representation**: Any set of rotations can be achieved by a single rotation about a suitable axis in space by a suitable angle. This is given by the formula:

$$R_{k,\theta} = \left[\begin{array}{c|c|c} k_x^2 v_\theta + c_\theta & k_x k_y v_\theta - k_z s_\theta & k_x k_z v_\theta + k_y s_\theta \\ k_x k_y v_\theta + k_z s_\theta & k_y^2 v_\theta + c_\theta & k_y k_z v_\theta - k_x s_\theta \\ k_x k_z v_\theta - k_y s_\theta & k_y k_z v_\theta + k_x s_\theta & k_z^2 v_\theta + c_\theta \end{array}\right]$$

k - unit vector defining the axis
θ - angle of rotation around k

Basic homogeneous transforms:

$$H = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix}$$

$$\text{Trans}_{x,a} = \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{Rot}_{x,\alpha} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_\alpha & -s_\alpha & 0 \\ 0 & s_\alpha & c_\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{Trans}_{y,b} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{Rot}_{y,\beta} = \begin{bmatrix} c_\beta & 0 & s_\beta & 0 \\ 0 & 1 & 0 & 0 \\ -s_\beta & 0 & c_\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{Trans}_{z,c} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{Rot}_{z,\gamma} = \begin{bmatrix} c_\gamma & -s_\gamma & 0 & 0 \\ s_\gamma & c_\gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

only describes translation

only describes rotation

The idea is that we can combine these basic homogeneous matrices to find the homogeneous transformations in another way than we did earlier. F.ex if we were to find H_1^0 from our example, we would have to: find the rotation angle and axis and the translation in x, y and z direction, put this info in the matrices and multiply.

$$H_1^0 = \text{Rot}_{x,-90}\ \text{Trans}_{x,300}\ \text{Trans}_{y,450}\ \text{Trans}_{z,100}$$

Tasks to do: 2.10, 2.11, 2.12, 2.13, 2.38, 2.39